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RESEARCH ARTICLE

Jordan tori for a torsion free abelian group

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Abstract We classify Jordan G-tori, where G is any torsion-free abelian group. Using the Zelmanov prime structure theorem, such a class divides into three types, the Hermitian type, the Clifford type, and the Albert type. We concretely describe Jordan G-tori of each type.

Keywords Jordan tori, extended affine Lie algebra, invariant affine reflection algebra
MSC 17B67, 17C50

1 Introduction

It is a well-known fact that the concept of a "$\mathbb{Z}^n$-torus" is of great importance in the context of classification of Lie tori. This concept was originally defined by Yoshii [12]. With the appearance of more general extensions of affine Kac-Moody Lie algebras, such as locally extended affine Lie algebras and invariant affine reflection algebras, one naturally extends the concept of a $\mathbb{Z}^n$-torus to a G-torus for an abelian group G, where for the algebras under consideration G is almost always torsion free. In this work, we classify Jordan G-tori, where G is a torsion free abelian group.

First, we discuss associative G-tori, using the concept of cocycles. Then we show that a Jordan G-torus is strongly prime, and so one can use the Zelmanov prime structure theorem [8]. Thus, such a class divides into three types, the Hermitian type, the Clifford type, and the Albert type. We classify each type using the result of associative G-tori and similar methods in [11].

This paper is organized as follows. In Section 2, we provide preliminary concepts, including direct limits and direct unions, pointed reflection subspaces,

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and (involutorial) associative G-tori. In Section 3, using a direct union approach, we show that a Jordan G-tori J of Hermitian type has one of involution, plus or extension types (see Definition 3.4) and that J is a direct union of Jordan tori of Hermitian type, where J and its direct union components have the same involution, plus or extension type, see Theorem 3.7. In Section 4, we show that a Jordan G-torus J of Clifford type with support S and central grading group Γ, is graded isomorphic to a Clifford G-torus $J(S, \Gamma, \{a_\varepsilon\}_{\varepsilon \in I})$, introduced explicitly in Example 4.2, for some nonempty index set I and choices of $a_\varepsilon \in \mathbb{F}^\times$, $\varepsilon \in I$, see Theorem 4.3. In Section 5, the final section, we first fully characterize associative G-tori of central degree 3. Then for two subgroups Δ and Γ of G satisfying

$$3G \subsetneq \Gamma \subseteq \Delta \subseteq G, \quad \dim_{\mathbb{Z}_3}(G/\Gamma) = 3, \quad \dim(\Delta/\Gamma) = 2,$$

we associate to the triple (G, Δ, Γ), a Jordan algebra $\mathbb{A}_t$ which turns out to be a Jordan G-torus of Albert type, called an *Albert G-torus* associated to the triple (G, Δ, Γ), see Example 5.8. Then we proceed with showing that given a Jordan G-torus J of Albert type with central grading group Γ, there exists a subgroup Δ of G such that the groups G, Δ, and Γ satisfy the above interactions and that J is graded isomorphic to the Albert G-torus $\mathbb{A}_t$, constructed from the triple (G, Δ, Γ), see Theorem 5.9.

2 Preliminaries

Throughout this work, $\mathbb{F}$ is a filed of characteristic zero and G is an abelian group. All algebras assumed over $\mathbb{F}$ and are unital, unless otherwise mentioned. For a subset X of an abelian group, by $\langle X \rangle$ we mean the subgroup generated by X. In a graded algebra we speak of invertible (homogeneous) elements, whenever this notion is defined. The support of a G-graded algebra T, denoted by $\mathrm{supp}(T)$, is by definition the set of those elements of G for which the corresponding homogeneous space is nonzero. For a set X, we denote by $\mathscr{M}_X$ the class of all finite subsets of X. For an associative algebra $\mathscr{A}$, we denote the corresponding plus algebra by $\mathscr{A}^+$; namely, $\mathscr{A}^+$ has $\mathscr{A}$ as its ground vector space, with Jordan product

$$a \circ b := \frac{1}{2}(ab + ba).$$

If $\mathscr{A}$ is equipped with an involution θ, then

$$H(\mathscr{A}, \theta) := \{a \in \mathscr{A} \mid \theta(a) = a\}$$

is a subalgebra of $\mathscr{A}^+$. The field of rational numbers will be denoted by $\mathbb{Q}$. To indicate that an example is concluded, we put the symbol $\diamondsuit$. We refer the reader to [7] or [13] for some terminologies on nonassociative algebras used in the sequel, such as prime, strongly prime, degree, Jordan domain.

2.1 A brief review of direct limits and direct unions

A set I together with a partially ordering $\preceq$, referred to $(I, \preceq)$, is called a *directed set* if for each two elements $i, j \in I$, there is $t \in I$ with $i \preceq t$ and $j \preceq t$. Suppose that $\mathscr{C}$ is a category and $(I, \preceq)$ is a directed set. A family $\{C_i \mid i \in I\}$ of objects of $\mathscr{C}$ together with a family $\{f_{i,j} \mid i, j \in I;\ i \preceq j\}$ of morphisms $f_{i,j}$ of C_i to C_j $(i, j \in I,\ i \preceq j)$ is called a *direct system* in $\mathscr{C}$ if for every pair (i, j) with $i \preceq j$, $f_{ii} = 1_{C_i}$ and $f_{k,i} = f_{k,j} \circ f_{j,i}$ for $i \preceq j \preceq k$. A *direct limit* of the direct system $(\{C_i\}_{i\in I}, \{f_{i,j}\}_{i\preceq j})$ is an object C together with morphisms $\varphi_i : C_i \to C$ $(i \in I)$ satisfying the following two conditions:

- $\varphi_i = \varphi_j \circ f_{i,j}$ for $i, j \in I$ with $i \preceq j$;
- for any other object D and morphisms ψ_i $(i \in I)$ from C_i to D with $\psi_i = \psi_j \circ f_{i,j}$ for $i, j \in I$ with $i \preceq j$, there exists a unique morphism ψ from C to D such that $\psi \circ \varphi_i = \psi_i$ for $i \in I$.

If a direct limit of a direct system $(\{C_i\}_{i\in I}, \{f_{i,j}\}_{i\preceq j})$ in a category $\mathscr{C}$ exists, it is unique up to equivalence, so we refer to as *the direct limit* and denote it by $\varinjlim C_i$. Suppose that C is the direct limit of a direct system $(\{C_i\}_{i\in I}, \{f_{i,j}\}_{i\preceq j})$ in a concrete category $\mathscr{C}$ such that each C_i is a subset of C and for $i, j \in I$ with $i \preceq j$, $f_{i,j}$ is the inclusion map, then we say that C is the *direct union* of $(\{C_i\}_{i\in I}, \{f_{i,j}\}_{i\preceq j})$ if $C = \cup_{i\in I} C_i$.

2.2 Pointed reflection subspaces

In this subsection, we recall the notion of a reflection subspace and record certain properties of reflection subspaces which will be needed in the sequel.

Definition 2.1 A *symmetric reflection subspace* of an additive abelian group G is a subset S of G satisfying $\langle S\rangle = G$ and $S - 2S \subseteq S$. A symmetric reflection subspace is called a *pointed reflection subspace* (PRS) if $0 \in S$. For details on symmetric reflection subspaces, we refer the interested reader to [6] and [2].

If the group G is free abelian of finite rank, a symmetric reflection subspace in G is also called a *translated semilattice* in G. In this case, a pointed reflection subspace is called a *semilattice*. A non-trivial interesting feature of semilattices is that any semilattice in G contains a $\mathbb{Z}$-basis of G (see [1, Proposition II.1.11]).

The following lemma, whose proof is straightforward, gives a characterization of a PRS in terms of its finitely generated pointed reflection subspaces.

Lemma 2.2 (i) *Let S be a PRS in G. Then the following hold.*

(a) *For $T \subseteq S$, $S_T := S \cap \langle T\rangle$ is a PRS in $\langle T\rangle$. In particular, if G is torsion free and T is finite, then S_T is a semilattice in $\langle T\rangle$.*

(b) *S is the union of $\{S_T\}_{T\in\mathscr{M}_S}$.*

(ii) *Let $\mathscr{S}$ be a family of subsets of G such that via the inclusion $\mathscr{S}$ is a directed set, and that each element of $\mathscr{S}$ is a PRS in its $\mathbb{Z}$-span in G. If $G = \cup_{S\in\mathscr{S}} \langle S\rangle$, then the union $\cup_{S\in\mathscr{S}} S$ is a PRS in G.*

2.3 G-tori

In this subsection, we study G-tori, where G is assumed to be a *torsion free* abelian group. Since G can be naturally imbedded in $G \otimes_{\mathbb{Z}} \mathbb{Q}$, we can make sense of σ/n for $\sigma \in G$ and $n \in \mathbb{Z} \setminus \{0\}$. We recall that since G is torsion free, it is an ordered group in the sense of [5, p. 94].

Definition 2.3 [11, Definition 3.1] A G-graded algebra $J = \sum_{\sigma \in G} J^\sigma$ satisfying conditions

(T1) $G = \langle \sigma \in G \mid J^\sigma \neq 0 \rangle$,

(T2) all nonzero homogeneous elements of J are invertible,

(T3) $\dim_{\mathbb{F}}(J^\sigma) \leqslant 1$ for all $\sigma \in G$,

is called a *G-torus*. It is called of *strong type*, if J is *strongly graded*, namely, $J^\sigma J^\tau = J^{\sigma+\tau}$ for all $\sigma, \tau \in G$. The G-torus J is called an *associative* or a *Jordan G-torus*, if J is associative or Jordan, respectively.

The proof of the following lemma is straightforward.

Lemma 2.4 *Suppose that G is an abelian group and Γ is a nonempty index set. Suppose that $\{G_\gamma \mid \gamma \in \Gamma\}$ is a class of subgroups of G such that $G = \cup_{\gamma \in \Gamma} G_\gamma$ and such that Γ is a directed set under the ordering "$\preccurlyeq$" defined by $\gamma \preccurlyeq \eta$ if G_γ is a subgroup of G_η $(\gamma, \eta \in \Gamma)$. If $(\{\mathscr{A}_\gamma\}, \{\varphi_{\gamma,\eta}\})$ is a direct system of associative algebras and algebra homomorphisms with direct limit $(\mathscr{A}, \{\varphi_\gamma\})$ such that*

- *each $\mathscr{A}_\gamma$ is equipped with a G-grading $\mathscr{A}_\gamma = \oplus_{g \in G} (\mathscr{A}_\gamma)^g$ with $\mathrm{supp}(\mathscr{A}_\gamma) = G_\gamma$ and $\dim((\mathscr{A}_\gamma)^g) \leqslant 1$ for all $g \in G$,*
- *each $\varphi_{\gamma,\eta}$ is a G-graded homomorphism,*
- *each φ_γ is monomorphism,*

then $\mathscr{A}$ as an algebra is equipped with a G-grading $\mathscr{A} = \oplus_{g \in G} \mathscr{A}^g$ with $\mathrm{supp}(\mathscr{A}) = G$ and $\dim(\mathscr{A}^g) = 1$ for all $g \in G$. Moreover, if each $\mathscr{A}_\gamma$ is an associative G_γ-torus, then $\mathscr{A}$ is an associative G-torus.

Lemma 2.5 *Let T be a Jordan or an associative G-torus. Let $\mathscr{M}$ be the set of all finite subsets of $\mathrm{supp}(T)$ containing a fixed finite subset $\mathfrak{m}_0$ of $\mathrm{supp}(T)$. For $\mathfrak{m} \in \mathscr{M}$, let*

$$G_{\mathfrak{m}} := \langle \mathfrak{m} \rangle, \quad T_{\mathfrak{m}} := \sum_{\sigma \in G_{\mathfrak{m}}} T^\sigma.$$

Then we have the following:

(i) $G = \cup_{\mathfrak{m} \in \mathscr{M}} G_{\mathfrak{m}}$;

(ii) *for $\mathfrak{m} \in \mathscr{M}$, $T_{\mathfrak{m}}$ is a $G_{\mathfrak{m}}$-torus, and $T = \cup_{\mathfrak{m} \in \mathscr{M}} T_{\mathfrak{m}}$*;

(iii) $\mathrm{supp}(T) = \cup_{\mathfrak{m} \in \mathscr{M}} \mathrm{supp}(T_{\mathfrak{m}})$;

(iv) $\mathrm{supp}(T)$ *is a PRS in G*;

(v) *T is domain, in particular, it has no nilpotents, and it is strongly prime if T is Jordan*;

(vi) *a nonzero element of T is invertible if and only if is homogeneous*;

(vii) *if $0 \neq x \in T$ and $x^m \in T^\sigma$ for some $\sigma \in G$, $m \in \mathbb{Z}$, then $\sigma \in mG$ and $x \in T^{\sigma/m}$.*

Proof The proof of (i)–(iii) is immediate. By [11, Lemma 3.5], for each $\mathfrak{m} \in \mathscr{M}$, $\mathrm{supp}(T_{\mathfrak{m}})$ is a PRS in $G_{\mathfrak{m}}$. Moreover, by (T1),

$$G = \langle \mathrm{supp}(T) \rangle = \bigcup_{\mathfrak{m} \in \mathscr{M}} G_{\mathfrak{m}}.$$

So by part (iii) and Lemma 2.2 (ii), $\mathrm{supp}(S)$ is a PRS in G, proving (iv). The proof of (v)–(vii) is well known. □

We know from Lemma 2.5 that the center $Z(T)$ of T is an associative commutative homogeneous subalgebra of T, as well as an integral domain. In particular, $\Gamma := \mathrm{supp}(Z(T))$ is a subgroup of G and $Z(T)$ is Γ-graded. It follows that $Z(T)$ is isomorphic to a commutative twisted group algebra. The group Γ is called the *central grading group* of T. Let $\overline{Z}$ be the field of fractions of Z, and consider $\overline{T} = \overline{Z} \otimes_Z T$. If T is an associative (Jordan) algebra, then by Lemma 2.5 (v) and [11, 2.6], $\overline{T}$ is an associative (Jordan) algebra over $\overline{Z}$ which is also an integral domain.

Here is a generalization of [11, Lemma 3.9] to torsion free case.

