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Factorizations in Universal Enveloping Algebras of Three Dimensional Lie Algebras and Generalizations

This paper is dedicated to Robert V. Moody on the occasion of his 60th birthday

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Abstract. We introduce the notion of Lie algebras with plus-minus pairs as well as regular plus-minus pairs. These notions deal with certain factorizations in universal enveloping algebras. We show that many important Lie algebras have such pairs and we classify, and give a full treatment of, the three dimensional Lie algebras with plus-minus pairs.

1 Introduction

All of our algebras will be over a field F of characteristic zero. We begin by recalling the well known fact that if L is a Kac-Moody Lie algebra with the usual Chevalley generators $\{e_i, f_i \mid 1 \leq i \leq l\}$ satisfying $L = [L, L]$ and l is finite, then every L-module on which the elements $e_i, f_i, 1 \leq i \leq l$ act locally nilpotently is integrable in the sense that the elements $h_i = [e_i, f_i], 1 \leq i \leq l$ are simultaneously diagonalizable. (*cf.* [10] Ex. 6.31, p. 585, or [11]). In other words, every weakly integrable module for such an algebra is integrable. The usual proof of this fact uses that the three dimensional Lie algebra with basis e_i, f_i, h_i is isomorphic to the Lie algebra sl_2 (so $\mathfrak{g}$ is generated by sl_2-triples) together with the result which says that if V is any module for sl_2 on which the standard generators e, f of sl_2 act locally nilpotently then the element $h = [e, f]$ is diagonalizable on V. One can see this last fact as follows. We denote by $M(W)$ the maximal integrable submodule of an sl_2-module W. In general, $M\big(W/M(W)\big) = 0$ for all sl_2-modules W. If a vector v of an sl_2-module W satisfies that $fv = 0$, $e^n v \neq 0$ and $e^{n+1}v = 0$, then we obtain $h(e^n v) = n(e^n v)$, since

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$fe^n = e^n f - ne^{n-1}(h+n-1)$ and

$$\begin{aligned} h(e^n v) &= (ef - fe)e^n v \\ &= efe^n v \\ &= e\big(e^n f - ne^{n-1}(h+n-1)\big)\, v \\ &= -ne^n(h+n-1)v \\ &= -n(h-n-1)e^n v \\ &= -nh(e^n v) + n(n+1)e^n v. \end{aligned}$$

This implies that $M(V)$ is nontrivial for every nonzero sl_2-module V on which the elements e and f are locally nilpotent operators. Therefore, $M(V) = V$ for such an sl_2-module V, that is, V is integrable.

We want to indicate another approach to the above fact about sl_2-modules which uses a factorization in the universal enveloping algebra. This method appears to be new and was the starting point of this paper. For any Lie algebra $\mathfrak{g}$ we let $U(\mathfrak{g})$ denote its universal enveloping algebra. Then one knows that for the algebra $\mathrm{sl}_2 = Fe \oplus Fh \oplus Ff$ we have the factorization

$$U(\mathrm{sl}_2) = U(Fe)\, U(Ff)\, U(Fe). \tag{1}$$

Using this it is easy to see that if V is an sl_2-module on which both e and f act locally nilpotently then any vector $v \in V$ generates a finite dimensional submodule. Thus h acts semisimply on this submodule and so we obtain h acts semisimply on V. Also, the proof of the factorization (1) is quite straightforward and follows easily from the following formula in $U(\mathrm{sl}_2)$:

$$f(e^i f^j e^k) = \frac{j-i+1}{j+1} e^i f^{j+1} e^k + \frac{i}{j+1} e^{i-1} f^{j+1} e^{k+1} + i(j-i+1)e^{i-1} f^j e^k, \tag{2}$$

for all $i > 0$, $j, k \geq 0$. This formula can be established using the following: for any $k \geq 0$ we have

(A_k) $fef^k = \frac{k}{k+1} ef^{k+1} + \frac{1}{k+1} f^{k+1} e + kf^k$,
(B_k) $f^k ef = \frac{1}{k+1} ef^{k+1} + \frac{k}{k+1} f^{k+1} e + kf^k$,

which is proved using $fe^i = e^i f - ie^{i-1}(h+i-1)$ for $i \geq 1$ and induction.

