LIE TORI OF RANK 1

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This article is based on a talk presented by the first author at the conference on Lie and Jordan Algebras, their Representations and Applications held in Guarujá, Brazil in May 2004. The article surveys some recent progress by a number of authors in the study of extended affine Lie algebras and some closely related Lie algebras called Lie tori.

Lie tori of rank 1 are coordinatized by algebras with involution called structurable tori. Recently the present authors have obtained a classification of structurable tori, and this article includes in Section 5 a description of part of that classification. Complete statements and proofs of our classification results will appear elsewhere.

Assumptions. Throughout the article we assume that F is a field of characteristic 0. We also assume that Δ is an irreducible root system (possibly nonreduced). Note that it is our convention that root systems contain 0 and so $\Delta^\times := \Delta \setminus \{0\}$ is an irreducible root system in the usual sense (see for example [MP, §3.2]). From the classification we know that

$$\Delta = A_\ell,\ B_\ell,\ C_\ell,\ D_\ell,\ E_6,\ E_7,\ E_8,\ F_4 \text{ or } G_2 \quad \text{(the reduced types)}$$

or

$$\Delta = BC_\ell \quad \text{(the nonreduced type)}.$$

In particular the rank one root systems are

$$A_1 = \{-\alpha, 0, \alpha\} \quad \text{and} \quad BC_1 = \{-2\alpha, -\alpha, 0, \alpha, 2\alpha\},$$

where $\alpha \neq 0$. Finally we assume that Λ is a finitely generated free abelian group of rank n, and so $\Lambda \simeq \mathbb{Z}^n$.

1. Lie tori

We begin with the definition of a Lie torus. Lie tori were defined first by Yoshii in [Y2]. The definition we give is an equivalent definition suggested by Neher in [N]. In this definition $\alpha^\vee$ will denote the *coroot* of α for $\alpha \in \Delta^\times$. That is, $\alpha^\vee$ is the element of the dual space of $\mathrm{span}_F(\Delta)$ so that $\beta \mapsto \beta - \langle \beta, \alpha^\vee \rangle \alpha$ is the reflection corresponding to α in the Weyl group of Δ, where $\langle\ ,\ \rangle$ is the natural pairing of $\mathrm{span}_F(\Delta)$ with its dual space [MP, § 3.2].

Definition 1.1. A *Lie torus* is a Lie algebra L over F satisfying:

(LT1): L has two algebra gradings

$$L = \bigoplus_{\alpha \in \Delta} L_\alpha \quad \text{and} \quad L = \bigoplus_{\lambda \in \Lambda} L^\lambda$$

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which are compatible in the sense that $L = \sum_{\alpha,\lambda} L_\alpha^\lambda$, where $L_\alpha^\lambda = L_\alpha \cap L^\lambda$. (So we assume that $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ with $L_{\alpha+\beta}$ interpreted as 0 if $\alpha+\beta \notin \Delta$, and we assume that $[L^\lambda, L^\mu] \subset L^{\lambda+\mu}$.)

(LT2): We have

(i) $L_\alpha^0 \neq 0$ for $\alpha \in \Delta^\times$, $\frac{1}{2}\alpha \notin \Delta^\times$.

(ii) If $\alpha \in \Delta^\times$, $\lambda \in \Lambda$ and $L_\alpha^\lambda \neq 0$, then $L_\alpha^\lambda = Fe_\alpha^\lambda$ and $L_{-\alpha}^{-\lambda} = Ff_\alpha^\lambda$, where

$$[[e_\alpha^\lambda, f_\alpha^\lambda], x_\beta^\mu] = \langle \beta, \alpha^\vee \rangle x_\beta^\mu$$

for $x_\beta^\mu \in L_\beta^\mu$, $\beta \in \Delta$, $\mu \in \Lambda$.

(LT3): L is generated as an algebra by the spaces L_α, $\alpha \in \Delta^\times$.

(LT4): Λ is generated as a group by $\mathrm{supp}_\Lambda(L)$, where $\mathrm{supp}_\Lambda(L) := \{\lambda \in \Lambda \mid L^\lambda \neq 0\}$.