Lemma 2.6 *Let G be a torsion free abelian group, and let $T = \oplus_{\alpha \in G} T_\alpha$ be a Jordan or an associative torus. Let $Z = Z(T)$ be the center of T with the central grading group Γ. Let $^{-}\colon G \to G/\Gamma$ be the canonical map. For $\alpha \in G$, let*

$$T_{\overline{\alpha}} := ZT_\alpha, \quad \overline{T}_{\overline{\alpha}} := \overline{Z} \otimes_Z ZT_\alpha.$$

Then

(i) *$ZT_\alpha = ZT_\beta$ for all $\alpha, \beta \in G$ with $\alpha \equiv \beta \bmod \Gamma$;*

(ii) *$T = \oplus_{\overline{\alpha} \in G/\Gamma} T_{\overline{\alpha}}$ is a free Z-module and a G/Γ-graded algebra over Z with $\mathrm{rank}\, T_{\overline{\alpha}} \leqslant 1$ for all $\overline{\alpha} \in G/\Gamma$;*

(iii) *$\overline{T} = \oplus_{\overline{\alpha} \in G/\Gamma} \overline{T}_{\overline{\alpha}}$ is a G/Γ-graded torus over $\overline{Z}$ with*

$$\dim_{\overline{Z}} \overline{T} = |(\mathrm{supp} T)/\Gamma|;$$

(iv) *the quotient group G/Γ cannot be a nontrivial cyclic group.*

Proof The proof of (i)–(iii) is straightforward, considering the fact that for $\gamma \in \Gamma$ and $\alpha \in G$, $T_\gamma \subseteq Z$ and $T_{\alpha+\gamma} = T_\gamma T_\alpha$. By (iii), $\overline{T}$ is a G/Γ-torus. If G/Γ is cyclic, $\overline{T}$ is also commutative and associative. Thus, T embeds in $\overline{T}$ and so $Z = T$ and $\Gamma = \Lambda$. This proves (iv). □

2.4 Associative G-tori

Let G be an abelian group. Symbols σ, τ, μ always denote elements of G. Let $\mathscr{A} = \oplus_{\sigma \in G} \mathscr{A}^\sigma$ be an associative G-torus. Since homogeneous non-zero elements of $\mathscr{A}$ are invertible, we have

$$\mathscr{A}^\sigma \mathscr{A}^\tau = \mathscr{A}^{\sigma+\tau}, \quad \sigma, \tau \in \mathrm{supp}(\mathscr{A}).$$

It follows that $\mathrm{supp}(\mathscr{A})$ is a subgroup of G and so by (T1), $\mathrm{supp}(\mathscr{A}) = G$. For $\sigma \in G$, we choose $0 \neq x^\sigma \in \mathscr{A}^\sigma$. Then $\mathscr{A} = \oplus_{\sigma\in G}\mathbb{F}x^\sigma$. Define $\lambda\colon G \times G \to \mathbb{F}^\times$ by

$$x^\sigma x^\tau = \lambda(\sigma,\tau)x^{\sigma+\tau}, \quad \sigma,\tau \in G. \tag{2.1}$$

Associativity of $\mathscr{A}$ implies that λ is a 2-cocycle, namely, for $\sigma,\tau,\mu \in G$,

$$\lambda(\sigma+\tau,\mu)\lambda(\sigma,\tau) = \lambda(\sigma,\tau+\mu)\lambda(\tau,\mu). \tag{2.2}$$

Conversely, let $\lambda\colon G\times G \to \mathbb{F}^\times$ be a 2-cocycle. Consider the abstract vector space $\mathscr{A} := \oplus_{\sigma\in G}\mathbb{F}x^\sigma$ with basis $\{x^\sigma \mid \sigma \in G\}$. Then the multiplication on $\mathscr{A}$ induced from (2.1) makes $\mathscr{A}$ into an associative G-torus with $\mathrm{supp}(\mathscr{A}) = G$. We denote $\mathscr{A}$ by $(\mathbb{F}^t(G), \lambda)$ and call it the *associative G-torus determined by the* 2-*cocycle* λ. We note that the associative G-torus $(\mathbb{F}^t[G], \lambda)$ can be characterized as the unital associative algebra defined by the set of generators $\{x^\sigma \mid \sigma \in g\}$ and relations (2.1). The associative G-torus $(\mathbb{F}^t[G], \lambda)$ is called *elementary* if $\mathrm{img}(\lambda) \subseteq \{1,-1\}$. In the literature, $\mathscr{A} = (\mathbb{F}^t[G], \lambda)$ is also known as the *twisted group algebra* determined by λ (see [10] or [9]). We summarize the above discussion as follows.

Lemma 2.7 *Let G be an abelian group, and let $\mathscr{A}$ be an associative algebra. Then $\mathscr{A}$ is a G-torus if and only if $\mathscr{A} \cong_G (\mathbb{F}^t[G], \lambda)$ for a 2-cocycle λ.*

Let $\mathscr{A} = (\mathbb{F}^t[G), \lambda)$ be an associative G-torus. Note that for $\sigma,\tau \in G$, x^σ and x^τ commute up to a "twisting", namely,

$$x^\sigma x^\tau = \lambda_t(\sigma,\tau)x^\tau x^\sigma, \tag{2.3}$$

where

$$\lambda_t(\sigma,\tau) := \lambda(\sigma,\tau)\lambda(\tau,\sigma)^{-1}. \tag{2.4}$$

We clearly have

$$\lambda_t(\sigma,\sigma) = 1, \quad \lambda_t(\sigma,\tau) = \lambda_t(\tau,\sigma)^{-1}.$$

Moreover, one can check that $\lambda_t\colon G\times G \to \mathbb{F}^\times$ is a group bihomomorphism.

Remark 2.8 Suppose in the above discussion that we replace the basis $\{x^\sigma \mid \sigma \in G\}$ of $\mathscr{A}$ by another basis $\{y^\sigma \mid \sigma \in G\}$. Then for $\sigma \in G$, $y^\sigma = d(\sigma)x^\sigma$, where $d\colon G \to \mathbb{F}^\times$ is a map. Denote the corresponding 2-cocycle as in (2.1) by $\hat\lambda\colon G\times G \to \mathbb{F}^\times$. Then we have

$$\hat\lambda(\sigma,\tau) = d(\sigma)d(\tau)d(\sigma+\tau)^{-1}\lambda(\sigma,\tau).$$

Therefore, λ and $\hat\lambda$ are *equivalent*, up to a coboundary. That is, the product on $\mathscr{A}$ is uniquely determined up to $H^2(G,\mathbb{F}^\times)$ (see [9, §1]).

Example 2.9 (Quantum tori) Let Λ be a free abelian group of rank $|I|$, where I is a nonempty index set with a fixed total ordering $<$. Let $\mathscr{A} = (\mathbb{F}^t[\Lambda], \lambda)$ be a Λ-torus determined by a 2-cocycle λ. We fix a basis $\{\sigma_i \mid i \in I\}$ of Λ, and set $q_{ij} = \lambda_t(\sigma_i, \sigma_j)$. Since λ_t is a bihomomorphism, for

$$\sigma = \sum_{i\in I} n_i\sigma_i, \quad \tau = \sum_{i\in I} m_i\sigma_i,$$

we have

$$\lambda_t(\sigma,\tau) = \prod_{i,j} q_{ij}^{n_i m_j},$$

with $q_{ij} = q_{ji}^{-1}$ and $q_{ii} = 1$ for all $i, j \in I$. We note that as n_i's and m_i's are zero almost for all i, the above product makes sense. In the literature, a matrix $(q_{ij})_{i,j\in I}$ (possibly of infinite rank), satisfying $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$ for all i, j, is called a *quantum matrix*. A quantum matrix $\mathbf{q}$ is called *elementary* if $q_{ij} \in \{\pm 1\}$ for all i, j. For $i \in I$, we set $y_i := x^{\sigma_i}$. Also for $\sigma = \sum_{i\in I} n_i\sigma_i$, we set $y^\sigma = 1$ if $\sigma = 0$ and if $\sigma \neq 0$, we set $y^\sigma := y_{i_1}^{n_{i_1}} \cdots y_{i_k}^{n_{i_k}}$, where $i_1 < \cdots < i_k$ are all indices for which $n_{ij} \neq 0$. Then we have

$$\mathscr{A} = \bigoplus_{\sigma\in\Lambda} \mathbb{F}y^\sigma,$$

and for all $i, j \in I$,

$$y_i y_j = q_{ij} y_j y_i, \quad y_i y_i^{-1} = y_i^{-1} y_i = 1. \tag{2.5}$$

The Λ-torus $\mathscr{A}$ can be described as the unital associative algebra defined by generators $1, y_i, y_i^{-1}$ and relations (2.5), induced from the quantum matrix $\mathbf{q} := (q_{ij})$. In this case, we denote $\mathscr{A}$ by $\mathscr{A} = (\mathbb{F}^t[\Lambda], \mathbf{q})$ and call it the *quantum torus* determined by the quantum matrix $\mathbf{q}$. $\diamondsuit$

Here is a generalization of [11, Lemma 4.6] to the torsion free case.

Lemma 2.10 *Let G be a torsion free abelian group, and let $\mathscr{A}$ be an associative algebra. If $\mathscr{A}^+$ is a Jordan G-torus, then $\mathscr{A} \cong_G (\mathbb{F}^t[G], \lambda)$ for some 2-cocycle λ. In particular, if G is free abelian, then $\mathscr{A} \cong_G (\mathbb{F}^t[G], \mathbf{q})$ for some quantum matrix $\mathbf{q}$.*

Proof By Lemma 2.7, we must show that $\mathscr{A}$ is an associative G-torus. Since $\mathscr{A}^+$ is a Jordan torus, we have

$$\mathscr{A}^+ = \mathscr{A} = \bigoplus_{\sigma\in g} \mathscr{A}^\sigma,$$

with $G = \langle \sigma \in G \mid \mathscr{A}^\sigma \neq 0\rangle$ and $\dim \mathscr{A}^\sigma \leqslant 1$ for all $\sigma \in G$. So it only remains to show that $\mathscr{A}$ is G-graded, namely, $\mathscr{A}^\sigma \mathscr{A}^\tau \subseteq \mathscr{A}^{\sigma+\tau}$ for all $\sigma, \tau \in G$. We proceed with showing this for fixed $\sigma, \tau \in G$. We may assume without loss of generality that both $\mathscr{A}^\sigma$ and $\mathscr{A}^\tau$ are non-zero. Let $0 \neq x \in \mathscr{A}^\sigma$ and $0 \neq y \in \mathscr{A}^\tau$. By Lemma 2.5, x and y are invertible in $\mathscr{A}^+$ and so they are invertible in $\mathscr{A}$.

Therefore, xy and yx are invertible in $\mathscr{A}$ and so in $\mathscr{A}^+$. Then by Lemma 2.5, both xy and yx are homogeneous in $\mathscr{A}$. Now, as

$$x \circ y = xy + yx \in \mathscr{A}^{\sigma+\tau},$$

we conclude that $xy \in \mathscr{A}^{\sigma+\tau}$ if $x \circ y \neq 0$. Suppose now that $x \circ y = 0$. Then $xy = -yx \in \mathscr{A}^{\delta}$ for some $\delta \in G$ and as $\mathscr{A}$ is associative,

$$(xy)^2 = -x^2y^2 = -y^2x^2.$$

Therefore,

$$0 \neq (xy)^2 = -\frac{1}{2}\,(x^2y^2 + y^2x^2) = -\frac{1}{2}\,(x^2 \circ y^2) \in \mathscr{A}^{2\delta} \cap \mathscr{A}^{2\sigma+2\tau}.$$

Thus, $\delta = \sigma + \tau$. The second statement follows immediately from Example 2.9. □

2.5 Involutorial associative G-tori

Let $\mathscr{A} = (\mathbb{F}^t[G], \lambda)$ be an associative G-torus. Assume further that $\mathscr{A}$ is equipped with a graded involution $^-$, namely, a period 2 anti-automorphism $^-$ satisfying $\overline{\mathscr{A}^\sigma} = \mathscr{A}^\sigma$, $\sigma \in G$. Then, for $\sigma \in G$, we have $\overline{x^\sigma} = a_\sigma x^\sigma$, where $a_\sigma \in \mathbb{F}^\times$ satisfies $a_\sigma^2 = 1$. So, for $\sigma \in G$,

$$\overline{x^\sigma} = (-1)^{q(\sigma)} x^\sigma,$$

where q is a map from G into the field $\mathbb{F}_2$ of 2 elements. We note that

$$(-1)^{q(\sigma+\tau)} x^{\sigma+\tau} = \overline{x^{\sigma+\tau}} = \lambda(\sigma,\tau)^{-1}\overline{x^\tau}\,\overline{x^\sigma} = \lambda(\sigma,\tau)^{-1}\lambda(\tau,\sigma)(-1)^{q(\sigma)+q(\tau)} x^{\sigma+\tau}.$$

Thus,

$$(-1)^{\beta_q(\sigma,\tau)} = \lambda_t(\sigma,\tau), \tag{2.6}$$

where $\beta_q\colon G \times G \to \mathbb{F}_2$ is defined by

$$\beta_q(\sigma,\tau) = q(\sigma) + q(\tau) - q(\sigma+\tau).$$

Now, λ_t being a bihomomorphism implies that β_q is also a group bihomomorphism. Therefore, by definition, $q\colon G \to \mathbb{F}_2$ is a quadratic map.

Conversely, starting from an associative G-torus $\mathscr{A} = (\mathbb{F}^t[G], \lambda)$ and a quadratic map $q\colon G \to \mathbb{F}_2$ satisfying $(-1)^{\beta_q(\sigma,\tau)} = \lambda_t(\sigma,\tau)$, one can define a graded involution $^-$ on $\mathscr{A}$ by $\overline{x^\sigma} = (-1)^{q(\sigma)} x^\sigma$. In fact, it is clear that $^-$ is a period 2 isomorphism of $\mathbb{F}$-vector spaces. Moreover, for $\sigma, \tau \in G$, we have

$$\begin{aligned}
\overline{x^\sigma x^\tau} &= \lambda(\sigma,\tau)\overline{x^{\sigma+\tau}} \\
&= \lambda(\sigma,\tau)(-1)^{q(\sigma+\tau)} x^{\sigma+\tau} \\
&= \lambda(\sigma,\tau)(-1)^{q(\sigma+\tau)}\lambda(\tau,\sigma)^{-1} x^\tau x^\sigma \\
&= (-1)^{\beta_q(\sigma,\tau)+q(\sigma,\tau)} x^\tau x^\sigma \\
&= (-1)^{q(\sigma)+q(\tau)} x^\tau x^\sigma \\
&= \overline{x^\tau}\,\overline{x^\sigma}.
\end{aligned}$$

Definition 2.11 Let $\mathscr{A} = (\mathbb{F}^t[G], \lambda)$ be a G-torus, and let $q\colon G \to \mathbb{F}_2$ be a quadratic map satisfying (2.6). We denote the induced involution on $\mathscr{A}$ by θ_q. We recall that in this case, $H(\mathscr{A}, \theta_q) = H((\mathbb{F}^t[G], \lambda), \theta_q)$ is a subalgebra of $\mathscr{A}^+$.

Let J be a Jordan G-torus. By Lemma 2.5, J is strongly prime, so by Zelmanov's Prime Structure Theorem [8, p. 200], J has one of the types, *Hermitian*, *Clifford*, or *Albert*. We recall that J is of Hermitian type if J is special and $q_{48}(J) \neq \{0\}$ (the term $q_{48}(J)$ will be explained in the next section). Also J is of Clifford type if the central closure $\overline{J}$ is a Jordan algebra over $\overline{Z}$ of a symmetric bilinear form. Finally, J is of Albert type if the central closure $\overline{J}$ is an Albert algebra over $\overline{Z}$. In the remaining sections, we study each of the mentioned types separately.

3 Jordan tori of Hermitian type

Throughout this section, G is a torsion free abelian group, unless otherwise mentioned. All associative algebras are assumed to be unital. We assume that any algebra homomorphism from a unital algebra to a unital algebra maps 1 to 1. We recall that a Jordan torus J is called a *Hermitian torus* if there exists an involutorial associative algebra $(\mathscr{A}, *)$ which is $*$-prime such that $\mathscr{A}$ is generated by J and $J = H(\mathscr{A}, *)$.