Next let $\mathfrak{H}$ be the three dimensional Heisenberg Lie algebra with a basis x, y, z satisfying $[x, y] = z$, $[x, z] = [y, z] = 0$. Then, in $U(\mathfrak{H})$, we obtain the following factorization

$$U(\mathfrak{H}) = U(Fx)\, U(Fy)\, U(Fx). \tag{3}$$

The proof of this is much like the sl_2 case. It follows easily from the following formula in $U(\mathfrak{H})$

$$y(x^i y^j x^k) = \frac{j-i+1}{j+1} x^i y^{j+1} x^k + \frac{i}{j+1} x^{i-1} y^{j+1} x^{k+1}, \tag{4}$$

for all $i > 0$, $j, k \geq 0$. Note that (4) is proved by establishing for any $k \geq 0$ we have

(A'_k) $yxy^k = \frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x$,
(B'_k) $y^kxy = \frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x$,

which in turn is proved using $yx^i = x^iy - ix^{i-1}z$ for $i \geq 1$ and induction. Thus, the picture is similar for both algebras sl_2 and $\mathfrak{H}$ in that they both have a pair of subalgebras P, M satisfying $P + M$ is not the whole algebra and $U(P)\,U(M)\,U(P)$ is the whole enveloping algebra. This prompts the following definition which singles out those Lie algebras having this type of factorization in their universal enveloping algebras.

Definition 1.1 (i) A Lie algebra L is said to have a *plus-minus pair* if it has two subalgebras P, M satisfying $P + M \neq L$ and

$$U(L) = U(P)\,U(M)\,U(P).$$

In this case we say L has a plus-minus pair (P, M).

(ii) Let (P, M) be a plus-minus pair of L. We say this is a *regular plus-minus pair* if $P \cap M = (0)$ and there is an automorphism σ of L of order two satisfying $\sigma(P) = M$. Note that in this case we then have $U(L) = U(P)\,U(M)\,U(P) = U(M)\,U(P)\,U(M)$.

It is clear that both Lie algebras sl_2 and $\mathfrak{H}$ have regular plus-minus pairs. Moreover if L is any three dimensional Lie algebra with a plus-minus pair (P, M) then each of P and M must be one dimensional. Indeed, $P + M$ cannot be 3 dimensional as $P{+}M \neq L$. Thus, $P{+}M$ is two dimensional and so if one of P, M is 2 dimensional then $P{+}M$ is a subalgebra of L and so $U(P{+}M) \neq U(L)$ but $U(P)\,U(M)\,U(P) \subseteq U(P{+}M)$ so (P, M) cannot be a plus-minus pair. Letting $P = Fx$, $M = Fy$ we have that

$$U(L) = \sum_{i,j,k\geq 0} Fx^iy^jx^k. \tag{5}$$

Moreover, the following result, which extends the situation discussed in the sl_2 case, is quite clear. Let L be a three dimensional Lie algebra with a plus-minus pair (P, M) where $P = Fx$, $M = Fy$. Let V be any L-module on which the action of the elements x, y is locally finite. Then any finitely generated submodule of V is finite dimensional. Thus, one is led to ask just which three dimensional Lie algebras have plus-minus pairs.

In Section 2 we will extend the methods used in the proofs for the sl_2 and $\mathfrak{H}$ cases above and show that any three dimensional Lie algebra which is generated by two elements has a plus-minus pair. Then we go on to see that there are only two isomorphism classes of three dimensional Lie algebras which do not have plus-minus pairs. We also go on to study, when the base field F is algebraically closed, which of these algebras have regular plus-minus pairs and are able to give a complete list of these. Here we use some results from [6]. In the third and final section of this paper we go on to investigate plus-minus pairs, or similar factorizations, in the universal enveloping algebras, of Borcherds Lie algebras as well as in some $\mathbf{Z}^n$-graded Lie algebras which satisfy some extra conditions.

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2 Three Dimensional Case

In this section we begin by showing a three dimensional Lie algebra has a plus-minus pair if and only if it is generated by two elements. We then go on to investigate some special cases as well as regular plus-minus pairs when the base field is algebraically closed.

Throughout we let L be a three dimensional Lie algebra unless mentioned otherwise. If (P, M) is a plus-minus pair for L then we know that each of P, M is one dimensional so we let $P = Fx$, $M = Fy$. If x and y do not generate L then it must be that $P + M$ is a proper subalgebra of L and so since $U(P)\,U(M)\,U(P) \subseteq U(P + M)$ we get a contradiction. Thus L is generated by x and y so is two-generated. We want to establish the converse of the above result. For the moment we let L be an arbitrary Lie algebra and x, y any two elements of L.

We define subspaces U_k for $k \geq 0$ of $U(L)$ by saying

$$U_k = \sum_{0 \leq m \leq k} (Fxy^m + Fy^m x). \tag{6}$$

Notice that $U_0 = Fx$ and that $U_k \subseteq U_{k+1}$ for all $k \geq 0$.