In that case Δ is called the *type* of L, the rank of Δ is called the *rank* of L, and n (the rank of the group Λ) is called the *nullity* of L.

Axioms (LT1) and (LT2) are of course the main axioms in this definition. In particular (LT2) tells us that we have a plentiful supply of sl_2-triples in L. (LT3) and (LT4) are less important. If (LT3) does not hold we can just replace L by the subalgebra generated by the root spaces L_α, $\alpha \in \Delta^\times$. Similarly, if (LT4) does not hold we can replace Λ by the subgroup of Λ generated by the support of L.

A Lie torus L will be said to be *centreless* if its centre $Z(L)$ is 0. We note that if L is an arbitrary Lie torus, then the quotient $L/Z(L)$ is a centreless Lie torus [Y2, Lemma 1.4]. For the remainder of this article, we will focus on centreless Lie tori.

2. Connection with EALA's

Yoshii and Neher were interested in Lie tori primarily because of their connection with extended affine Lie algebras. We now describe this connection in order to motivate the study of Lie tori. For convenience of reference, we assume in this section and the next that F is the field of complex numbers. (This assumption can be dropped if the statements are suitably modified [N].)

Suppose that $\mathcal{G}$ is a tame extended affine Lie algebra (EALA). Thus by definition $\mathcal{G}$ has a split toral subalgebra $\mathcal{H}$ and a nondegenerate invariant symmetric bilinear form satisfying certain natural axioms generalizing the properties of affine Kac-Moody Lie algebras [HT, AABGP].

Let $\mathcal{G}_c$ be the *core* of $\mathcal{G}$; that is, let $\mathcal{G}_c$ be the subalgebra of $\mathcal{G}$ generated by the nonisotropic root spaces of $\mathcal{G}$. Let $\mathcal{G}_{cc} = \mathcal{G}_c/Z(\mathcal{G}_c)$, the *centreless core* of $\mathcal{G}$. Then $\mathcal{G}_{cc}$ is a centreless Lie torus.

Conversely, Yoshii showed that any centreless Lie torus occurs as the centreless core of some tame EALA [Y2, Theorem 7.3]. Moreover, Neher described a procedure for constructing all tame EALA's with a given centreless Lie torus as centreless core [N, Theorem 14]. (See also [BGK, Section 3] and [BGKN, Section 3] in the case when Δ is of type A_ℓ, $\ell \geq 2$.)

In particular, if $\mathcal{G}$ is an affine Kac-Moody Lie algebra (= tame EALA of nullity 1 [ABGP]), then $\mathcal{G}_c = \mathcal{G}'$ and $\mathcal{G}_{cc} = \mathcal{G}'/Z(\mathcal{G}')$ is a centreless Lie torus of nullity 1. In this case, it is well known that $\mathcal{G}$ can constructed by double extension from $\mathcal{G}_{cc}$.

3. Examples

With this connection with EALA's as motivation, we now go on to discuss the status of the classification problem for Lie tori. For this we need a couple of examples of Lie tori.

Example 3.1. Let $\dot{\mathcal{G}}$ be a finite dimensional simple Lie algebra of type Δ (and so Δ is reduced). Let

$$L = \dot{\mathcal{G}} \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm 1}],$$

where $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ denotes the algebra of Laurent polynomials over F. L (or often its universal central extension) is called a *toroidal* Lie algebra. Then L has a root grading coming from the root grading of $\dot{\mathcal{G}}$ (obtained by choosing a Cartan subalgebra for $\dot{\mathcal{G}}$), and L has a Λ-grading coming from the Λ-grading of the Laurent polynomials (obtained by choosing a basis for Λ). Using these gradings, L is a centreless Lie torus. (In fact this example is one of the explanations for the term Lie torus.)

In particular, if $\dot{\mathcal{G}} = \mathrm{sl}_{\ell+1}(F)$, then

$$L = \mathrm{sl}_{\ell+1}(F) \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = \mathrm{sl}_{\ell+1}(F[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$$

is a centreless Lie torus of type A_ℓ and nullity n.