We make a convention that for two elements x, y of an associative algebra, by $[x, y]$, we mean $xy - yx$ and by $x \circ y$, we mean $xy + yx$. Suppose that X is an infinite set and $\mathfrak{a}(X)$ is the free associative algebra on X. We consider the special Jordan algebra $\mathfrak{a}(X)^+$ and take $\mathfrak{fsj}(X)$ to be the subalgebra of $\mathfrak{a}(X)^+$ generated by X. This is the *free special Jordan algebra on* X. We recall that an ideal I of $\mathfrak{fsj}(X)$ is called *formal* if for each polynomial $p(x_1, \ldots, x_n) \in I$ with $x_1, \ldots, x_n \in X$, and each permutation σ of X, one has $p(\sigma(x_1), \ldots, \sigma(x_n)) \in I$. A formal ideal H of $\mathfrak{fsj}(X)$ is called *Hermitian* if it is closed under n-tads for each natural number n greater than 3, i.e., for $n \in \mathbb{N}$ with $n > 3$ and $x_1, \ldots, x_n \in H$,

$$\{x_1, \ldots, x_n\} := x_1 \cdots x_n + x_n \cdots x_1 \in H.$$

Now, suppose that $H(X)$ is a Hermitian ideal of $\mathfrak{fsj}(X)$. For an i-special Jordan algebra (i.e., a quotient algebra of a special Jordan algebra) J, by $H(J)$, we mean the evaluation of $H(X)$ on J; $H(J)$ is an ideal of J and called a *Hermitian part* of J.

Now, for $x, y, z, w \in X$, we take

$$D_{x,y}(z) := [[x, y], z]$$

and set

$$p_{16}(x, y, z, w) := [[D^2_{x,y}(z)^2, D_{x,y}(w)], D_{x,y}(w)].$$

Then

$$q_{48} := [[p_{16}(x_1, y_1, z_1, w_1), p_{16}(x_2, y_2, z_2, w_2)], p_{16}(x_3, y_3, z_3, w_3)]$$

is a polynomial in the free associative algebra on X in 12 variables x_i, y_i, z_i, w_i, $1 \leqslant i \leqslant 3$. Take Q_{48} to be the linearization-invariant T-ideal of $\mathfrak{fsj}(X)$ generated by q_{48}. It means that Q_{48} is the smallest ideal of $\mathfrak{fsj}(X)$ containing q_{48} with the following two properties. If p is a polynomial in Q_{48}, then each linearization of p is also an element of Q_{48} and that Q_{48} is invariant under all algebra endomorphisms of $\mathfrak{fsj}(X)$. We note that for 12 variables x_i, y_i, z_i, w_i, $1 \leqslant i \leqslant 3$, each monomial of q_{48} is a product of 12 variables x_i, y_i, z_i, w_i, $1 \leqslant i \leqslant 3$, and monomials have the same number of $x \in \{x_i, y_i, z_i, w_i \mid 1 \leqslant i \leqslant 3\}$. So each polynomial in Q_{48} is a summation of monomials having the same partial degree.

Lemma 3.1 *Suppose that J is a Jordan G-torus of Hermitian type. Then $J = H(P, *)$ for an associative algebra P with an involution $*$ such that P is $*$-prime and is generated by J.*

Proof Since J is of Hermitian type, we have $q_{48}(J) \neq \{0\}$. Fix a basis B of J consisting of homogeneous elements. If $Q_{48}(B) = \{0\}$, we get $q_{48}(J) = \{0\}$, which is a contradiction. So there is a polynomial $p \in Q_{48}$ and $b_1, \ldots, b_m \in B$ such that $p(b_1, \ldots, b_m) \neq 0$. We know that p is a linear combination of monomials having the same partial degree. This together with the fact that $b_1, \ldots, b_m$ are homogeneous elements implies that $p(b_1, \ldots, b_m)$ is homogeneous and so it is invertible. So $Q_{48}(J) = J$. $\square$

Lemma 3.2 *Suppose that J is a Jordan G-torus. Suppose that $\mathbb{E}$ is a quadratic field extension of $\mathbb{F}$, $\sigma_{\mathbb{E}}$ is the nontrivial Galois automorphism, and $\lambda\colon G \times G \to \mathbb{E}^\times$ is a 2-cocycle. Assume $\mathbb{E} \otimes_{\mathbb{F}} J \simeq_G (\mathbb{E}^t[G], \lambda)^+$, say via φ. Then either there is a 2-cocycle $\mu\colon G \times G \to \mathbb{F}^\times$ such that $(\mathbb{E}^t[G], \lambda) \simeq_G (\mathbb{E}^t[G], \mu)$ and $J \simeq_G (\mathbb{F}^t[G], \mu)^+$, or $J \simeq_G H((\mathbb{E}^t[G], \mu), \theta)$ for some 2-cocycle μ satisfying $\sigma_{\mathbb{E}}(\mu(g_1, g_2)) = \mu(g_2, g_1)$ for $g_1, g_2 \in G$, and an $\sigma_{\mathbb{E}}$-semilinear anti-automorphism θ, where $(\mathbb{E}^t[G], \mu)$ is considered as an $\mathbb{F}$-algebra.*

Proof Since $\mathbb{E}$ is a quadratic field extension of $\mathbb{F}$, there is an irreducible polynomial on $\mathbb{F}$ of degree 2 with distinct roots e, f. Then $\mathbb{E} = \mathbb{F} + e\mathbb{F}$ and $\sigma_{\mathbb{E}}\colon \mathbb{E} \to \mathbb{E}$ is the Galois automorphism mapping e to f. Set

$$\tau := \sigma_{\mathbb{E}} \otimes id\colon \mathbb{E} \otimes J \to \mathbb{E} \otimes J.$$

Since for $x, y \in E$ and $a \in J$, we have

$$\tau(xy \otimes a) = \sigma_{\mathbb{E}}(xy) \otimes a = \sigma_{\mathbb{E}}(x)\sigma_{\mathbb{E}}(y) \otimes a = \sigma_{\mathbb{E}}(x)(\sigma_{\mathbb{E}}(y) \otimes a),$$

we get that τ is a $\sigma_{\mathbb{E}}$-semilinear automorphism of the Jordan algebra $\mathbb{E} \otimes J$. Consider the $\mathbb{E}$-Jordan algebra isomorphism $\varphi\colon \mathbb{E} \otimes_{\mathbb{F}} J \to (\mathbb{E}^t[G], \lambda)^+$. Then $\theta := \varphi\tau\varphi^{-1}$ is a Jordan $\sigma_{\mathbb{E}}$-semilinear automorphism on $(\mathbb{E}^t[G], \lambda)^+$. Next, we note that as $\theta = \varphi\tau\varphi^{-1}$ is a Jordan $\sigma_{\mathbb{E}}$-semilinear automorphism on $(\mathbb{E}^t[G], \lambda)^+$, it is also an $\mathbb{F}$-linear automorphism of the $\mathbb{F}$-Jordan algebra $(\mathbb{E}^t[G], \lambda)^+$. So by [3, Lemma 1.1.7] either θ is a $\sigma_{\mathbb{E}}$-semilinear associative algebra automorphism of $(\mathbb{E}^t[G], \lambda)$ or it is a $\sigma_{\mathbb{E}}$-semilinear anti-automorphism of the associative algebra $(\mathbb{E}^t[G], \lambda)$ which is not an automorphism. We know that

$$J \simeq \mathrm{span}_{\mathbb{F}}\{r \otimes x \mid r \in \mathbb{F},\ x \in J\} = H(\mathbb{E} \otimes J, \tau)$$

and the restriction of φ to $H(\mathbb{E}\otimes J,\tau)$ is an $\mathbb{F}$-Jordan algebra isomorphism from $J \simeq H(\mathbb{E}\otimes J,\tau)$ to $H((\mathbb{E}^t[G],\lambda),\theta)$. So to complete the proof, we show that either

$$H((\mathbb{E}^t[G],\lambda),\theta) = (\mathbb{F}^t[G],\mu)^+$$

for a 2-cocycle $\mu\colon G\times G\to\mathbb{F}$ or

$$H((\mathbb{E}^t[G],\lambda),\theta) = H((\mathbb{E}^t[G],\mu),\theta)$$

for some 2-cocycle μ satisfying $\sigma_{\mathbb{E}}(\mu(g_1,g_2)) = \mu(g_2,g_1)$ for $g_1,g_2\in G$.

We fix $0\neq x^g\in J_g$ $(g\in G)$, so $\{x^g \mid g\in G\}$ is an $\mathbb{F}$-basis for J and an $\mathbb{E}$-basis for $\mathbb{E}\otimes_{\mathbb{F}} J$ (here, we identify J with $\{1\otimes x \mid x\in J\}\subseteq \mathbb{E}\otimes J$). Now, as for $g\in G$, $\tau(x^g)=x^g$ and φ is a G-graded isomorphism, $\{y^g := \varphi(x^g) \mid g\in G\}$ is a basis for the $\mathbb{E}$-vector space $\mathbb{E}^t[G]$ consisting of homogeneous elements fixed by θ. Now, let $\mu\colon G\times G\to\mathbb{E}$ be the 2-cocycle corresponding to this new basis; see Remark 2.8. We note that

$$(\mathbb{E}^t[G],\lambda) = \Big(\bigoplus_{g\in G}\mathbb{E}y^\sigma,\mu\Big) = \bigoplus_{g\in G}\mathbb{F}y^\sigma + \bigoplus_{g\in G}e\mathbb{F}y^\sigma$$

and

$$H((\mathbb{E}^t[G],\lambda),\theta) = \bigoplus_{g\in G}\mathbb{F}y^\sigma.$$

Now, we consider the two cases that either θ is a $\sigma_{\mathbb{E}}$-semilinear associative algebra automorphism of $(\mathbb{E}^t[G],\lambda) = (\mathbb{E}^t[G],\mu)$ or it is a $\sigma_{\mathbb{E}}$-semilinear anti-automorphism of the associative algebra $(\mathbb{E}^t[G],\lambda) = (\mathbb{E}^t[G],\mu)$ which is not an automorphism. In the former case, for $g_1,g_2\in G$, we have

$$0 = \theta(y^{g_1}y^{g_2} - \mu(g_1,g_2)y^{g_1+g_2}) = y^{g_1}y^{g_2} - \sigma_{\mathbb{E}}(\mu(g_1,g_2))y^{g_1+g_2}.$$

So we have

$$\mu(g_1,g_2)y^{g_1+g_2} = y^{g_1}y^{g_2} = \sigma_{\mathbb{E}}(\mu(g_1,g_2))y^{g_1+g_2}.$$

Therefore, $\mu(g_1,g_2)\in\mathbb{F}$. Now, we have

$$(\mathbb{E}^t[G],\lambda) = \bigoplus_{g\in G}\mathbb{E}y^\sigma = \bigoplus_{g\in G}\mathbb{F}y^\sigma + \bigoplus_{g\in G}e\mathbb{F}y^\sigma.$$

Since $\mu(G,G)\subseteq\mathbb{F}$, $\oplus_{g\in G}\mathbb{F}y^\sigma$ is closed under the associative product on $(\mathbb{E}^t[G],\mu)$ and

$$H((\mathbb{E}^t[G],\lambda)) = H((\mathbb{E}^t[G],\mu),\theta) = \bigoplus_{g\in G}\mathbb{F}y^\sigma$$

can be identified with $(\mathbb{F}^t[G],\mu)^+$. In the latter case, for $g_1,g_2\in G$, we have

$$0 = \theta(y^{g_1}y^{g_2} - \mu(g_1,g_2)y^{g_1+g_2}) = y^{g_2}y^{g_1} - \sigma_{\mathbb{E}}(\mu(g_1,g_2))y^{g_1+g_2}.$$

So $\sigma_{\mathbb{E}}(\mu(g_1,g_2)) = \mu(g_2,g_1)$. This completes the proof. □

The following generalizes [11, Proposition 4.7] to the torsion free case.

Proposition 3.3 *Let $\mathscr{A}$ be an involutorial associative algebra and assume that $J := H(\mathscr{A}, *)$ is a Jordan G-torus generating $\mathscr{A}$.*

(a) *Suppose that there exists $a \in \mathscr{A}$ such that $aa^* = 0$ and $a + a^*$ is invertible in J. Then $J \cong_G (\mathbb{F}^t[G], \lambda)^+$ for some 2-cocycle λ.*

(b) *Suppose that there exists an invertible element $a \in \mathscr{A}$ such that $a^* = -a$ and $0 \neq y \in J_\gamma$ for some $\gamma \in G$ such that $a^2 \in J_{2\gamma}$, $ay^{-1}a \in J_\gamma$, and $[a, y] \in J_{2\gamma}$. Then $J \cong_G (\mathbb{F}^t[G], \lambda)^+$ or $E \otimes_{\mathbb{F}} J \cong_G (\mathbb{E}^t[G], \lambda)^+$ for some 2-cocycle λ.*

Proof (a) By Lemma 2.5 (v), J is domain. By [11, Lemma 4.5], $J \cong \mathscr{A}^+$ for some associative algebra $\mathscr{A}$. Then by Lemma 2.10, we are done. The proof of part (b) is exactly the same as [11, Proposition 4.7 (b)]. □

Definition 3.4 A Jordan G-torus J is said to be of *involution type* if we have $J \cong_G H((\mathbb{F}^t[G], \lambda), \theta_q)$ (λ a 2-cocycle and q a quadratic map), it is said to be of *plus type* if $J \cong_G (\mathbb{F}^t[G], \lambda)^+$ (λ a 2-cocycle), and it is said to be of *extension type* if $J \cong_G H((\mathbb{E}^t[G], \lambda), \sigma)$ ($\mathbb{E}$ a quadratic field extension of $\mathbb{F}$, λ a 2-cocycle, and σ an involution). If G is free abelian of finite rank, we call a Jordan G-torus of one of the above types, simply a Jordan torus of that type.