Lemma 2.1 *Let L be an arbitrary Lie algebra and x, y any two elements of L. For $k \geq 0$ the following statements hold,*

(A_k) $yxy^k \equiv \frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x \mod U_k$,
(B_k) $y^kxy \equiv \frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x \mod U_k$,
(C_k) $yU_k \subseteq U_{k+1}$, $U_k y \subseteq U_{k+1}$.

Proof We prove this by induction on k noting that for $k = 0$ both (A_0) and (B_0) are clear. Next, we show $(\mathrm{A}_0), \ldots, (\mathrm{A}_k), (\mathrm{B}_0), \ldots, (\mathrm{B}_k)$ imply (C_k). Indeed, by definition we have that

$$yU_k = \sum_{0 \leq m \leq k} (Fyxy^m + Fy^{m+1}x)$$

and so by $(\mathrm{A}_0), \ldots, (\mathrm{A}_k)$ we get that this is contained in U_{k+1}. Similarly we have

$$U_k y = \sum_{0 \leq m \leq k} (Fxy^{m+1} + Fy^m xy)$$

and so by $(\mathrm{B}_0), \ldots, (\mathrm{B}_k)$ we get this is contained in U_{k+1}. Hence (C_k) holds.

Next we show that (A_k), (B_k), (C_k) imply (A_{k+1}), (B_{k+1}). Now $yxy^{k+1} = (yxy^k)y$ so that (A_k) implies that the difference

$$yxy^{k+1} - \left(\frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x\right) y \in U_k y.$$

But (C_k) implies that $U_k y \subseteq U_{k+1}$ so we get that

$$yxy^{k+1} \equiv \left(\frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x\right) y \mod U_{k+1}.$$

Similarly, using (B_k) and (C_k) we get that

$$y^{k+1}xy \equiv y\left(\frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x\right) \quad \bmod U_{k+1}.$$

Thus, we finally get

$$yxy^{k+1} \equiv \frac{k}{k+1}xy^{k+2} + \frac{1}{(k+1)^2}yxy^{k+1} + \frac{k}{(k+1)^2}y^{k+2}x \quad \bmod U_{k+1}$$

which implies that

$$\frac{k(k+2)}{(k+1)^2}yxy^{k+1} \equiv \frac{k}{k+1}xy^{k+2} + \frac{k}{(k+1)^2}y^{k+2}x \quad \bmod U_{k+1}.$$

Therefore, we obtain

$$yxy^{k+1} \equiv \frac{k+1}{k+2}xy^{k+2} + \frac{1}{k+2}y^{k+2}x \quad \bmod U_{k+1},$$

and we see that (A_{k+1}) holds.

Using a similar type of argument we have that

$$\begin{aligned} y^{k+1}xy &= y(y^kxy) \\ &\equiv y\left(\frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x\right) \quad \bmod U_{k+1} \\ &\equiv \frac{1}{k+1}yxy^{k+1} + \frac{k}{k+1}y^{k+2}x \quad \bmod U_{k+1} \\ &\equiv \frac{1}{k+1}\left(\frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x\right)y + \frac{k}{k+1}y^{k+2}x \quad \bmod U_{k+1} \\ &\equiv \frac{k}{(k+1)^2}xy^{k+2} + \frac{1}{(k+1)^2}y^{k+1}xy + \frac{k}{k+1}y^{k+2}x \quad \bmod U_{k+1} \end{aligned}$$

and

$$\frac{k(k+2)}{(k+1)^2}y^{k+1}xy \equiv \frac{k}{(k+1)^2}xy^{k+2} + \frac{k}{k+1}y^{k+2}x \quad \bmod U_{k+1}.$$

Therefore, we obtain

$$y^{k+1}xy \equiv \frac{1}{k+2}xy^{k+2} + \frac{k+1}{k+2}y^{k+2}x \quad \bmod U_{k+1},$$

and we see that (B_{k+1}) holds. This completes our induction. ■

We apply this lemma to the three dimensional case in our next result.

Theorem 2.2 *Let L be a three dimensional Lie algebra. Then L has a plus-minus pair if and only if L is two generated. Moreover, if x and y generate L then (P, M) is a plus-minus pair for L where $P = Fx$, $M = Fy$.*