There is an important deformation of this last example introduced by Berman, Gao and Krylyuk in [BGK].

Example 3.2. Let $\mathbf{q} = (q_{ij}) \in M_n(F)$ be a *quantum matrix*; that is suppose that $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$. Let $F_{\mathbf{q}} = F_{\mathbf{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be the unital associative algebra with basis consisting of the monomials $t_1^{i_1} \dots t_n^{i_n}$, $i_1, \dots, i_n \in \mathbb{Z}$, and multiplication determined by the relations

$$t_j t_i = q_{ij} t_i t_j.$$

This associative algebra $F_{\mathbf{q}}$ is called the *quantum torus* determined by $\mathbf{q}$. (Alternatively, $F_{\mathbf{q}}$ is a twisted group ring of the group $\mathbb{Z}^n$.) If we choose a basis $B = \{\sigma_1, \dots, \sigma_n\}$ for Λ, we can give $F_{\mathbf{q}}$ a unique Λ-grading, called the *toral* Λ*-grading* determined by B, by assigning the degree σ_i to the generator t_i of $F_{\mathbf{q}}$. Let

$$L = \mathrm{sl}_{\ell+1}(F_{\mathbf{q}}) := \{X \in M_{n\times n}(F_{\mathbf{q}}) \mid \mathrm{tr}(X) \in [F_{\mathbf{q}}, F_{\mathbf{q}}]\}.$$

Then L is a Lie algebra under the commutator product, L has a natural A_ℓ-grading, and L has a Λ-grading coming from the toral Λ-grading of the coordinate algebra $F_{\mathbf{q}}$. Once again, using these gradings, L is a centreless Lie torus of type A_ℓ and nullity n. (This example is another explanation for the term Lie torus.)

Furthermore, Berman, Gao and Krylyuk showed in [BGK] (although they didn't use this language) that any centreless Lie torus of type A_ℓ, where $\ell \geq 3$, is isomorphic to $\mathrm{sl}_{\ell+1}(F_{\mathbf{q}})$ for some $\mathbf{q}$ as in Example 3.2. This then is a classification result for centreless Lie tori of type A_ℓ, $\ell \geq 3$. Similar classification results have been proved in recent years for all types of rank ≥ 2 except type BC_2.

We do not have anything to say in this article about the type BC_2 and so we now turn our attention to the rank 1 types.

4. Coordinatization of rank 1 Lie tori

Assume once again that F is an arbitrary field of characteristic 0. It was shown by Allison and Yoshii in [AY, Theorem 5.6] that the centreless core of any EALA of rank 1 is coordinatized by an algebra with involution called a structurable torus. This argument can be easily modified to show the same result for centreless rank 1 Lie tori. In this section, we outline that argument. First we need two definitions.

Definition 4.1. A *structurable algebra* is a pair $(A, *)$ consisting of a unital (in general nonassociative) algebra A together with an involution $*$ (an anti-automorphism of period 2) so that the following 5-linear identity holds:

$$\{xy\{zwq\}\} - \{zw\{xyq\}\} = \{\{xyz\}wq\} - \{z\{yxw\}q\},$$

where

$$\{xyz\} := (xy^*)z + (zy^*)x - (zx^*)y.$$

Definition 4.2. A *structurable torus* is a structurable algebra $(A, *)$ satisfying:

(ST1): $A = \bigoplus_{\lambda \in \Lambda} A^\lambda$ is Λ-graded as an algebra with involution (so the product and the involution are graded).

(ST2): If $\lambda \in \Lambda$ and $A^\lambda \neq 0$, then $A^\lambda = Fx$ and $A^{-\lambda} = Fx^{-1}$, where

$$xx^{-1} = x^{-1}x = 1$$

and

$$[L_x, L_{x^{-1}}] = 0 \quad \text{and} \quad [R_x, R_{x^{-1}}] = 0$$

(ST3): Λ is generated as a group by $\mathrm{supp}_\Lambda(A)$

In that case the integer n (the rank of Λ) is called the *nullity* of $(A, *)$.