Lemma 3.5 *Suppose that G is a free abelian group of finite rank and $J = \oplus_{g \in G} J^g$ is a Jordan G-torus. Suppose that P is an associative algebra with involution $*$ and $J = H(P, *)$. For $g \in \operatorname{supp}(J)$, fix $0 \neq x_g \in J^g$. If for all $g, h \in \operatorname{supp}(J)$, $x^g x^h = \pm x^h x^g$ (product in P), then one of the following occurs.*

(a) *P is isomorphic to $(\mathbb{F}^t[G], \lambda)$ for a 2-cocycle $\lambda\colon G \times G \to \mathbb{F}$; in particular, P is a G-graded algebra. Moreover, $*$ is a G-graded involution and J is graded isomorphic to $H((\mathbb{F}^t[G], \lambda), \theta_q)$, where $q\colon G \times G \to \mathbb{F}_2$ is the quadratic map arising from the involution on $(\mathbb{F}^t[G], \lambda)$ induced via the isomorphism form P to $(\mathbb{F}^t[G], \lambda)$ (see Section 2.5); in particular, $P^g = J^g$ for all $g \in \operatorname{supp}(J)$.*

(b) *There are an invertible element u of P and a nonzero element y of J^γ for some $\gamma \in G$ such that the following four conditions hold:*

$$u^* = -u, \quad u^2 \in J^{2\gamma}, \quad uy^{-1}u \in J^\gamma, \quad [u, y] \in J^{2\gamma}. \tag{3.1}$$

Proof One knows from Lemma 2.5 and [1, Proposition II.1.11] that there is a basis $B = \{\sigma_1, \ldots, \sigma_n\} \subseteq \operatorname{supp}(J)$ for G. If J is generated by r-tads $\{x_{\sigma_{i_1}}^{\varepsilon_1} \cdots x_{\sigma_{i_r}}^{\varepsilon_r}\}$ for $r \in \mathbb{Z}^{>0}$, $1 \leqslant i_1, \ldots, i_r \leqslant n$, $\varepsilon_1, \ldots, \varepsilon_r \in \{\pm 1\}$, conditions (A) and (B) of the proof of [11, Thm. 4.11] hold and so by the proof of the same theorem, all the statements in (a) are fulfilled. But if J is not generated by r-tads $\{x_{\sigma_{i_1}}^{\varepsilon_1} \cdots x_{\sigma_{i_r}}^{\varepsilon_r}\}$ for $r \in \mathbb{Z}^{>0}$, $1 \leqslant i_1, \ldots, i_r \leqslant n$, $\varepsilon_1, \ldots, \varepsilon_r \in \{\pm 1\}$, condition (A) but not (B) of the proof of [11, Thm. 4.11] holds and again by the proof of the same theorem, there are an invertible element u of P and a nonzero element y of J^γ for some $\gamma \in G$ such that (3.1) is satisfied. □

Proposition 3.6 *Suppose that P is an associative algebra with an involution $*$. Suppose that $J := H(P, *)$ generates P. Suppose that $\{G_i \mid i \in \mathscr{I}\}$ is a* *class of free abelian subgroups of G such that $G = \cup_{i\in\mathscr{I}} G_i$. Also assume that $J = \oplus_{g\in G} J^g$ is a Jordan G-torus. Set*

$$J_i := \bigoplus_{g\in G_i} J^g, \quad i \in \mathscr{I}.$$

Assume that $*_i$, the restriction of $*$ to the subalgebra P_i $(i \in \mathscr{I})$ of P generated by J_i is an involution of P_i and that $J_i = H(P_i, *)$. If P is $*$-prime, then one of the following holds for J:

- *$J \simeq H((\mathbb{F}^t[G], \lambda), \theta_q)$, λ a 2-cocycle and q a quadratic map;*
- *$J \simeq (\mathbb{F}^t[G], \lambda)^+$, λ a 2-cocycle;*
- *$J \simeq H((\mathbb{E}^t[G], \lambda), \sigma)$, $\mathbb{E}$ a quadratic field extension of $\mathbb{F}$, λ a 2-cocycle and σ an involution.*

Moreover, if J is of involution (resp. plus or extension) type, it is a direct union of Jordan tori of involution (resp. plus or extension) type.

Proof We know that $J = \oplus_{g\in G} J^g$ is a Jordan G-torus. For $g \in \text{supp}(J)$, we fix $0 \neq x_g \in J^g$. We consider the following two cases.

Case 1 For all $g, h \in \text{supp}(J)$, $x_g x_h = \pm x_h x_g$.

By Lemma 3.5, one of the following occurs:

(a) for all $i \in \mathscr{I}$,

- P_i is equipped with a G_i-grading $\oplus_{g\in G} P_i^g$ with $P_i^g = J_i^g$ for all $g \in \text{supp}(J_i)$,
- $P_i = \oplus_{g\in G_i} P_i^g$ is an associative G_i-torus,
- $*_i = * \mid_{P_i}$ is a G_i-graded involution;

(b) there is $i \in \mathscr{I}$ for which there are an invertible element u of P_i and a nonzero element y of J_i^γ for some $\gamma \in G_i$ such that (3.1) holds.

We now assume that (a) is satisfied, $i, j \in \mathscr{I}$ with $i \preccurlyeq j$, and $g \in G_i$. If $g \in \text{supp}(J_i)$, then

$$P_i^g = J_i^g = J_j^g = P_j^g.$$

Also we know that P_i is generated by J_i and so P_i is generated by

$$\bigcup_{g\in G_i} J_i^g = \bigcup_{g\in\text{supp}(J_i)} J_i^g.$$

In particular, for $g \in G_i$, there are $\tau_1, \ldots, \tau_t \in \text{supp}(J_i)$ such that $g = \tau_1 + \cdots + \tau_t$ and

$$P_i^g = J_i^{\tau_1} \cdots J_i^{\tau_t} = P_i^{\tau_1} \cdots P_i^{\tau_t} = P_j^{\tau_1} \cdots P_j^{\tau_t} \subseteq P_j^g.$$

Therefore, we have proved

$$P_i^g = P_j^g, \quad i, j \in \mathscr{I},\ i \preccurlyeq j,\ g \in G_i.$$

So by Lemma 2.4, P is an associative G-torus with $P^g = J^g$ for all $g \in \text{supp}(J)$. Therefore, J is graded isomorphic to $H((\mathbb{F}^t[G], \lambda), \theta_q)$ for a 2-cocycle $\lambda\colon G\times G \to \mathbb{F}^\times$ and a quadratic map $q\colon G \times G \to \mathbb{F}_2$.

Next, we assume that (b) is satisfied. Then we are done by Proposition 3.3.

Case 2 There are $g, h \in \mathrm{supp}(J)$ such that $x_g x_h \neq \pm x_h x_g$.

Set

$$u := [x_g, x_h] \neq 0, \quad d := x_g \circ x_h \neq 0.$$

Then we have one of the following conditions.

- $u^2 = 0$.

We have $u = -u^*$ and so $uu^* = 0$, then there exists $y \in J$ such that for $v := yu$, $v + v^* \neq 0$. Otherwise, for all $y \in J$, we have $v + v^* = 0$ in which $v := yu$. So we have

$$yu = v = -v^* = -u^* y^* = uy.$$

Therefore, for $w \in P$, we have

$$(uy)(uw) = u(yu)w = u(uy)w = u^2 w = 0.$$

This implies that $(uJ)(uP) = \{0\}$. Now, as J generates P, we get $(uP)^2 = \{0\}$. So we have

$$(PuP)^2 = PuPPuP \subseteq PuPuP = P(uP)^2 = \{0\}.$$

Also $(PuP)^* = PuP$ and so PuP is a nonzero $*$-ideal of P with $(PuP)^2 = \{0\}$, a contradiction, as P is $*$-prime. Therefore, there exists $y \in J$ such that for $v := yu$, $v + v^* \neq 0$. Since $y \in J$, we have

$$y = \sum_{g \in G} y_g, \quad y_g \in J_g.$$

For $g \in G$, set $v_g := y_g u$. If for all $g \in G$, $v_g + v_g^* = 0$, then we get

$$v + v^* = [y, u] = \bigoplus_{g \in G} [y_g, u] = \bigoplus_{g \in G} (y_g u - u y_g) = \bigoplus_{g \in G} v_g + v_g^* = 0,$$

which is a contradiction. So there is $g_* \in G$ with $v_{g_*} + v_{g_*}^* \neq 0$. Now, as $v_{g_*} + v_{g_*}^*$ is a homogeneous element of J (see [11, (2.7)]), it is invertible. Also $v_{g_*} v_{g_*}^* = 0$. So setting $c := v_{g_*} \in P$, we get that $cc^* = 0$ and $c + c^*$ is invertible in J. Now, fix $r_* \in \mathscr{I}$ with $y \in J_{r_*}$ and $u \in P_{r_*}$. For all $i \in \mathscr{I}$ with $r_* \preccurlyeq i$, we have $c \in J_{r_*}$, $cc^* = 0$, and $c + c^*$ is invertible in J_{r_*}. We know that $P = \cup_{i \in \mathscr{I}_*} P_i$ in which $\mathscr{I}_* = \{i \in \mathscr{I} \mid r_* \preccurlyeq i\}$ and $J = \cup_{i \in \mathscr{I}_*} J_i$. So J is the direct union of Jordan tori J_i's, each of which contains the element c. Since $c + c^*$ is invertible in J_{r_*}, this is invertible in each J_i ($i \in \mathscr{I}_*$). Now, as $cc^* = 0$, we get that J as well as each J_i, $i \in \mathscr{I}$, is of plus type by Proposition 3.3.

- $u^2 \neq 0$.

We have $d = x_g \circ x_h \in J^{g+h}$. So there is $j_* \in \mathscr{I}$ with $d \in J_{i_*}^{g+h}$. Now, we have

$$u^2 \in (J_{i_*})_{2\gamma}, \quad ud^{-1}u \in (J_{i_*})_{\gamma}, \quad [u, d] \in (J_{i_*})_{2\gamma}$$

(see [11, Page 24]). Then P is the direct union of P_i's ($i \in \mathscr{I}_*$), where $\mathscr{I}_* := \{i \in \mathscr{I} \mid i_* \preccurlyeq i\}$ and J is the direct union of J_i's for $i \in \mathscr{I}_*$. Now, using the proof of [11, Prop. 4.7] together with [11, Prop. 4.9] either each J_i ($i \in \mathscr{I}_*$) is of plus type or each J_i ($i \in \mathscr{I}_*$) is of extension type. Moreover, by Proposition 3.3 and Lemma 3.2, either J is of plus type or of extension type, respectively. □

Theorem 3.7 *Suppose that J is a Jordan G-torus of Hermitian type. Then J is a direct union of Jordan tori of Hermitian type and it is of one of involution, plus, or extension types. Moreover, if J is of involution (resp. plus, extension) type, it is a direct union of Jordan tori of involution (resp. plus, extension) type.*

Proof The group G is a torsion free abelian group and $J = \oplus_{g\in G} J^g$ is a Jordan G-torus of Hermitian type. Let S be the support of J. Since J is of Hermitian type, $q_{48}(J) \neq 0$. Fix $x_1, \ldots, x_{12} \in J$ such that $q_{48}(x_1, \ldots, x_{12}) \neq 0$. Since $J = \oplus_{\sigma\in G} J^\sigma$, there are $\sigma_1, \ldots, \sigma_n \in G$ such that $x_1, \ldots, x_{12} \in J^{\sigma_1} \oplus \cdots \oplus J^{\sigma_n}$. Now, let

$$\mathscr{I} := \{T \subseteq S \mid \sigma_1, \ldots, \sigma_n \in T, |T| < \infty\}.$$

Set

$$G_T := \langle T\rangle, \quad S_T := S \cap G_T, \quad T \in \mathscr{I}.$$

Then

$$S = \bigcup_{T\in\mathscr{I}} S_T, \quad G_T = \langle S_T\rangle.$$

Next, set

$$J_T := \bigoplus_{\sigma\in G_T} J^\sigma, \quad T \in \mathscr{I}.$$

One has $J = \cup_{T\in\mathscr{I}} J_T$ and that each J_T is a Jordan G_T-torus. Since $x_1, \ldots, x_{12} \in J_T$ for all $T \in \mathscr{I}$, we get that $q_{48}(J_T) \neq 0$ and so J_T is of Hermitian type. So J is a direct union of Jordan tori of Hermitian type.

We know that J is special, so by [8], there is an associative algebra $\mathscr{A}$ equipped with an involution $*$ such that

- $J \subseteq H(\mathscr{A}, *)$,
- $\mathscr{A}$, as an associative algebra, is generated by J,
- if I is a $*$-ideal of $\mathscr{A}$, then $I \cap J \neq \{0\}$.

Also by the Special Hermitian Structure Theorem [8, §1.6] and Lemma 3.1, the associative subalgebra P of $\mathscr{A}$ generated by J is $*$-prime and $J = H(P, *)$. Now, if for $T \in \mathscr{I}$, $\mathscr{P}_T$ is the associative subalgebra of $\mathscr{A}$ generated by J_T, then we have $J_T = H(\mathscr{P}_T, *)$. We also have

$$\mathscr{P} = \bigcup_{T\in\mathscr{I}} \mathscr{P}_T.$$

We next note that for $T \in \mathscr{I}$, G_T is a finitely generated torsion free abelian group and so it is a free abelian group of finite rank. Now, we get the result by using Proposition 3.6. □

4 Jordan tori of Clifford type

Let R be a unital commutative associative ring, and let V be an R-module. Let $(\cdot,\cdot)\colon V\times V\to R$ be a symmetric R-bilinear form. Define a linear R-algebra structure on $J:=R1\oplus V$ by having 1 as the identity element and requiring $v\cdot w=(v,w)1$ for $v,w\in V$. Then J is a Jordan algebra called the *Jordan algebra of the bilinear form* $(\cdot,\cdot)$ (or a *Jordan spin factor* if R is a field). We recall that a Jordan algebra is called of Clifford type if its central closure is a Jordan algebra of a symmetric bilinear form.

The following example is a generalization of a Clifford torus that appeared in [1, Theorem III.2.9] as the coordinate algebra of an extended affine Lie algebra of type A_1. The setting is based on [11, Example 5.2] and [12].

Definition 4.1 Let G be an abelian group, let S be a pointed reflection subspace of G, and let Γ be a subgroup of G such that

$$2G\subseteq\Gamma\subsetneq S\subseteq G,\quad S+\Gamma=S. \tag{4.1}$$

Let I be a set of coset representatives for $\{\sigma+\Gamma\mid\sigma\in S\}\setminus\{\Gamma\}$. Then for a collection $\{a_\varepsilon\}_{\varepsilon\in I}$, $a_\varepsilon\in\mathbb{F}^\times$, we call the triple $(S,\Gamma,\{a_\varepsilon\})$ a *Clifford triple.*

Example 4.2 Let G be an abelian group, not necessarily torsion free, and let $(S,\Gamma,\{a_\varepsilon\})$ be a Clifford triple. Let Z be a commutative associative Γ-torus (a commutative twisted group algebra) with basis $\{z^\gamma\mid\gamma\in\Gamma\}$. Let V be a free Z-module with basis $\{t_\varepsilon\}_{\varepsilon\in I}$. Define a Z-bilinear form $f\colon V\times V\to Z$ by Z-linear extension of

$$f(t_\varepsilon,t_\eta)=\begin{cases}a_\varepsilon z^{2\varepsilon}, & \varepsilon=\eta,\\ 0, & \text{otherwise},\end{cases} \tag{4.2}$$

for all $\varepsilon,\eta\in I$ (here, we note that $2\varepsilon\in\Gamma$ by (4.1)). Let

$$J:=J(S,\Gamma,\{a_\varepsilon\}_{\varepsilon\in I}):=Z\oplus V$$

be the Jordan algebra over Z of f. We note that

$$V=\bigoplus_{\varepsilon\in I}Zt_\varepsilon=\bigoplus_{\varepsilon\in I,\gamma\in\Gamma}\mathbb{F}z^\gamma t_\varepsilon.$$

We also note that for $\sigma\in S$, there exists a unique $\varepsilon_\sigma\in I\cup\{0\}$ such that $\sigma-\varepsilon_\sigma\in\Gamma$. Set $t_0:=1\in J$. Now, for $\sigma\in G$, we set

$$J_\sigma:=\begin{cases}\mathbb{F}z^{\sigma-\varepsilon_\sigma}t_{\varepsilon_\sigma}, & \sigma\in S,\\ 0, & \text{otherwise}.\end{cases}$$

Then $J=\oplus_{\sigma\in G}J_\sigma$ and $\mathrm{supp}(J)=S$.