Proof We need only show L has a plus-minus pair if L is generated by two elements x, y. Let $z = [x, y]$. Now we want to show $U(L) = \sum_{i,j,k\geq 0} Fx^i y^j x^k$. Put $\mathfrak{X} = \sum_{i,j,k\geq 0} Fx^i y^j x^k \subseteq U(L)$ and let U_k be defined as above. Clearly $x\mathfrak{X} \subseteq \mathfrak{X}$, $\mathfrak{X}x \subseteq \mathfrak{X}$ and $U_k \subseteq \mathfrak{X}$ for all $k \geq 0$. We claim

$$y(x^\ell y^m x^n) \in \mathfrak{X},$$
$$z(x^\ell y^m x^n) \in \mathfrak{X}.$$

and show this by induction on ℓ. If $\ell = 0$, then we see $y(y^m x^n) \in \mathfrak{X}$ and using (A_m) we get

$$\begin{aligned} z(y^m x^n) &= (xy - yx)(y^m x^n) \\ &= xy^{m+1}x^n - yxy^m x^n \\ &\in Fxy^{m+1}x^n + (Fxy^{m+1} + Fy^{m+1}x + U_m)x^n \subseteq \mathfrak{X}. \end{aligned}$$

Let $\ell > 0$. Then, we obtain, using our inductive assumption, that

$$\begin{aligned} y(x^\ell y^m x^n) &= (xy - z)(x^{\ell-1} y^m x^n) \\ &\in x\mathfrak{X} + \mathfrak{X} \subseteq \mathfrak{X} \end{aligned}$$

and, letting $[z, x] = ax + by + cz$ for $a, b, c \in F$, we also get using our inductive assumption that

$$\begin{aligned} z(x^\ell y^m x^n) &= (xz + ax + by + cz)(x^{\ell-1} y^m x^n) \\ &\in x\mathfrak{X} + \mathfrak{X} + \mathfrak{X} + \mathfrak{X} \subseteq \mathfrak{X}. \end{aligned}$$

Hence, $y\mathfrak{X} \subseteq \mathfrak{X}$. Since $\mathfrak{X}$ is a left ideal of $U(L)$ containing 1, we obtain $\mathfrak{X} = U(L)$. Therefore, (P, M) with $P = Fx$ and $M = Fy$ is a plus-minus pair for L. ■

If our three dimensional Lie algebra L is abelian it clearly does not have a plus-minus pair. Also, we let $\mathfrak{g}$ be the three dimensional Lie algebra with basis x, y, z satisfying

$$[x, y] = 0, \quad [x, z] = x, \quad [y, z] = y.$$

Then for any elements $a, b, c, \alpha, \beta, \gamma \in F$ we have the very special identity

$$[ax + by + cz, \alpha x + \beta y + \gamma z] = \gamma(ax + by + cz) - c(\alpha x + \beta y + \gamma z).$$

This clearly implies that $\mathfrak{g}$ is not two generated so does not have a plus-minus pair. Our next result shows that these are the only two kinds of three dimensional Lie algebras which do not have plus-minus pairs.

Theorem 2.3 *Let L be a three dimensional Lie algebra which is not two generated. Then L is either abelian or is isomorphic to the algebra $\mathfrak{g}$ above.*

Proof Assume L is not abelian. Choose a 1-dimensional subspace Fz of L which is not an ideal. Every 2-dimensional subspace of L is a subalgebra. Hence there exist x, y in L such that $\{x, y, z\}$ is a basis of L and $[z, x] = ax$, $[z, y] = by$ for some a, b in F. As $[z, x + y]$ belongs to $Fx + Fy$ and $F(x + y) + Fz$, we must have $a = b$. As Fz is not an ideal, a is not 0. We may assume that $a = 1$. As $[x, y] = [x + z, y] - y$ belongs to $Fx + Fy$ and $F(x + z) + Fy$, we deduce that $[x, y]$ is in Fy. Similarly, it is in Fx. Hence $[x, y] = 0$ and L is isomorphic to $\mathfrak{g}$. ■

The special case when $L = L_{-1} \oplus L_0 \oplus L_1$ is a three graded Lie algebra of dimension three with a plus-minus pair will be used in the final section of this work so will be discussed now. Put $L_1 = Fx$, $L_{-1} = Fy$, $L_0 = Fz$. We can assume first that $[x, y]$ is either 0 or z. Suppose $[x, y] = 0$. If $[x, z] = [y, z] = 0$, then L is abelian so has no plus-minus pair. If $[x, z] = 0$ and $[y, z] \neq 0$, then we can also suppose $[z, y] = y$ and hence, $P = F(x + y)$ and $M = Fz$ give a plus minus pair. If $[x, z] \neq 0$ and $[y, z] = 0$, then we can suppose $[z, x] = x$ and hence, again $P = F(x + y)$ and $M = Fz$ becomes a plus-minus pair. If $[x, z] = ax$ and $[y, z] = by$ with $ab \neq 0$, then we can suppose $a = 1$. In this case, $P = F(x + y)$ and $M = Fz$ give a plus-minus pair when $b \neq 1$. Otherwise we have $[x, z] = x$ and $[y, z] = y$ and there is no plus-minus pair. Next we suppose $[x, y] = z$. If $[x, z] = [y, z] = 0$, then L is a Heisenberg Lie algebra, and hence, L has a plus-minus pair. If $[x, z] = ax$ and $[y, z] = by$ with $a \neq 0$ or $b \neq 0$, then $0 = [z, z] = \big[[x, y], z\big] = \big[[x, z], y\big] + \big[x, [y, z]\big] = a[x, y] + b[x, y] = (a + b)z$ and $a + b = 0$. Put $x' = x$, $y' = -2y/a$, $z' = -2z/a$. Then, $[x', y'] = -2[x, y]/a = -2z/a = z'$, $[z', x'] = -[x', z'] = 2[x, z]/a = 2x = 2x'$ and $[z', y'] = -[y', z'] = -4[y, z]/(a^2) = 4y/a = -2y'$. This means that L is isomorphic to sl_2. Therefore we obtain the following result which gives a characterization of sl_2 and $\mathfrak{H}$.