Suppose now that L is a centreless rank 1 Lie torus. Since the root system of type A_1 is contained in the root system of type BC_1, we can assume that $\Delta = BC_1$. Thus

$$\Delta = \{-2\alpha, -\alpha, 0, \alpha, 2\alpha\}.$$

Hence the root grading of L becomes

$$L = L_{-2\alpha} \oplus L_{-\alpha} \oplus L_0 \oplus L_\alpha \oplus L_{2\alpha}. \tag{1}$$

That is, we have a 5-*grading* for L. Further, if we set

$$e = e^0_\alpha, \ f = f^0_\alpha \quad \text{and} \quad h = [e, f],$$

then $\{e, h, f\}$ is an sl_2-triple and the 5-grading (1) is obtained from this triple as the eigenspace decomposition for $\mathrm{ad}(h)$ (corresponding to the eigenvalues $-4, -2, 0, 2, 4$ respectively).

Now 5-gradings obtained from sl_2-triples in this way have been studied, first by Kantor in [K1, K2] and subsequently by Allison in [A1, A2] and Benkart and Smirnov in [BS]. It follows from this work that the Lie torus L can be constructed from a structurable algebra.

More precisely, it follows that the vector space $A = L_\alpha$ can be given a multiplication and involution $*$ so that $(A, *)$ is a structurable algebra and that

$$L = K(A, *),$$

where $K(A, *)$ is the 5-graded Lie algebra obtained from $(A, *)$ by means of a construction called the *Kantor construction* [K2, A2].

So far this analysis has taken into account only the root graded structure of L. It has not exploited the existence of the Λ-graded structure on L or the existence of the plentiful supply of sl_2-triples described in (LT2). In fact, using these tools, it follows that the coordinate algebra $(A, *)$ is a structurable torus.

Conversely one shows easily that given a structurable torus, the Lie algebra $K(A, *)$ is naturally a centreless Lie torus of rank 1. In this way, the problem of classifying centreless Lie tori of rank 1 becomes equivalent to the problem of classifying structurable tori.

5. Structurable tori

When discussing the classification problem for structurable tori, it is natural to first consider the case when the involution is the identity.

Indeed, if $(A, *)$ is a structurable torus and $* = \mathrm{id}$, then the corresponding Lie torus $L = K(A, *)$ satisfies

$$L_{2\alpha} = L_{-2\alpha} = 0,$$

and so L has type A_1. In that case A is a Jordan torus, which is defined as a unital Jordan algebra satisfying axioms the axioms (ST1)–(ST3) (with $* = \mathrm{id}$), and the construction $A \mapsto K(A) := K(A, \mathrm{id})$ is the classical Tits-Kantor-Koecher construction.

Jordan tori, and hence centerless Lie tori of type A_1, were classified by Yoshii in [Y1]. It turns out that Jordan tori are strongly prime Jordan algebras, and that examples of Jordan tori exist of hermitian type, Clifford type and Albert type (in the terminology of McCrimmon and Zelmanov [McZ]). In fact (if $n \geq 3$) there is just one torus of Albert type, an algebra called the *Albert torus*.

We now turn our attention to structurable tori with nonidentity involution. These algebras were studied by Allison and Yoshii in [AY], where a number of basic properties were developed resulting in a classification in nullities 1 and 2. In recent work by the three authors of this article, we have obtained a full classification of structurable tori with nonidentity involution. The rest of this section will discuss that work, beginning with an example.