We next show that J is G-graded. Let $\sigma, \tau \in S$. If $\varepsilon_\sigma = \varepsilon_\tau = 0$, then

$$J_\sigma J_\tau = \mathbb{F} z^\sigma z^\tau = \mathbb{F} z^{\sigma+\tau} = J_{\sigma+\tau}.$$

If $\varepsilon_\sigma = 0$ and $\varepsilon_\tau \neq 0$, then

$$J_\sigma J_\tau = \mathbb{F} z^\sigma z^{\tau-\varepsilon_\tau} t_{\varepsilon_\tau} = \mathbb{F} z^{\sigma+\tau-\varepsilon_\tau} t_{\varepsilon_\tau} = J_{\sigma+\tau}.$$

Finally, suppose $\varepsilon_\sigma \neq 0$ and $\varepsilon_\tau \neq 0$. We note that if $\varepsilon_\sigma = \varepsilon_\tau$, then $\sigma + \tau \in \Gamma \subseteq S$ and $J_{\sigma+\tau} = \mathbb{F} z^{\sigma+\tau}$. Then

$$J_\sigma J_\tau = \mathbb{F} z^{\sigma-\varepsilon_\sigma} t_{\varepsilon_\sigma} z^{\tau-\varepsilon_\tau} t_{\varepsilon_\tau} = \mathbb{F} z^{\sigma+\tau-\varepsilon_\sigma-\varepsilon_\tau} f(t_{\varepsilon_\sigma}, t_{\varepsilon_\tau}) z^{2\varepsilon_\sigma} = \begin{cases} \mathbb{F} z^{\sigma+\tau}, & \varepsilon_\sigma = \varepsilon_\tau, \\ 0, & \text{otherwise.} \end{cases}$$

So $J_\sigma J_\tau = J_{\sigma+\tau}$ if $\varepsilon_\sigma = \varepsilon_\tau$, and $J_\sigma J_\tau = \{0\}$ otherwise. This completes the proof that J is a G-graded Jordan algebra over Z. Thus, J is a Jordan G-torus with $Z(J) = Z$. If G is torsion free, then we can consider the central closure $\overline{J}$ of J. If $\overline{V} := \overline{Z} \otimes_Z V$, then $\overline{J}$ can be identified with $\overline{Z} \oplus \overline{V}$. Extending f to $\overline{f} \colon \overline{V} \times \overline{V} \to \overline{Z}$ by

$$\overline{f}(\alpha \otimes v, \beta \otimes w) := \alpha\beta f(v, w),$$

one can see that $\overline{J}$ is the Jordan algebra of the extended bilinear form $\overline{f}$. Hence, J is of Clifford type, which we call it the *Clifford G-torus* associated to the Clifford triple $(S, \Gamma, \{a_\varepsilon\})$. ♢

Theorem 4.3 *Let G be a torsion free abelian group, and let J be a Jordan G-torus of Clifford type with support S and central grading group Γ. Let I be a set of coset representatives for $\{\sigma + \Gamma \mid \sigma \in S\} \setminus \{\Gamma\}$. Then for each $\varepsilon \in I$, there exists $a_\varepsilon \in \mathbb{F}^\times$ such that $(S, \Gamma, \{a_\varepsilon\})$ is a Clifford triple and J is graded isomorphic to the Clifford G-torus $J(S, \Gamma, \{a_\varepsilon\}_{\varepsilon \in I})$ associated to the Clifford triple $(S, \Gamma, \{a_\varepsilon\})$.*

Proof By assumption, the central closure $\overline{J} = \overline{Z} \otimes_Z J$ is a Jordan algebra over $\overline{Z}$ of a symmetric bilinear form, where $\overline{Z}$ is the field of fractions of the center $Z = Z(J)$ of J. Thus, $\overline{J}$ has degree less than or equal 2 over $\overline{Z}$, that is, there exists a $\overline{Z}$-linear form $\mathrm{tr} \colon \overline{J} \to \overline{Z}$ and a $\overline{Z}$-quadratic map $n \colon \overline{J} \to \overline{Z}$ with $n(1) = 1$ such that for all $x \in \overline{J}$,

$$x^2 - \mathrm{tr}(x)x + n(x)1 = 0.$$

Let $n \colon \overline{J} \times \overline{J} \to \overline{Z}$ be the symmetric $\overline{Z}$-bilinear form associated to the quadratic map n. Let $W := \{x \in \overline{J} \mid \mathrm{tr}(x) = 0\}$. Then $\overline{J} = \overline{Z}1 \oplus W$ is the Jordan algebra over $\overline{Z}$ of the bilinear form

$$h := -\frac{1}{2} n(\cdot, \cdot)_{|W \times W}.$$

If $\dim_{\overline{Z}} \overline{J} = 1$, then by Lemma 2.6 (iii), $\mathrm{supp}(J) = \Gamma = G$ and so $J = Z$. Hence, J is a commutative associative torus and so is G-graded isomorphic to the group algebra of G over $\mathbb{F}$.

We assume from now on that $\dim_{\overline{Z}} \overline{J} \neq 1$. The same argument as in [11, Claim 1] shows that

$$\operatorname{tr}(\overline{J}_{\overline{\alpha}}) = \{0\} \ (\alpha \in G \backslash \Gamma), \quad 2G \subseteq \Gamma \subsetneq \operatorname{supp}(\mathrm{J}), \quad \operatorname{supp}(J) + \Gamma = \operatorname{supp}(J). \quad (4.3)$$

Moreover,

$$G/\Gamma \text{ cannot be a nontrivial cyclic group.} \quad (4.4)$$

Recall from Lemma 2.6 (ii) that $J = \oplus_{\overline{\alpha} \in G/\Gamma} J_{\overline{\alpha}}$ is a G/Γ-graded algebra over Z. Then $\operatorname{tr}(J_{\overline{\alpha}}) \subseteq \operatorname{tr}(\overline{J}_{\overline{\alpha}}) = \{0\}$ for $\overline{\alpha} \neq \overline{0}$, by (4.3). So

$$V := \bigoplus_{\overline{\alpha} \neq \overline{0}} J_{\overline{\alpha}} = \bigoplus_{\alpha \in G \backslash \Gamma} Z J_\alpha \subseteq W.$$

Then

$$J = \bigoplus_{\overline{\alpha} \in G/\Gamma} J_{\overline{\alpha}} = \bigoplus_{0 \neq \overline{\alpha} \in G/\Gamma} J_{\overline{\alpha}} + J_{\overline{0}} = V \oplus Z,$$

as a direct sum of Z-modules. For $x, y \in V$,

$$x \cdot y = h(x, y) \cdot 1 \in J \cap \overline{Z} \cdot 1 = J \cap Z(\overline{J}) = Z.$$

Therefore, $J = Z \oplus V$ is the Jordan algebra over Z of $f := h_{|_{V \times V}}$. Let $S := \operatorname{supp}(J)$. By Lemma 2.5, S is a pointed reflection space in G. By (4.3), Γ is a proper subset of S and the pair (S,Γ) satisfies (4.1). Next, let I be a set of coset representatives for $\{\sigma + \Gamma \setminus \sigma \in S\} \setminus \{\Gamma\}$, namely,

$$S = \bigcup_{\varepsilon \in I \cup \{0\}} (\varepsilon + \Gamma).$$

For $\varepsilon \in I$, let $0 \neq t_\varepsilon \in J_\varepsilon$. Then using Lemma 2.6 (ii), we have

$$V = \bigoplus_{\overline{\alpha} \neq 0} J_{\overline{\alpha}} = \bigoplus_{\varepsilon \in I} Z t_\varepsilon,$$

as direct sum of Z-modules. We note that $Z = \oplus_{\gamma \in \Gamma} J_\gamma$ is a commutative associative Γ-torus. If $\varepsilon \neq \varepsilon' \in I$, we have $e + \varepsilon' \notin \Gamma$ (since ε and ε' are distinct coset representatives of Γ in S). Therefore,

$$t_\varepsilon t_{\varepsilon'} = f(t_\varepsilon, t_{\varepsilon'}) \in J_{\overline{\varepsilon + \varepsilon'}} \cap J_{\overline{0}} = \{0\}.$$

Also,

$$0 \neq t_\varepsilon^2 = f(t_\varepsilon, t_\varepsilon) \in J_{2\varepsilon} = \mathbb{F} z^{2\varepsilon},$$

say $f(t_\varepsilon, t_\varepsilon) = a_\varepsilon$ for some $0 \neq a_\varepsilon \in \mathbb{F}$. (We note that $2\varepsilon \in 2G \subseteq \Gamma$.) Now, since $V = \oplus_{\varepsilon \in I} Z t_\varepsilon$, it is clear that the bilinear form f here coincides with the one given in Example 4.2 (see (4.2)). Thus, J is graded isomorphic to the Clifford G-torus $J(S, \Gamma, \{a_\varepsilon\}_{\varepsilon \in I})$ of Example 4.2 associated to $(S, \Gamma, \{a_\varepsilon\})$. □

5 Jordan tori of Albert type

Throughout this section, we assume that G is a torsion free abelian group. We recall that an Albert algebra is by definition a form of a 27-dimensional central simple exceptional Jordan algebra of degree 3. We also recall that a Jordan torus of Albert type is by definition a Jordan torus whose central closure is an Albert algebra.

Definition 5.1 [11, Definition 6.4] A prime Jordan or associative algebra P over $\mathbb{F}$ is said to have *central degree* 3, if the central closure $\overline{P} = \overline{Z} \otimes_Z P$ is finite dimensional and has (generic) degree 3.

The following is a generalization of [11, Proposition 6.7] to our case. Its proof is almost the same, but for completeness, we present the proof here.

Proposition 5.2 *Let G be a torsion free abelian group, and let $T = \oplus_{\alpha \in G} T_\alpha$ be a Jordan or an associative G-torus over $\mathbb{F}$ of central degree* 3. *Let* tr *be the generic trace of the central closure $\overline{T}$, and let Γ be the central grading group of T. Then $3G \subseteq \Gamma \subsetneq G$ and* $\operatorname{supp}(T) = G$. *Moreover, for any $\alpha \in G \setminus \Gamma$, we have* $\operatorname{tr}(T_\alpha) = \{0\}$.

Proof If $G = \Gamma$, then $\dim_{\overline{Z}} \overline{T} = 1$, and hence, T does not have central degree 3. Therefore, $\Gamma \neq G$ and $\operatorname{supp}(T)/\Gamma \neq \{0\}$. Let

$$\overline{0} \neq \overline{\beta} \in \operatorname{supp}(T)/\Gamma, \quad 0 \neq x \in \overline{T}_{\overline{\beta}}.$$

Since $\overline{T} = \oplus_{\overline{\alpha} \in G/\Gamma} \overline{T}_{\overline{\alpha}}$ (see Lemma 2.6 (iii)) has generic degree 3, we have

$$x^3 + z_1 x^2 + z_2 x + z_3 1 = 0$$

for some $z_1, z_2, z_3 \in \overline{Z}$ and $z_1 = -\operatorname{tr}(x)$. If $2\overline{\beta} = \overline{0}$, then $3\overline{\beta} = \overline{\beta}$, and therefore,

$$x^3 + z_2 x = -z_1 x^2 - z_3 1 \in \overline{T}_{\overline{\beta}} \cap \overline{T}_{\overline{0}} = \{0\}.$$

Hence, we get

$$x^3 + z_2 x = x(x^2 + z_2 1) = 0.$$

Since $\overline{T}$ is a Jordan or an associative domain, the subalgebra $\overline{Z}[x]$ of $\overline{T}$ generated by x is a commutative associative algebra domain over $\overline{Z}$ and so $x^2 + z_2 1 = 0$. Since $x \notin \overline{T}_{\overline{0}}$, the polynomial $f(\lambda) = \lambda^2 + z_2$ is the minimal polynomial of x over $\overline{Z}$. If $f(\lambda)$ is reducible over $\overline{Z}$, say

$$f(\lambda) = (\lambda - a)(\lambda - b), \quad a, b \in \overline{Z},$$

then

$$(x - a1)(x - b1) = 0$$

in $\overline{Z}[x]$. Hence, $x = a1$ or $x = b1$, and so $x \in \overline{Z}1 = \overline{T}_{\overline{0}}$, that is, $\overline{\beta} = \overline{0}$, which is absurd. Therefore, $f(\lambda)$ is irreducible over $\overline{Z}$. Note that the minimal polynomial and the generic minimal polynomial of an element have the same irreducible factors. Since $f(\lambda)$ is the irreducible minimal polynomial of x, the generic minimal polynomial of x has to be a power of $f(\lambda)$. However, this is impossible since the degree of the generic minimal polynomial of x is 3. Therefore, $2\overline{\beta} \neq \overline{0}$. This implies that $3\overline{\beta} \neq \overline{\beta}$. Since $\overline{\beta} \neq \overline{0}$, we have $3\overline{\beta} \neq 2\overline{\beta}$. Hence,

$$\{3\overline{\beta}, \overline{0}\} \cap \{2\overline{\beta}, \overline{\beta}\} = \emptyset.$$

So

$$(\overline{T}_{3\overline{\beta}} + \overline{T}_{\overline{0}}) \cap (\overline{T}_{2\overline{\beta}} \oplus \overline{T}_{\overline{\beta}}) = \{0\}.$$

Since

$$x^3 + z_3 1 = -z_1 x^2 - z_2 x \in (\overline{T}_{3\overline{\beta}} + \overline{T}_{\overline{0}}) \cap (\overline{T}_{2\overline{\beta}} \oplus \overline{T}_{\overline{\beta}}),$$

we get two equalities

$$x^3 + z_3 1 = 0, \quad -z_1 x^2 - z_2 x = 0.$$

From the first equality, we have

$$0 \neq x^3 = -z_3 1 \in \overline{T}_{3\overline{\beta}} \cap \overline{T}_{\overline{0}},$$

and hence, $3\overline{\beta} = \overline{0}$. Thus, $3G \subseteq \Gamma$, and so the exponent of G/Γ is 3. Also, we have $3G \subseteq \mathrm{supp}(T)$. Since $\mathrm{supp}(T)$ is a pointed reflection space,

$$G = 3G - 2G \subseteq \mathrm{supp}(T) + 2G \subseteq \mathrm{supp}(T).$$

Thus, $G = \mathrm{supp}(T)$.

From the second equality and by the same reason above, we have

$$-z_1 x - z_2 1 = 0.$$

Then

$$-z_1 x = z_2 1 \in \overline{T}_{\overline{\beta}} \cap \overline{T}_{\overline{0}} = \{0\}.$$

Hence, $z_1 = 0$; that is, $\mathrm{tr}(x) = 0$. Therefore, for any $\alpha \in G/\Gamma$, we have $\mathrm{tr}(T_\alpha) = \{0\}$. □

The following example gives a construction of an associative algebra which will be crucial in the classification of Jordan tori of Albert type. In what follows, for $n \in \mathbb{Z}_{\geqslant 0}$, we let $\varepsilon(n) \in \{0, 1, 2\}$ be congruent mod 3 of n and $\eta(n) := n - \varepsilon(n)$.