Proposition 2.4 *Let $L = L_1 \oplus L_0 \oplus L_{-1}$ be a three graded Lie algebra of dimension three with* $\dim L_{\pm 1} = \dim L_0 = 1$.

(1) *If L has a plus-minus pair, then L is isomorphic to one of $s\ell_2$, $\mathfrak{H}$ and $K(a, b)$, where $K(a, b) = Fx \oplus Fy \oplus Fz$ is the Lie algebra having the relations: $[x, y] = 0$, $[x, z] = ax$, $[y, z] = by$ with $a \neq b$.*
(2) *If L has (L_1, L_{-1}) for a plus-minus pair, then L is isomorphic to either* sl_2 *or* $\mathfrak{H}$.

Remark If $a = b$ is non-zero then we have $K(a, b) = K(a, a) \simeq K(1, 1)$ and this is nothing but our algebra $\mathfrak{g}$ of Theorem 2.3 which does not have a plus-minus pair.

Next we will briefly discuss isomorphism classes among the Lie algebras $K(a, b)$. For this we will freely use the classification in Jacobson's book [6] on page 12 where he classifies the three dimensional Lie algebras having a two dimensional derived algebra. This is listed there as (d) of his general classification. We have $K(0, c) \simeq K(c, 0) \simeq K(0, 1)$ for nonzero $c \in F$. Thus if a or b is 0 then $K(a, b) \simeq K(0, 1)$. Next we suppose that both a, b are nonzero. Then, we also see $K(a, b) \simeq K(a/b, 1)$. The only isomorphisms between the algebras $K(c, 1)$ for c non-zero are $K(c, 1) \simeq K(1/c, 1)$ and none of these are isomorphic to $K(0, c)$. Thus, the isomorphism classes of the Lie algebras $K(a, b)$, having plus-minus pairs, are parametrized by the set

$$\mathfrak{P}(F) = \big\{ \{u, u^{-1}\} \mid u \in F, u \neq 0, 1 \big\} \cup \big\{ \{0\} \big\}.$$

Here the isomorphism class of $K(0,a)$ corresponds to $\{0\}$ while that of $K(a,1)$ to $\{a,a^{-1}\}=\{a^{-1},a\}$ for $a\in F$, $a\neq 0$.

We next assume that F is an algebraically closed field of characteristic 0, and will study the three dimensional Lie algebras over F having a regular plus-minus pair. Let L be such an algebra and let (P,M) be a regular plus-minus pair of L. Then we can choose nonzero elements $x\in P$ and $y\in M$ as well as an involutive automorphism σ of L such that $[x,y]\neq 0$ and $\sigma(x)=y$. Put $z=[x,y]$, and set $u=x+y$ and $v=x-y$. Let $L_1=Fu$ (the 1-eigenspace of σ) and $L_{-1}=Fv\oplus Fz$ (the -1-eigenspace of σ). Then, $[u,v]=[x+y,x-y]=-2[x,y]=-2z$. We write $[z,x]=ax+by+cz$. Then we obtain $[z,y]=-\sigma([z,x])=-\sigma(ax+by+cz)=-bx-ay+cz$ and so the Jacobi identity implies $[bx+ay-cz,x]+[ax+by+cz,y]=0$. Therefore,

$$\{-az-c(ax+by+cz)\}+\{az+c(-bx-ay+cz)\}=0$$

and $c(a+b)x+c(a+b)y=0$, which implies $c=0$ or $a+b=0$.

Case 1 $c=0$.