Example 5.1. Any alternative torus with involution (defined as a unital alternative algebra with involution satisfying (ST1)–(ST3)) is a structurable torus. In particular, suppose that $n = 1, 2, 3$, and let

$$A(n) = \mathrm{CD}(F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}], t_1, \ldots, t_n)$$

be the algebra obtained by n applications of the Cayley-Dickson process starting from the ring $F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ of Laurent polynomials and using the scalars $t_1, \ldots, t_n$. These algebras were introduced by Berman, Gao, Krylyuk and Neher [BGKN] in their study of EALA's of type A_2. ($A(3)$ is called the *octonion torus*, and $A(2)$ is called the *quaternion torus*. $A(1)$ could similarly be called the binarion torus following the terminology in [Mc].) If we choose a basis $B = \{\sigma_1, \ldots, \sigma_n\}$ for Λ, then $A(n)$ has unique Λ-grading, called the *toral* Λ*-grading* determined by B, obtained by assigning the degrees $\sigma_1, \ldots, \sigma_n$ to the canonical generators of $A(n)$. Also, $A(n)$ has the *standard involution* $\natural$ which anti-fixes the canonical generators. In this way, $(A(n), \natural)$ is a alternative torus with involution and hence a structurable torus for $n = 1, 2, 3$.

Note also that the quaternion torus $A(2)$ has a nonstandard involution $*_m$, called the *main involution*, that fixes the two canonical generators. Thus if $n = 2$, we obtain another important structurable torus $(A(2), *_m)$.

We have just considered alternative tori with involution as examples of structurable tori. It is natural to ask whether any structurable torus with nontrivial involution is alternative, but examples constructed from hermitian forms [AY, Example 4.6] tell us that this is not true. However one does know that *skew elements* (elements s so that $s^* = -s$) in a structurable algebra behave very much like elements in an alternative algebra. For example there is an analog due to Smirnov [S] of Artin's theorem for skew elements in a structurable algebra (any two such elements generate an associative algebra). It is reasonable to expect then that a structurable torus generated by skew elements is in some sense close to alternative. This philosophy is supported by the following classification theorem.

Theorem 5.2. *Suppose $(A, *)$ is a structurable torus that is generated as an algebra by skew-elements. Then $(A, *)$ is the tensor product of tori from the following list:*

$$(A(1), \natural),\ (A(2), \natural),\ (A(3), \natural),\ (A(2), *_m),\ (F[t_1^{\pm 1}, \dots, t_p^{\pm 1}], \mathrm{id}).$$

More precisely

(a) *If A is associative, then there is an internal direct sum decomposition $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_{k+2}$ of Λ so that*

$$(A, *) \simeq (A_1, *) \otimes \ldots \otimes (A_{k+2}, *), \tag{2}$$

as Λ-graded algebras, where $k \geq 0$, $(A_i, *) = (A(2), \natural)$ for $1 \leq i \leq k$, $(A_{k+1}, *) = (F, \mathrm{id})$, $(A(1), \natural)$ or $(A(2), *_m)$, $(A_{k+2}, *) = (F[t_1^{\pm 1}, \dots, t_q^{\pm 1}], \mathrm{id})$ for some $q \geq 0$, and $(A_i, *)$ has a toral Λ_i-grading for $1 \leq i \leq k+2$.

(b) *If A is not associative, then there is an internal direct sum decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$ of Λ so that*

$$(A, *) \simeq (A_1, *) \otimes (A_2, *) \otimes (A_3, *), \tag{3}$$

as Λ-graded algebras, where $(A_1, *) = (A(3), \natural)$, $(A_2, *) = (A(p), \natural)$ for some $p = 1, 2, 3$, $(A_3, *) = (F[t_1^{\pm 1}, \dots, t_q^{\pm 1}], \mathrm{id})$ for some $q \geq 0$, and $(A_i, *)$ has a toral Λ_i-grading for $i = 1, 2, 3$.

The tensor products on the right hand sides of (2) and (3) require some further explanation. The underlying vectors spaces for these algebras are respectively $A_1 \otimes \ldots \otimes A_{k+2}$ and $A_1 \otimes A_2 \otimes A_3$. In each case a product of pure tensors is obtained by multiplying the corresponding factors, and the involution is the tensor product of the involutions on the factors. Finally in each case the degree of a pure tensor with homogeneous factors is defined to be the sum of the degrees of the homogeneous factors.