Example 5.3 Consider the pair (G, Γ), where G is a torsion free abelian group and Γ is a subgroup of G satisfying $3G \subseteq \Gamma$ and $|G/\Gamma| = 9$. Let $\mu\colon \Gamma \times \Gamma \to \mathbb{F}^\times$ be a symmetric 2-cocycle, that is, a 2-cocycle with $\mu(\sigma, \tau) = \mu(\tau, \sigma)$ for all σ, τ. Assume that $\mathbb{F}$ contains a primitive 3rd root of unity q. We fix σ_1 and σ_2 in G such that $\{\sigma_1 + G, \sigma_2 + G\}$ is a basis for G/Γ over the field of 3 elements. Then

$$G = \bigcup_{0 \leqslant i,j \leqslant 2} (i\sigma_1 + j\sigma_2 + \Gamma).$$

Define $\lambda\colon G\times G\to\mathbb{F}^\times$ by

$$\begin{aligned}\lambda(i\sigma_1+j\sigma_2+\gamma,i'\sigma_1+j'\sigma_2+\gamma')=q^{ji'}\mu\big(\eta(i+i')\sigma_1,\eta(j+j')\sigma_2\big)\mu(\gamma,\gamma')\\ \cdot\mu(\eta(i+i')\sigma_1+\eta(j+j')\sigma_2,\gamma+\gamma')\end{aligned}\quad(5.1)$$

for $0\leqslant i,j,i',j'\leqslant 2$, $\gamma,\gamma'\in\Gamma$. We claim that λ is a 2-cocycle on G. To see this, we must show that for any three fixed elements

$$\sigma:=i\sigma_1+j\sigma_2+\gamma,\quad \tau:=i'\sigma_1+j'\sigma_2+\gamma',\quad \delta:=i''\sigma_1+j''\sigma_2+\gamma''$$

of the above form, the 2-cocycle identity (2.2) holds, namely,

$$q^{\varepsilon(j+j')i''}q^{ji'}A=q^{j\varepsilon(i'+i'')}q^{j'i''}B,$$

where

$$\begin{aligned}A:={}&\mu(\eta(\varepsilon(i+i')+i'')\sigma_1,\eta(\varepsilon(j+j')+j'')\sigma_2)\cdot\mu(\gamma+\gamma'+\eta(i+i')\sigma_1\\&+\eta(j+j')\sigma_2,\gamma'')\cdot\mu(\eta(\varepsilon(i+i')+i'')\sigma_1+\eta(\varepsilon(j+j')+j'')\sigma_2,\gamma+\gamma'\\&+\eta(i+i')\sigma_1+\eta(j+j')\sigma_2+\gamma'')\cdot\mu(\eta(i+i')\sigma_1,\eta(j+j')\sigma_2)\mu(\gamma,\gamma')\\&\cdot\mu(\eta(i+i')\sigma_1+\eta(j+j')\sigma_2,\gamma+\gamma'),\\ B:={}&\mu(\eta(i+\varepsilon(i'+i''))\sigma_1,\eta(j+\varepsilon(j'+j''))\sigma_2)\cdot\mu(\gamma'+\gamma''+\eta(i'+i'')\sigma_1\\&+\eta(j'+j'')\sigma_2,\gamma)\cdot\mu(\eta(i+\varepsilon(i'+i''))\sigma_1+\eta(j+\varepsilon(j+j''))\sigma_2,\gamma'+\gamma''\\&+\eta(i'+i'')\sigma_1+\eta(j'+j'')\sigma_2+\gamma)\cdot\mu(\eta(i'+i'')\sigma_1,\eta(j'+j'')\sigma_2)\\&\cdot\mu(\gamma',\gamma'')\cdot\mu(\eta(i'+i'')\sigma_1+\eta(j'+j'')\sigma_2,\gamma'+\gamma'').\end{aligned}$$

Since

$$q^{\varepsilon(j+j')i''}q^{ji'}=q^{j\varepsilon(i'+i'')}q^{j'i''},$$

λ is a 2-cocycle if and only if $A=B$. Let

$$\begin{aligned}a:={}&\eta(\varepsilon(i+i')+i'')\sigma_1+\eta(\varepsilon(j+j')+j'')\sigma_2\\&+\eta(i+i')\sigma_1+\eta(j+j')\sigma_2+\gamma+\gamma'+\gamma'',\\ b:={}&\eta(i+\varepsilon(i'+i''))\sigma_1+\eta(j+\varepsilon(j'+j''))\sigma_2\\&+\eta(i'+i'')\sigma_1+\eta(j'+j'')\sigma_2+\gamma'+\gamma''+\gamma.\end{aligned}$$

Then in the commutative associative torus $(\mathbb{F}^t[\Gamma]:=\oplus_{\gamma\in\Gamma}\mathbb{F}x^\gamma,\mu)$, we have

$$\begin{aligned}(x^{\eta(\varepsilon(i+i')+i'')\sigma_1}x^{\eta(\varepsilon(j+j')+j'')\sigma_2})(x^{\eta(i+i')\sigma_1}x^{\eta(j+j')\sigma_2})(x^\gamma x^{\gamma'})x^{\gamma''}=Ax^a,\\ (x^{\eta(i+\varepsilon(i'+i''))\sigma_1}x^{\eta(j+\varepsilon(j'+j''))})(x^{\eta(i'+i'')\sigma_1}x^{\eta(j'+j'')\sigma_2})(x^{\gamma'}x^{\gamma''})x^{\gamma}=Bx^b.\end{aligned}$$

Therefore, if we show that $a=b$, then we get $A=B$ if and only if

$$\begin{aligned}&(x^{\eta(\varepsilon(i+i')+i'')\sigma_1}x^{\eta(\varepsilon(j+j')+j'')\sigma_2})(x^{\eta(i+i')\sigma_1}x^{\eta(j+j')\sigma_2})\\&\quad=(x^{\eta(i+\varepsilon(i'+i''))\sigma_1}x^{\eta(j+\varepsilon(j'+j''))})(x^{\eta(i'+i'')\sigma_1}x^{\eta(j'+j'')\sigma_2})\end{aligned}\quad(5.2)$$

for any $0 \leqslant i, i', i'', j, j', j'' \leqslant 2$. Now, $a = b$ if and only if

$$\begin{aligned}
&\eta(\varepsilon(i+i')+i'')\sigma_1 + \eta(\varepsilon(j+j')+j'')\sigma_2 \\
&\qquad + \eta(i+i')\sigma_1 + \eta(j+j')\sigma_2 + \gamma + \gamma' + \gamma'' \\
&\quad = \eta(i+\varepsilon(i'+i''))\sigma_1 + \eta(j+\varepsilon(j'+j'')\sigma_2 \\
&\qquad + \eta(i'+i'')\sigma_1 + \eta(j'+j'')\sigma_2 + \gamma' + \gamma'' + \gamma,
\end{aligned}$$

which in turn holds if and only if for any i, i', i'',

$$\eta(\varepsilon(i+i')+i'') + \eta(i+i') = \eta(i+\varepsilon(i'+i'')) + \eta(i'+i''). \tag{5.3}$$

To see that this last equality holds, we note that

$$\varepsilon(i+i') + i'' + \eta(i+i') = (i+i') + i'' = i + (i'+i'') = i + \varepsilon(i'+i'') + \eta(i'+i'')$$

and so

$$\begin{aligned}
\eta(\varepsilon(i+i')+i'') + \eta(i+i') &= \eta(\varepsilon(i+i') + i'' + \eta(i+i')) \\
&= \eta(i + \varepsilon(i'+i'') + \eta(i'+i'')) \\
&= \eta(i+\varepsilon(i'+i'')) + \eta(i'+i'').
\end{aligned}$$

Then (5.3) holds and $a = b$. Thus, $A = B$ if and only if (5.2) holds. Now, the left-hand side in (5.2) is equal to

$$\mu(\eta(\varepsilon(i+i')+i'')\sigma_1, \eta(i+i')\sigma_1)x^{c_1\sigma_1}\mu(\eta(\varepsilon(j+j')+j'')\sigma_2, \eta(j+j')\sigma_2)x^{c_2\sigma_2}$$

where

$$c_1 := \eta(\varepsilon(i+i')+i'') + \eta(i+i'), \quad c_2 := \eta(\varepsilon(j+j')+j'') + \eta(j+j').$$

Also the right hand side in (5.2) is equal to

$$\mu(\eta(i+\varepsilon(i'+i''))\sigma_1, \eta(i'+i'')\sigma_2)x^{c_1'\sigma_1}\mu(\eta(j+\varepsilon(j'+j''))\sigma_2, \eta(j'+j'')\sigma_2)x^{c_2'\sigma_2},$$

where

$$c_1' := \eta(i+\varepsilon(i'+i'') + \eta(i'+i''), \quad c_2' := \eta(j+\varepsilon(j'+j'')) + \eta(j'+j'').$$

By (5.3), $c_1 = c_2$ and $c_1' = c_2'$. So $A = B$ if and only if

$$\begin{aligned}
&\mu(\eta(\varepsilon(i+i')+i'')\sigma_1, \eta(i+i')\sigma_1)\mu(\eta(\varepsilon(j+j')+j'')\sigma_2, \eta(j+j')\sigma_2) \\
&\quad = \mu(\eta(i+\varepsilon(i'+i'')\sigma_1), \eta(i'+i'')\sigma_1)\mu(\eta(j+\varepsilon(j'+j'')\sigma_2), \eta(j'+j'')\sigma_2)
\end{aligned}$$

for all $0 \leqslant i, i', i'', j, j', j'' \leqslant 2$. But clearly, the latter holds if and only if

$$\mu(\eta(\varepsilon(i+i')+i'')\sigma, \eta(i+i')\sigma) = \mu(\eta(i+\varepsilon(i'+i'')\sigma, \eta(i'+i'')\sigma) \tag{5.4}$$

for all $\sigma \in G$ and $0 \leqslant i, i', i'' \leqslant 2$. Let

$$\alpha := \varepsilon(i+i') + i'', \quad \beta := i+i', \quad \alpha' := i + \varepsilon(i'+i''), \quad \beta' := i'+i''.$$

Then

$$\eta(\alpha) + \eta(\beta) = \eta(\alpha + \eta(\beta)) = \eta(i+i'+i'').$$

Similarly,

$$\eta(\alpha') + \eta(\beta') = \eta(i+i'+i'').$$

So

$$\eta(\alpha) + \eta(\beta) = \eta(\alpha') + \eta(\beta').$$

From this and the fact that $\eta(\alpha), \eta(\beta), \eta(\alpha'), \eta(\beta') \in \{0, 3\}$, we get

$$\eta(\alpha)\eta(\beta) = \eta(\alpha')\eta(\beta') \in \{0, 9\}.$$

Therefore, (5.4) holds. Hence, $A = B$ and λ is a 2-cocycle.

We denote the 2-cocycle λ by $\lambda := \lambda(q, \mu)$ and the corresponding associative G-torus by $(\mathbb{F}^t[G], \lambda(q, \mu))_\Gamma$, and call it the associative G-torus associated to the pair (G, Γ). Let $T = (\mathbb{F}^t[G], \lambda(q, \mu))_\Gamma$. Note that if we fix $x_i := x^{\sigma_i} \in T_{\sigma_i}$, $i = 1, 2$, and $x^\gamma \in T^\gamma$, for each $\gamma \in \Gamma$, then the elements $x_1^{i_1} x_2^{i_2} x^\gamma$, $0 \leqslant i_1, i_2 \leqslant 2$, $\gamma \in \Gamma$ form a basis of T over $\mathbb{F}$. Moreover, we have

$$(x_1^i x_2^j x^\gamma)(x_1^{i'} x_2^{j'} x^{\gamma'}) = \lambda(i\sigma_1 + j\sigma_2 + \gamma, i'\sigma_1 + j'\sigma_2 + \gamma') x_1^{\varepsilon(i+i')} x_2^{\varepsilon(j+j')} x^{\gamma''},$$

where $0 \leqslant i, j, i', j' \leqslant 2$, $\gamma, \gamma' \in \Gamma$, and $\gamma'' = \gamma + \gamma' + \eta(i+i') + \eta(j+j')$. Using this, it is easy to see that

$$Z(T) = \bigoplus_{\gamma \in \Gamma} \mathbb{F} x^\gamma, \quad x_2 x_1 = q x_1 x_2, \quad x_1^i x_2^j \in Z(T) \tag{5.5}$$

for all $i, j \in \mathbb{Z}$ with $i \equiv j \equiv 0 \pmod 3$. It follows that the central closure of T is 9-dimensional, namely,

$$\overline{J} := \overline{Z} \otimes_Z T \cong \bigoplus_{0 \leqslant i,j \leqslant 2} \mathbb{F} x_1^i x_2^j.$$

Since $\overline{T}$ is domain, it is a division algebra and so is an associative algebra of central degree 3 (see Definition 5.1). Note that $Z(T^+) = Z(T)$. Then

$$\overline{T^+} = T^+ \otimes_{Z(T^+)} \overline{Z(T^+)} = T^+ \otimes_{Z(T)} \overline{Z(T)} = \overline{T}^+.$$

So $\overline{T^+}$ is a 9-dimensional central special Jordan division algebra over $\overline{Z(T)}$. Hence, by [11, Lemma 2.11], it has degree 3. ♢

The following proposition gives a characterization of associative G-tori of central degree 3.

Proposition 5.4 *Let G be a torsion free abelian group, and let T be an associative G-torus over $\mathbb{F}$ with central grading group Γ. Then T has central degree 3 if and only if $3G \subseteq \Gamma$, $\mathrm{supp}(T) = G$, and G/Γ is a vector space of dimension 2 over the field of 3 elements. If T has central degree 3, then $\mathbb{F}$ contains a primitive third root of unity, say ω, and $T \cong (\mathbb{F}^t[G], \lambda(\omega, \mu))_\Gamma$, where μ is a symmetric 2-cocycle on Γ. Moreover, if Γ is free abelian or $\mathbb{F}$ is algebraically closed, then $T \cong (\mathbb{F}^t[G], \lambda(\omega, 1))$.*

Conversely, suppose that $\mathbb{F}$ contains a primitive third root of unity ω. Also suppose that G is a torsion free abelian group and Γ is a subgroup satisfying $3G \subseteq \Gamma$ and $|G/\Gamma| = 9$. Let μ be a symmetric 2-cocycle on Γ. Then $(\mathbb{F}^t[G], \lambda(\omega, \mu))_\Gamma$ is an associative G-torus of central degree 3 with central grading group Γ.

Proof Let $T = \oplus_{\alpha\in G} T^\alpha$ be an associative torus over $\mathbb{F}$ of central degree 3, and let $\overline{T}$ be its central closure over $\overline{Z}$. By Proposition 5.2, $\mathrm{supp}(T) = G$ and G/Γ is a nontrivial vector space over the field of 3 elements. By Lemma 2.6 (iii), we have $\dim_{\overline{Z}} \overline{T} = |G/\Gamma|$. Since, by definition, $\overline{T}$ is finite dimensional over $\overline{Z}$, we have $\dim_{\overline{Z}} \overline{T} = 3^m$ for some positive integer m. Now, $\overline{T}$ as a finite-dimensional associative domain is a division algebra, by Wedderburn's structure theorem. So as $\overline{T}_{\overline{Z}}$ is a central simple associative algebra with $\dim \overline{T}_{\overline{Z}} = 3^m$, we have $m = 2$. It is also clear that an associative torus whose central grading group Γ satisfies $|G/\Gamma| = 9$ has central degree 3. In fact, $\overline{T}$ has dimension 9 over $\overline{Z}$ and is a division associative algebra. So by Lemma [11, Lemma 2.11], it has degree 3.