In this case, we have $[x,y]=z$, $[z,x]=ax+by$, $[z,y]=-bx-ay$. We write the matrix of $\mathrm{ad}(z)$ restricted to the space $Fx\oplus Fy$ as

$$\mathrm{ad}\, z=\begin{pmatrix} a & -b \\ b & -a \end{pmatrix}.$$

Then, its characteristic polynomial is $t^2-a^2+b^2$. Hence, $\mathrm{ad}\, z\,|_{Fx\oplus Fy}$ is similar to one of

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where $\lambda=(a^2-b^2)^{1/2}$. If $\lambda\neq 0$, that is $a^2-b^2\neq 0$, then we have certain elements $x',y'\in Fx\oplus Fy$ such that $[z',x']=\lambda' x'$, $[z',y']=-\lambda' y'$ with $z'=[x',y']\neq 0$ and $\lambda'\neq 0$. This means that $L\simeq \mathrm{sl}_2$. If $a=b=0$, then we see $L\simeq \mathfrak{H}$. Now we suppose $a=b\neq 0$. Thus, we have $[z,u]=0$, $[z,v]=2au$, $[u,v]=-2z$. Put $\mu=(-a)^{1/4}$, and set $z'=z/\mu$, $u'=\mu u$, $v'=v/(2\mu^2)$. Then,

$$[v',u']=[v,u]/(2\mu)=z/\mu=z',$$
$$[v',z']=(-a)u/(\mu^3)=u',$$
$$[u',z']=0.$$

As is easy to check, again from Jacobson's book, [6], on page 12 under heading (d) one finds our algebra here is just the one with $\alpha=1$ and we have

$$\sigma(v')=-v', \quad \sigma(u')=u', \quad \sigma(z')=-z'.$$

We denote this algebra by $L(\alpha=1)$.

Next we suppose $a = -b \neq 0$. Thus, we have $[z, v] = 0$, $[z, u] = 2av$, $[u, v] = -2z$. Put $\nu = a^{1/4}$, and set $z' = z/\nu$, $u' = -u/(2\nu^2)$, $v' = \nu v$. Then,

$$[u', v'] = -[u, v]/(2\nu) = z/\nu = z',$$
$$[u', z'] = -[u, z]/(2\nu^3) = \nu v = v',$$
$$[v', z'] = 0.$$

Here we have once again that our algebra is just $L(\alpha = 1)$ and

$$\sigma(u') = u', \quad \sigma(v') = -v', \quad \sigma(z') = -z'.$$

Case 2 $c \neq 0, a + b = 0.$

In this case, we have $[x, y] = z$, $[z, x] = [z, y] = ax - ay + cz$, and hence $[u, v] = -2z$, $[z, u] = 2av + 2cz4$, $[z, v] = 0$. Set $u' = -u/(2c)$, $v' = cv$, $z' = z$. Then,

$$[u', v'] = -[u, v]/2 = z = z',$$
$$[u', z'] = -[u, z]/(2c) = av/c + z = av'/(c^2) + z,$$
$$[v', z'] = 0.$$

This time we get that our algebra is the one from [6] page 12 having $\beta = a/c^2$, which we denote as $L(\beta = a/c^2)$, and so have

$$\sigma(u') = u', \quad \sigma(v') = -v', \quad \sigma(z') = -z'$$

With the notation developed above we see that the preceding arguments, together with the results in [6] about isomorphisms between these algebras, establish the following result.

Theorem 2.5 *Let F be an algebraically closed field of characteristic* 0*. Let L be a three dimensional Lie algebra with a regular plus-minus pair. Then L is isomorphic to one of* sl_2, $\mathfrak{H}$, $L(\alpha = 1)$ *or* $L(\beta = r)$ *for any r in F. Moreover no two distinct algebras in this list are isomorphic.*

Remark Finally we want to point out that the Lie algebra $K(u, 1)$ with $u \in F$ is isomorphic to $L(\alpha = 1)$ if $u = -1$, and is isomorphic to $L(\beta = -\frac{u}{(u+1)^2})$ if $u \neq -1$. As a consequence of our work we see that a three dimensional three graded Lie algebra has a regular plus-minus pair or no plus-minus pair at all.

3 Plus-Minus Pairs in Some General Classes of Lie Algebras

In this section we will show that Borcherds Lie algebras have plus-minus pairs. Since these generalize the well-known Kac-Moody Lie algebras our results apply to these as well. We next go on to investigate $\mathbf{Z}^n$-graded Lie algebras and see that with certain other assumptions these also have plus-minus pairs. Our method is to establish slightly more general factorization results in the universal enveloping algebras of these Lie algebras and then show how this gives rise to plus-minus pairs. The techniques are quite general and no doubt apply to other situations as well.