Next, we assume that $T = \oplus_{\alpha\in G} T^\alpha$ is an associative torus whose central grading group satisfies $3G \subseteq \Gamma \subsetneq G$, $|G/\Gamma| = 9$, and $\mathrm{supp}(T) = G$. We fix σ_1, σ_2 in G such that $\{\sigma_i + \Gamma \mid i = 1, 2\}$ is a basis for the vector space G/Γ. Then

$$G = \bigcup_{0\leqslant i,j\leqslant 2} (i\sigma_1 + j\sigma_2 + \Gamma).$$

We fix $x_i := x^{\sigma_i} \in T^{\sigma_i}$, $i = 1, 2$, and $x^\gamma \in T^\gamma$ for each $\gamma \in \Gamma$. Then $x_1x_2 \neq x_2x_1$ and the elements $x_1^{i_1} x_2^{i_2} x^\gamma$, $0 \leqslant i_1, i_2 \leqslant 2$, $\gamma \in \Gamma$ form a basis for T over $\mathbb{F}$. Moreover, as $3G \subseteq \Gamma$,

$$(x_1^{i_1} x_2^{i_2})^3 x^\gamma \in Z(T) \tag{5.6}$$

for all $0 \leqslant i_1, i_2 \leqslant 2$ and $\gamma \in \Gamma$. Since $x_1x_2, x_2x_1 \in T^{\sigma_1+\sigma_2}$, there exists $q \in \mathbb{F}^\times$ such that $x_2x_1 = qx_1x_2$. Then as x_1^3 is central, we get $q^3 = 1$. Thus, $\mathbb{F}$ must contain a primitive third root of unity, say ω. Then $q = \omega$ or ω^2. Let $\lambda\colon G \times G \to \mathbb{F}^\times$ be the corresponding 2-cocycle for T with respect to the basis mentioned above. Then we have

$$(x_1^i x_2^j x^\gamma)(x_1^{i'} x_2^{j'} x^{\gamma'}) = \lambda(i\sigma_1 + j\sigma_2 + \gamma, i'\sigma_1 + j'\sigma_2 + \gamma') x_1^{\varepsilon(i+i')} x_2^{\varepsilon(j+j')} x^{\gamma''},$$

where $0 \leqslant i, j, i', j' \leqslant 2$, $\gamma, \gamma' \in \Gamma$, $\gamma'' = \gamma + \gamma' + \eta(i + i') + \eta(j + j')$, and ε and η are defined as in Example 5.3. Denote by $\mu\colon \Gamma \times \Gamma \to \mathbb{F}^\times$ the symmetric

2-cocycle obtained from λ by restriction to Γ. Then using (5.6), the facts that $x_2x_1 = qx_1x_2$, and $\eta(n)G \subseteq 3G \subseteq \Gamma$ for all $n \in \mathbb{Z}$, we see that

$$\lambda(i\sigma_1 + j\sigma_2 + \gamma, i'\sigma_1 + j'\sigma_2 + \gamma') = q^{ji'}\mu(\eta(i+i')\sigma_1, \eta(j+j')\sigma_2)\mu(\gamma,\gamma') \cdot \mu(\eta(i+i')\sigma_1 + \eta(j+j')\sigma_2, \gamma+\gamma') \quad (5.7)$$

for $0 \leqslant i, j, i', j' \leqslant 2$, $\gamma, \gamma' \in \Gamma$. Then, in the notation of Example 5.3, we have $T = (\mathbb{F}^t[G], \lambda(q,\mu))_\Gamma$. But one can see that the corresponding associative tori $(\mathbb{F}^t[G], \lambda(\omega,\mu))_\Gamma$ and $(\mathbb{F}^t[G], \lambda(\omega^2,\mu))_\Gamma$ are isomorphic, under the isomorphism induced by $x_1^{i_1}x_2^{i_2}x^\gamma \mapsto x_2^{i_1}x_1^{i_2}x^\gamma$. So we may assume that $q = \omega$, namely, $T = (\mathbb{F}^t[G], \lambda(\omega,\mu))_\Gamma$. We recall from [9, Lemma 1.1] that if Γ is free abelian or $\mathbb{F}$ is algebraically closed, then any commutative twisted group algebra on Γ is isomorphic to the commutative untwisted group algebra. Thus, if Γ is free abelian or $\mathbb{F}$ is algebraically closed, then μ can be taken to be 1. The converse part follows from Example 5.3. □

Remark 5.5 In the notation of Proposition 5.4, let G be free abelian of rank $\geqslant 2$ with a basis indexed by a set, say J. Assume, $1, 2 \in J$. By Proposition 5.4, $T \cong (\mathbb{F}^t[G], \lambda(\omega, 1))$. However, by Example 2.9, we may assume $\lambda(\omega,1) = \mathbf{q}_\omega$, where $\mathbf{q}_\omega = (q_{ij})_{i,j\in J}$ is the quantum matrix satisfying

$$q_{ij} = \begin{cases} \omega, & i=1,\ j=2, \\ \omega^{-1}, & i=2,\ j=1, \\ 1, & \text{otherwise.} \end{cases} \quad (5.8)$$

Using our earlier results and a modified reasoning of [11, Proposition 6.13], we get the following result. To be precise, we provide details of the proof.

Proposition 5.6 *Let ω be a third root of unity. Let J be a special Jordan G-torus over $\mathbb{F}$ of central degree 3 with central grading group Γ. Then $3G \subseteq \Gamma \subsetneq G$ and $|G/\Gamma| = 9$. Also,*

$$J \cong_G \begin{cases} (\mathbb{F}^t[G], \lambda(\omega,\mu))_\Gamma^+, & \omega \in \mathbb{F}, \\ H((\mathbb{E}^t[G], \lambda(\omega,\mu))_\Gamma, \sigma), & \omega \notin \mathbb{F}, \end{cases}$$

where μ is a 2-cocycle on Γ, $\mathbb{E} = \mathbb{F}(\omega) = \mathbb{F}(\sqrt{-3})$, and σ is the unique non-trivial Galois automorphism of E.

Proof Since J is special, it is either a Hermitian torus or a Clifford torus. We have already seen that if J is a Clifford torus, then $\deg(\overline{J}) \leqslant 2$ (see § 4). So J can only be a Hermitian torus. By Proposition 5.2, $\mathrm{supp}(J) = G$. Therefore, by Theorem 3.7, we have one of the following three possibilities:

$J \cong H((\mathbb{F}^t[G], \lambda), \theta_q)$, λ a 2-cocycle, and q a quadratic map,

$J \cong (\mathbb{F}^t[G], \lambda)^+$, λ a 2-cocycle,

$J \cong H((\mathbb{E}^t[G], \lambda), \theta)$, $\mathbb{E}$ a quadratic field extension of $\mathbb{F}$, λ a 2-cocycle, and θ an involution, as defined in Lemma 3.2.

We begin by showing that the first possibility cannot happen. Consider the center Z of $J = H((\mathbb{F}^t[G], \lambda), \theta_q)$. By Proposition 5.2,

$$3G \subseteq \Gamma \subsetneq G = \mathrm{supp}(J).$$

But as q is a quadratic map, we have $(x^\sigma)^2$ is central for any $\sigma \in G$ implying that $2G \subseteq \Gamma$. Now, $2G \cup 3G \subseteq \Gamma$ implies $\Gamma = G$, which is absurd.

We now consider the second and the third possibilities. By definition, $\overline{J}$ is a finite-dimensional central special Jordan division algebra over $\overline{Z}$ of degree 3. By [11, 2.11] and Proposition 2.6 (iii), we have $\dim_{\overline{Z}} \overline{J} = 9$ and G/Γ is a 2-dimensional vector space over the field of 3 elements.

If $J \cong_G (\mathbb{F}^t[G], \lambda)^+$, λ a 2-cocycle, then taking this isomorphism as an identification, we get

$$Z(J) = Z((\mathbb{F}^t[G], \lambda)^+) = Z((\mathbb{F}^t[G], \lambda)),$$

and so Γ is the central grading group of $(\mathbb{F}^t[G], \lambda)$. Then by Proposition 5.4, $\mathbb{F}$ contains a primitive third root of unity ω and

$$(\mathbb{F}^t[G], \lambda) \cong (\mathbb{F}^t[G], \lambda(\omega, \mu))_\Gamma,$$

where μ is a symmetric 2-cocycle on Γ. Thus,

$$J \cong (\mathbb{F}^t[G], \lambda(\omega, \mu))_\Gamma^+.$$

Finally, we suppose that the third possibility holds and we take it as an identification. Then

$$Z((\mathbb{E}^t[G], \lambda)) = Z((\mathbb{E}^t[G], \lambda)^+) = Z(J \otimes_{\mathbb{F}} \mathbb{E}) \cong Z(J) \otimes_{\mathbb{F}} \mathbb{E}.$$

So $(\mathbb{E}^t[G], \lambda)$ is an associative G-torus with central grading group Γ such that $3G \subseteq \Gamma \subsetneq G$ and G/Γ is a 2-dimensional vector space over $\mathbb{Z}_3$. Then by Proposition 5.4, $(\mathbb{E}^t[G], \lambda)$ has central degree 3 and $\mathbb{E}$ contains a primitive third root of unity ω such that

$$(\mathbb{E}^t[G], \lambda) \cong (\mathbb{E}^t[G], \lambda(\omega, \mu))_\Gamma,$$

where μ is a 2-cocycle on Γ. It follows from Lemma 3.2 that $\theta(x_i) = x_i$ for $i = 1, 2$ and that θ acts as an anti-automorphism on $(\mathbb{E}^t[G], \lambda(\omega, \mu))_\Gamma$. Therefore,

$$x_1 x_2 = \theta(x_2 x_1) = \theta(\omega x_1 x_2) = \theta(\omega)\omega x_1 x_2.$$

Thus, $\theta(\omega) = \omega^{-1} \neq \omega$ and so $\omega \notin \mathbb{F}$. Finally, as $[\mathbb{E} \colon \mathbb{F}] = 2$, we have

$$\mathbb{E} = \mathbb{F}(\omega) = \mathbb{F}(\sqrt{-3}). \qquad \square$$

Definition 5.7 Let G be a torsion free abelian group, and let Δ, Γ be two subgroups of G satisfying

$$3G \subsetneq \Gamma \subseteq \Delta \subseteq G, \quad \dim_{\mathbb{Z}_3}(G/\Gamma) = 3, \quad \dim_{\mathbb{Z}_3}(\Delta/\Gamma) = 2.$$

Then we call the triple (G, Δ, Γ) an *Albert triple*.

Example 5.8 Let (G, Δ, Γ) be an Albert triple. We take $\sigma_1, \sigma_2, \sigma_3 \in G$ such that $\{\sigma_i + \Gamma \mid 1 \leqslant i \leqslant 3\}$ is a basis for G/Γ and $\{\sigma_i + \Gamma \mid 1 \leqslant i \leqslant 2\}$ is a basis for Δ/Γ. Then

$$G = \bigcup_{0 \leqslant i,j,k \leqslant 2} (i\sigma_1 + j\sigma_2 + k\sigma_3 + \Gamma), \quad \Delta = \bigcup_{0 \leqslant i,j \leqslant 2} (i\sigma_1 + j\sigma_2 + \Gamma).$$

Let

$$\mathscr{A} := (\mathbb{F}^t[\Delta], \lambda(\omega, \mu))_\Gamma = \bigoplus_{\sigma \in \Delta} \mathscr{A}^\sigma$$

be the Δ-tori associated to the pair (Δ, Γ) (see Example 5.3), where μ is a 2-cocycle on Γ and ω is a third root of unity. Let $Z = Z(\mathscr{A})$, and let tr be the generic trace of the central closure $\overline{\mathscr{A}}$. We fix nonzero elements $u_1 \in \mathscr{A}^{\sigma_1}$, $u_2 \in \mathscr{A}^{\sigma_2}$, and $u_3 \in \mathscr{A}^{3\sigma_3}$. We note that $\mathscr{A}$ is a free Z-module with free basis $\{u_1^i u_2^j \mid 0 \leqslant i, j \leqslant 2\}$. Since tr is Z-linear, for any $z \in Z$ and a basis element $u_1^i u_2^j$, we have

$$\operatorname{tr}(u_1^i u_2^j z) = z \operatorname{tr}(u_1^i u_2^j) = 0 \quad ((i,j) \neq (0,0))$$

by Proposition 5.2, and so $\operatorname{tr}(\mathscr{A}) \subseteq Z$. Since u_3 is an invertible element of Z, we consider the first Tits construction $\mathbb{A}_t = (\mathscr{A}, u_3)$ (see [11, 6.5]). We call $\mathbb{A}_t$ the *Jordan algebra associated to the Albert triple* (G, Δ, Γ).

Claim $\mathbb{A}_t$ is a Jordan G-torus of strong type.

To see this, we first give a G-grading to $\mathbb{A}_t$ as follows. Recall that $u_i \in \mathscr{A}^{\sigma_i}$ for $i = 1, 2$ and $u_3 \in \mathscr{A}^{3\sigma_3}$. We now fix $u_0 = 1 \in \mathbb{F} = \mathscr{A}^0$ and nonzero elements $u_\gamma \in \mathscr{A}^\gamma$ for $\gamma \in \Gamma \setminus \{0, 3\sigma_3\}$. For $\alpha = i\sigma_1 + j\sigma_2 + \gamma \in \Delta$, $0 \leqslant i, j \leqslant 2$, $\gamma \in \Gamma$, set $u_\alpha := u_1^i u_2^j u_\gamma$. Then we have

$$\mathscr{A} = \bigoplus_{\alpha \in \Delta} \mathbb{F} u_\alpha.$$

Next, for $\alpha = i\sigma_1 + j\sigma_2 + k\sigma_3 + \gamma \in G$, $0 \leqslant i, j, k \leqslant 2$, $\gamma \in \Gamma$, we set

$$t_\alpha := \begin{cases} (u_\alpha, 0, 0), & k = 0, \\ (0, u_{\alpha - \sigma_3}, 0), & k = 1, \\ (0, 0, u_{\alpha + \sigma_3}), & k = 2. \end{cases}$$

We have

$$t_{\sigma_3} = (0, 1, 0), \quad t_{2\sigma_3} = (0, 0, u_3), \quad t_{-\sigma_3} = t_{\sigma_3}^{-1} = (0, 0, 1).$$

One easily checks that, as a vector space, we have

$$\mathbb{A}_t = \bigoplus_{\alpha \in G} \mathbb{F} t_\alpha.$$

Moreover, considering the multiplication rule in $\mathbb{A}_t$, it is not hard, even though tedious, to see that $\mathbb{A}_t$ is strongly G-graded as a Jordan algebra and so $\mathbb{A}_t$ is a G-torus of strong type. To be more precise on this, we give a rough outline of the argument as follows. Let us recall that, as a vector space, we have

$$\mathbb{A}_t = \mathscr{A} \oplus \mathscr{A} \oplus \mathscr{A}.$$

Now, for $a \in \mathscr{A}$, we set

$$a^{(0)} := (a, 0, 0), \quad a^{(1)} := (0, a, 0), \quad a^{(2)} := (0, 0, a).$$

Also for $\alpha = i\sigma_1 + j\sigma_2 + k\sigma_3 + \gamma \in G$ of the above form, we set

$$(\alpha) := \begin{cases} 0, & k = 0, \\ -1, & k = 1, \\ 1, & k = 2. \end{cases}$$

Then we have

$$t_\alpha = u^{(k)}_{\alpha+(\alpha)\sigma_3}.$$

Now, if $\alpha' = i'\sigma_1 + j'\sigma_2 + k'\sigma_3 + \gamma'$ is another element in G of the above form, then it is easy to see that

$$u^{(k)}_{\alpha+(\alpha)\sigma_3} \times u^{(k')}_{\alpha'+(\alpha')\sigma_3} = r u^{(k)}_{\alpha+(\alpha)\sigma_3} \cdot u^{(k')}_{\alpha'+(\alpha')\sigma_3}$$

for some $s \in \mathbb{Z}/2$. Therefore,

$$t_\alpha t_{\alpha'} = r(u_3^{(\alpha)(\alpha')(\alpha+\alpha')} u^{(k)}_{\alpha+(\alpha)\sigma_3} \cdot u^{(k')}_{\alpha'+(\alpha')\sigma_3})^{(\varepsilon(k+k'))}.$$

But

$$u_3^{(\alpha)(\alpha')(\alpha+\alpha')} u^{(k)}_{\alpha+(\alpha)\sigma_3} \cdot u^{(k')}_{\alpha'+(\alpha')\sigma_3}$$

is a homogeneous element of degree

$$3(\alpha)(\alpha')(\alpha+\alpha')\sigma_3 + \alpha + \alpha' + (\alpha)\sigma_3 + (\alpha')\sigma_3 = \alpha + \alpha' + (\alpha+\alpha')\sigma_3.$$

It follows that

$$t_\alpha t_{\alpha'} = r u^{\varepsilon(k+k')}_{\alpha+\alpha'+(\alpha+\alpha')} = r t_{\alpha+\alpha'}$$

for some scalar r. This shows that $\mathbb{A}_t$ is G-graded. To see that it is of strong type, we need to show that r is nonzero, or equivalently,

$$a \times b \neq 0$$

if

$$a := u^{(k)}_{\alpha+(\alpha)\sigma_3}, \quad b := u^{(k')}_{\alpha'+(\alpha')\sigma_3}.$$

Suppose to the contrary that $a \times b = 0$. Then we must have

$$\mathrm{tr}(a \cdot b) = \mathrm{tr}(a)\mathrm{tr}(b).$$

Now, if both a and b are central, then this gives $ab = 3ab$ as $\mathrm{tr}(1) = 3$, which is absurd. If a is central but b not, then we get $\mathrm{tr}(a \cdot b) = 0$, which in turn implies $ab = 0$, which is again absurd. Finally, if both a and b are non-central, then again we get $\mathrm{tr}(a \cdot b) = 0$, which together with $a \times b = 0$ implies $a \cdot b = 0$, or equivalently, $ab = -ba$. Then

$$ab = -ba = \omega^t ba$$

for some integer t, which is absurd as ω is a third root of unity.

By [11, Lemma 6.5], the central closure $\overline{\mathbb{A}_t}$ of $\mathbb{A}_t$ is an Albert algebra over $\overline{Z}$, and so $\mathbb{A}_t$ is a Jordan G-torus of Albert type. We refer to $\mathbb{A}_t$ as an *Albert G-torus* constructed from an Albert triple (G, Δ, Γ). ♢

Theorem 5.9 *Let J be a Jordan G-torus of Albert type over $\mathbb{F}$ with central grading group Γ. Then G contains a subgroup Δ such that (G, Δ, Γ) is an Albert triple and J is graded isomorphic to the Albert G-torus $\mathbb{A}_t$, constructed from the Albert triple (G, Δ, Γ) (see Example 5.8). Conversely, given an Albert triple (G, Δ, Γ), the associated Jordan algebra $\mathbb{A}_t$ is an Albert G-torus.*

Proof Let $J = \oplus_{\sigma \in G} J^\sigma$ be a Jordan G-torus as in the statement. Then the central closure $\overline{J}$ is an Albert algebra over $\overline{Z}$, $Z := Z(J)$. We recall that an Albert algebra is a 27-dimensional central simple exceptional Jordan algebra of degree 3. By Proposition 5.2,

$$3G \subsetneq \Gamma \subseteq G, \quad \mathrm{supp}(J) = G.$$

Moreover, by Lemma 2.6,

$$27 = \dim_{\overline{Z}} \overline{J} = |G/\Gamma|.$$

Since G/Γ is a vector space over the field of 3 elements, we have

$$\dim_{\mathbb{Z}_3}(G/\Gamma) = 3.$$

Fix $\sigma_1, \sigma_2, \sigma_3 \in G$ such that $\{\sigma_i + \Gamma \mid i = 1, 2, 3\}$ is a basis for G/Γ. Then

$$G = \bigcup_{0 \leqslant i,j,k \leqslant 2} (i\sigma_1 + j\sigma_2 + k\sigma_3 + \Gamma).$$

Set

$$\Delta := \bigcup_{1 \leqslant i,j \leqslant 2} (i\sigma_1 + j\sigma_2 + \Gamma), \quad U := \bigoplus_{\sigma \in \Delta} J_\sigma.$$

Since $3G \subseteq \Gamma$, Δ is a subgroup of G and so U is a subalgebra of J. We now show that $Z(U) = Z(J)$. Since $\Gamma \subseteq \Delta$, we have $Z(J) \subseteq Z(U)$. Thus, we must show $Z(U) \subseteq Z(J)$. Let Δ_1 be the central grading group of U. Then

$$3G \subseteq \Gamma \subseteq \Delta_1 \subseteq \Delta \subseteq G.$$

We now note that $\Delta_1 \subsetneq \Delta$, because otherwise U is commutative and associative and as J is an Albert division algebra, the subfield $\overline{Z} \otimes_Z U$ of J is 9-dimensional, since $|\Delta/\Gamma| = 9$. But it follows from [4, Lemma 1] that this is impossible.

By Lemma 2.6, the central closure

$$\overline{U} := \overline{Z(U)} \otimes_{Z(U)} U$$

is (Δ/Δ_1)-graded and Δ/Δ_1 cannot be a non-trivial cyclic group. Thus,

$$2 \leqslant \dim(\Delta/\Delta_1) \leqslant \dim(\Delta/\Gamma) = 2.$$

This gives

$$\dim(\Delta/\Delta_1) = 2, \quad \Delta_1 = \Gamma.$$

That is,

$$Z(U) = Z = Z(J).$$

Since $Z(U) = Z(J)$, we have

$$\overline{U} = \overline{Z} \otimes_Z U \hookrightarrow \overline{J}.$$

By [11, 2.6 (ii)], $\overline{U}$ is central. Thus, $\overline{U}$ is a central subalgebra of the division algebra $\overline{J}$ and is 9-dimensional as $|\Delta/\Gamma| = 9$. So by the classification of finite-dimensional central simple Jordan algebras, $\overline{U}$ is special (see [4, Corollary 2, pp. 204–207]). Then by [11, 2.11], $\overline{U}$ has degree 3. Thus, U is a special Jordan G-torus of central degree 3. So we may use the characterization given in Proposition 5.6 for U, in terms of a primitive third root of unity ω and a 2-cocycle μ on Γ, namely,

$$U \cong_G \begin{cases} (\mathbb{F}^t[\Delta], \lambda(\omega, \mu))^+_\Gamma, & \omega \in \mathbb{F}, \\ H((\mathbb{E}^t[\Delta], \lambda(\omega, \mu))_\Gamma, \sigma), & \omega \notin \mathbb{F}, \end{cases}$$

where $\mathbb{E} = \mathbb{F}(\omega)$ and σ is the non-trivial Galois automorphism of E.

We assume first that $\omega \in \mathbb{F}$. Then $U = (\mathbb{F}^t[\Delta], \lambda(\omega, \mu))^+_\Gamma$. We fix nonzero elements

$$u_1 := u_{\sigma_1} \in J^{\sigma_1}, \quad u_2 := u_{\sigma_2} \in J^{\sigma_2}, \quad x \in J^{\sigma_3}.$$

Set

$$u_3 := u_{3\sigma_3} := x^3 \in J^{3\sigma_3}.$$

Let tr be the generic trace of $\overline{J}$. We have

$$\begin{aligned} U &= \bigoplus_{\sigma\in\Delta} J^\sigma \\ &= \bigoplus_{0\leqslant i,j\leqslant 2,\, \gamma\in\Gamma} J^{i\sigma_1+j\sigma_2+\gamma} \\ &= \bigoplus_{0\leqslant i,j\leqslant 2,\, \gamma\in\Gamma} J^\gamma J^{i\sigma_1+j\sigma_2} \\ &= \bigoplus_{0\leqslant i,j\leqslant 2} Z J^{i\sigma_1+j\sigma_2}, \end{aligned}$$

where the second equality follows from Lemma 2.6 (i). Thus, U is a free Z-module with basis $\{u_1^i u_2^j \mid 0 \leqslant i, j \leqslant 2\}$. Now, for $z \in Z$ and $0 \leqslant i, j \leqslant 2$,

$$\mathrm{tr}(z u_1^i u_2^j) = z\mathrm{tr}(u_1^i u_2^j) = 0 \quad ((i,j) \neq (0,0))$$

by Proposition 5.2, and is equal to $z\mathrm{tr}(1)$ if $i = j = 0$. Thus,

$$\mathrm{Tr}(\mathbb{F}^t[\Delta], \lambda(\omega, \mu))_\Gamma \subseteq Z.$$

Since $x^3 = u_3$ is an invertible element of Z, we may consider the first Tits construction $\mathbb{A}_t := (\mathscr{A}, u_3)$ over Z, where $\mathscr{A} := (\mathbb{F}^t[\Delta], \lambda(\omega, \mu))_\Gamma$ (see [11, 6.5]). As we have seen in Example 5.8, $\mathbb{A}_t$ is a Jordan G-torus of strong type.

Next, let

$$U^\perp := \{y \in J \mid \mathrm{Tr}(Uy) = 0\}.$$

We show that $J^{\sigma_3}, J^{2\sigma_3} \subseteq U^\perp$. Now, for $0 \leqslant i, j \leqslant 2$ and $k = 1, 2$, we have $(u_1^i u_2^j)x^k \in G \setminus \Gamma$, so $\mathrm{Tr}((u_1^i u_2^j)x^k) = 0$, again by Proposition 5.2. Since tr is Z-linear, we are done.

Now, setting

$$\mathscr{J} := J, \quad \mathscr{U} := \mathscr{A}^+, \quad z := u_3,$$

we see that the conditions of [11, 6.14] hold for the mentioned elements. Therefore, J contains a subalgebra J' such that one of the following holds:

(I) there exists a Z-isomorphism $\varphi\colon (\mathscr{A}, u_3) \to J'$, which acts as identity on $\mathscr{A}$ and $\varphi((0,1,0)) = x$;

(II) there exists a Z-isomorphism $\varphi\colon (\mathscr{A}, u_3^{-1}) \to J'$, which acts as identity on $\mathscr{A}$ and $\varphi((0,0,1)) = x$.

We assume first that (I) holds and take $\sigma \in G$. Then

$$\sigma = i\sigma_1 + j\sigma_2 + k\sigma_3 + \gamma,$$

where $0 \leqslant i, j, k \leqslant 2$ and $\gamma \in \Gamma$. Since $\mathbb{A}_t$ is of strong type,

$$u_0 := t_{\sigma_1}^i \cdot (t_{\sigma_2}^j \cdot (t_{\sigma_3}^k \cdot t_\gamma))$$

is a nonzero element of $\mathbb{A}_t$ and

$$0 \neq \varphi(u_0) = u_1^i u_2^j x^k u_\gamma \in J^\alpha.$$

Thus, φ is an isomorphism over Z, in particular, $J \cong_G \mathbb{A}_t$.

Next, we assume that (II) holds. We note that the $\mathbb{F}$-linear map

$$f\colon \mathscr{A} = \bigoplus_{\alpha \in \Delta} \mathbb{F}u_\alpha \to \mathscr{A}^{op}$$

induced by

$$u_1^i u_2^j u_\gamma \mapsto u_2^i u_1^j u_\gamma$$

is an algebra isomorphism over $\mathbb{F}$. We note that $f(u_3) = u_3$ and $\mathrm{Tr} \circ f = f \circ \mathrm{Tr}$. It follows that $J \cong_G \mathbb{A}_t$. This takes care of the case $\omega \in \mathbb{F}$.

Finally, we consider the case $\omega \notin \mathbb{F}$. Then we have

$$U \cong_G H((\mathbb{E}^t[\Delta], \lambda(\omega, \mu))_\Gamma, \sigma),$$

where $\mathbb{E} = \mathbb{F}(\omega)$ and σ is the non-trivial Galois automorphism of $\mathbb{E}$ over $\mathbb{F}$. Let $J_\mathbb{E} = \mathbb{E} \otimes_\mathbb{F} J$ be the Jordan torus over E. Let $\tau := \sigma \otimes \mathrm{id}$ be a σ-semilinear involution of $J_\mathbb{E}$ over $\mathbb{F}$. Then $U_\mathbb{E} = \mathbb{E} \otimes_\mathbb{F} U$ is a subalgebra of $J_\mathbb{E}$. Since J is exceptional, so is $J_\mathbb{E}$. Hence, the Jordan G-torus $J_\mathbb{E}$ is of Albert type since the other two types are special. Then taking $u_3 := x^3$, where

$$0 \neq x \in J^{\sigma_3} \subseteq \mathbb{E} \otimes_\mathbb{F} J^{\sigma_3},$$

we can consider, as in the previous case, the Albert torus $\widetilde{\mathbb{A}}_t := (B, u_3)$, where $B := (\mathbb{E}^t[\Delta], \lambda(\omega, \mu))_\Gamma$ for $t = (0, 1, 0) \in \widetilde{\mathbb{A}}_t$, and corresponding two isomorphisms $\varphi_1 \colon J_\mathbb{E} \to \widetilde{\mathbb{A}}_t$ with ${\varphi_1}_{|_{U_\mathbb{E}}} = {\varphi_1}_{|_B} = \mathrm{id}$, $\varphi_1(x) = t$; and $\varphi_2 \colon J_\mathbb{E} \to \widetilde{\mathbb{A}}_t$ with $\varphi_2(x) = t$, $\varphi_2(u_1) = u_2$, $\varphi_2(u_2) = u_1$, ${\varphi_2}_{|_{U_\mathbb{E}}} = {\varphi_2}_{|_B}$ is an automorphism of the associative algebra B. Now, considering these isomorphisms as identifications and using the fact that

$$\tau(u_1 u_2) = \sigma(u_1 u_2) = u_2 u_1,$$

we get

$$(u_1 u_2) \cdot t = (\omega u_1 u_2) \cdot t = \omega (u_1 u_2) \cdot t,$$

which contradicts $(u_1 u_2) \cdot t \neq 0$. Thus, $\omega \notin \mathbb{F}$ cannot happen and so there is no second Tits construction in this case. □

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