Let $\mathfrak{g}$ be a rank l Borcherds Lie algebra over F with the standard Cartan subalgebra $\mathfrak{h}$ and Chevalley generators $\{e_1, \ldots, e_l, f_1, \ldots, f_l\}$, and let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ the derived subalgebra of $\mathfrak{g}$. Put $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$, and take a complement $\mathfrak{h}''$ of $\mathfrak{h}'$ in $\mathfrak{h}$ with $\mathfrak{h} = \mathfrak{h}'' \oplus \mathfrak{h}'$. Then, $\mathfrak{g} = \mathfrak{h}'' \oplus \mathfrak{g}'$. Let $\mathfrak{g}_+$ be the subalgebra of $\mathfrak{g}$ generated by $e_1, \ldots, e_l$, and $\mathfrak{g}_-$ the subalgebra of $\mathfrak{g}$ generated by $f_1, \ldots, f_l$ (*cf.* [4], [7], [8], [9], [10], [12]).

Proposition 3.1 *Let $\mathfrak{g}$ be a rank l Borcherds Lie algebra, and let $I \cup J = \{1, 2, \ldots, l\}$ be a partition of $\{1, 2, \ldots, l\}$ into disjoint subsets. Then,*

$$U(\mathfrak{g}) = \Big(\prod_{i \in I} U(Fe_i)\Big) U(\mathfrak{g}_-) U(\mathfrak{h}'') U(\mathfrak{g}_+) \Big(\prod_{j \in J} U(Ff_j)\Big).$$

Proof Let $\mathfrak{g}_+^i$ be the standard homogeneous complementary subalgebra of Fe_i in $\mathfrak{g}_+$, and $\mathfrak{g}_-^i$ the standard homogeneous complementary subalgebra of Ff_i in $\mathfrak{g}_-$. For each $k = 1, \ldots, l$ we put $h_k = [e_k, f_k]$ and $\mathfrak{h}_k = Fh_{k+1} \oplus \cdots \oplus Fh_l$, and we set $I_k = I \cap \{1, \ldots, k\}$ and $J_k = J \cap \{1, \ldots, k\}$. We make free use of the PBW Theorem as well as the fact that $Fe_i \oplus Ff_i \oplus Fh_i$ is either sl_2 or $\mathfrak{H}$ so has a regular plus-minus pair.

If $1 \in I$, then

$$\begin{aligned}
U(\mathfrak{g}) &= U(\mathfrak{g}_-)\, U(\mathfrak{h})\, U(\mathfrak{g}_+) \\
&= U(\mathfrak{g}_-^1)\, U(Ff_1)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1)\, U(Fh_1) U(Fe_1) U(\mathfrak{g}_+^1) \\
&= U(\mathfrak{g}_-^1)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1)\, U(Ff_1)\, U(Fh_1) U(Fe_1) U(\mathfrak{g}_+^1) \\
&= U(\mathfrak{g}_-^1)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1)\, U(Fe_1)\, U(Ff_1) U(Fe_1) U(\mathfrak{g}_+^1) \\
&= U(Fe_1) U(\mathfrak{g}_-^1)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1)\, U(Ff_1) U(Fe_1) U(\mathfrak{g}_+^1) \\
&= U(Fe_1) U(\mathfrak{g}_-) U(\mathfrak{h}'')\, U(\mathfrak{h}_1) U(\mathfrak{g}_+).
\end{aligned}$$

In the other case when $1 \in J$ by using the same type of argument we have

$$U(\mathfrak{g}) = U(\mathfrak{g}_-)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1) U(\mathfrak{g}_+)\, U(Ff_1).$$

If we began with

$$U(\mathfrak{g}) = \Big(\prod_{i \in I_k} U(Fe_i)\Big)\, U(\mathfrak{g}_-) U(\mathfrak{h}'')\, U(\mathfrak{h}_k)\, U(\mathfrak{g}_+) \Big(\prod_{j \in J_k} U(Ff_j)\Big),$$

then, again using the same method, we can obtain

$$U(\mathfrak{g}) = \Big(\prod_{i \in I_{k+1}} U(Fe_i)\Big)\, U(\mathfrak{g}_-) U(\mathfrak{h}'')\, U(\mathfrak{h}_{k+1})\, U(\mathfrak{g}_+) \Big(\prod_{j \in J_{k+1}} U(Ff_j)\Big).$$

Thus after several applications of this process we reach the stated result. ■

We next see that this gives the desired plus-minus pair.

Corollary 3.2 *Let $\mathfrak{g}$ be a Borcherds Lie algebra of finite rank. Then,*

$$U(\mathfrak{g}) = U(\mathfrak{g}_\pm)U(\mathfrak{g}_\mp)U(\mathfrak{h}'')U(\mathfrak{g}_\pm).$$

Hence, Borcherds Lie algebras have plus-minus pairs. In particular, perfect Kac-Moody Lie algebras or, more generally, perfect Borcherds Lie algebras have regular plus-minus pairs.

Proof We just take one of I and J to be empty. This leads to the result. Then, for example, let $P = \mathfrak{h}'' \oplus \mathfrak{g}_+$ and $M = \mathfrak{g}_-$. This gives a plus-minus pair. ■

We next generalize the previous discussion by considering $\mathbf{Z}^n$-graded Lie algebras. Thus, let $Q = \bigoplus_{i=1}^n \mathbf{Z}\alpha_i$ be a free abelian group of rank n generated by $\alpha_1, \dots, \alpha_n$, and let $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ be a Lie algebra graded by Q. Put $\Delta = \{\alpha \in Q \mid \mathfrak{g}_\alpha \neq 0\}$. We also assume that $\mathbf{Z}\alpha_1 \cap \Delta = \{0, \pm\alpha_1\}$, and that $L = \mathfrak{g}_{\alpha_1} \oplus [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}] \oplus \mathfrak{g}_{-\alpha_1}$ is a subalgebra with a plus-minus pair $(\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1})$ in L. (Thus, if L is three dimensional Proposition 2.4 implies L is isomorphic to either sl_2 or $\mathfrak{H}$.) We also suppose that there exists a complementary subalgebra $\mathfrak{g}_0'$ of $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}]$ in $\mathfrak{g}_0$ with $\mathfrak{g}_0 = [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}] \oplus \mathfrak{g}_0'$. An element $\alpha = \sum_{i=1}^n c_i\alpha_i \in Q$ is called *positive* (resp. *negative*), that is $\alpha > 0$ (resp. $\alpha < 0$), if there is an index i satisfying $c_i > 0$ (resp. $c_i < 0$) and $c_{i+1} = c_{i+2} = \cdots = c_n = 0$. Put $\Delta_+ = \{\alpha \in \Delta \mid \alpha > 0\}$ and $\Delta_- = \{\alpha \in \Delta \mid \alpha < 0\}$. Let $\mathfrak{g}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$, and $\mathfrak{g}_\pm' = \bigoplus_{\alpha \in \Delta_\pm \setminus \{\alpha_1\}} \mathfrak{g}_\alpha$. Then, $\mathfrak{g}_\pm = \mathfrak{g}_{\pm\alpha_1} \oplus \mathfrak{g}_\pm'$, and we see that $\mathfrak{g}_{\pm\alpha_1} \oplus \mathfrak{g}_\mp'$ are subalgebras. In this situation we have the following result.

Proposition 3.3 *Let $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ be a graded Lie algebra with the extra conditions as above. Then,*

$$U(\mathfrak{g}) = U(\mathfrak{g}_{\alpha_1})\, U(\mathfrak{g}_-)U(\mathfrak{g}_0')U(\mathfrak{g}_+).$$

Moreover, letting $P = \mathfrak{g}_+ \oplus \mathfrak{g}_0'$ and $M = \mathfrak{g}_-$ gives a plus-minus pair for $\mathfrak{g}$.

Proof Using our assumptions we see that

$$\begin{aligned}
U(\mathfrak{g}) &= U(\mathfrak{g}_-)U(\mathfrak{g}_0)\, U(\mathfrak{g}_+)\\
&= U(\mathfrak{g}_-')U(\mathfrak{g}_{-\alpha_1})U(\mathfrak{g}_0')U([\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}])U(\mathfrak{g}_{\alpha_1})\, U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_-')U(\mathfrak{g}_0')\, U(\mathfrak{g}_{-\alpha_1})U([\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}])\, U(\mathfrak{g}_{\alpha_1})\, U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_-')U(\mathfrak{g}_0')\, U(\mathfrak{g}_{\alpha_1})U(\mathfrak{g}_{-\alpha_1})\, U(\mathfrak{g}_{\alpha_1})\, U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_{\alpha_1})U(\mathfrak{g}_-')\, U(\mathfrak{g}_0')U(\mathfrak{g}_{-\alpha_1})\, U(\mathfrak{g}_{\alpha_1})\, U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_{\alpha_1})U(\mathfrak{g}_-')\, U(\mathfrak{g}_{-\alpha_1})U(\mathfrak{g}_0')U(\mathfrak{g}_{\alpha_1})\, U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_{\alpha_1})U(\mathfrak{g}_-)\, U(\mathfrak{g}_0')U(\mathfrak{g}_+).
\end{aligned}$$

■

Remark It can be seen that many EALA's and some of the root-graded Lie algebras (*cf.* [1], [2], [3], [13]) satisfy the hypothesis of Proposition 3.3.

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