Conversely, the tensor products on the right hand sides of (2) and (3) are structurable tori and, with two exceptions, they are generated by skew-elements. (The exceptions occur in (2) when $k = 0$ and $(A_{k+1}, *)$ is either (F, id) or $(A(2), *_m)$.)

Some features of the proof. The proof of Theorem 5.2 has some interesting features, and we now discuss some of them briefly.

First, using the fact that A is generated by skew elements, one can show that A has no homogeneous zero divisors:

$$0 \neq x \in A^\lambda,\ 0 \neq y \in A^\mu \implies xy \neq 0.$$

Thus $\lambda, \mu \in \mathrm{supp}_\Lambda(A) \implies \lambda + \mu \in \mathrm{supp}_\Lambda(A)$. So since $\mathrm{supp}_\Lambda(A)$ generates Λ as a group, $\mathrm{supp}_\Lambda(A) = \Lambda$. That is A has *full support*.

If $\lambda \in \Lambda$, we therefore have $A^\lambda = Fx$, where $x \neq 0$. Now $*$ stabilizes A^λ and has period 2, and so $x^* = (-1)^{\xi(\lambda)}x$, where $\xi(\lambda) \in \{0,1\}$. Thus we have a function $\xi : \Lambda \to \{0,1\}$.

One next shows that ξ is constant on cosets of 2Λ, and so ξ induces a function $\xi : \Lambda/2\Lambda \to \{0,1\} = \mathbb{F}_2$. We regard $\Lambda/2\Lambda$ as a vector space over $\mathbb{F}_2$, and from now on we think of ξ as a function defined on this vector space.

Suppose now that A is associative. Then $\xi : \Lambda/2\Lambda \to \mathbb{F}_2$ turns out to be a quadratic form over $\mathbb{F}_2$. So, using the classification of quadratic forms, we obtain an orthogonal decomposition of ξ which leads to the tensor product decomposition in (a).

Suppose next that A is not associative. Then ξ is not a quadratic form. In fact the obstruction to ξ being a quadratic form turns out to be exactly the obstruction to A being associative. Nonetheless we can still use the language of quadratic forms, and we obtain an orthogonal decomposition of ξ which leads to the tensor product decomposition in (b). As one might suspect, obtaining the appropriate orthogonal decomposition for ξ in part (b) is the most difficult part of the proof. $\square$

Theorem 5.2 gives a complete classification of structurable tori that are generated by skew-elements. Very recently the authors have also obtained a classification of structurable tori with nonidentity involution that are not generated by skew-elements. These algebras are closer in their behavior to Jordan tori than they are to alternative tori. In particular, they do not in general have full support and so new techniques are needed for their classification. It turns out that the classification includes an infinite family of algebras constructed from graded hermitian forms over a quantum torus (see [AY, Example 4.6] for a description of these). In addition there are 5 new exceptional tori that occur.

6. Central closure

We conclude this article with a brief discussion, based on work of Neher [N, Theorem 7], about the central closure of a centreless Lie torus.

Theorem 6.1 (Neher)**.** *Suppose that L is a centreless Lie torus of nullity n. Let Z be the centroid of L. Then*

$$Z \simeq F[t_1^{\pm 1}, \dots, t_p^{\pm 1}]$$

for some $0 \le p \le n$, and hence

$$\widetilde{Z} := \text{quotient field of } Z \simeq F(t_1, \dots, t_p).$$

Suppose further that L has type $\neq A_\ell$. Then L is a finitely generated free Z-module, and if we set

$$\widetilde{L} := \widetilde{Z} \otimes_Z L,$$

$\widetilde{L}$ is a finite dimensional central simple Lie algebra over $\widetilde{Z}$.

The Lie algebra $\widetilde{L}$ occurring in the second part of this theorem is called the *central closure* of L. Now L embeds in its central closure. Hence (except in type A) a centreless Lie torus L can be regarded as a Z-form of a finite dimensional central simple (in general nonsplit) Lie algebra $\widetilde{L}$ over a rational function field $\widetilde{Z}$.

Example 6.2. Let $L = K(A, *)$, where $(A, *) = (A(3), \natural) \otimes (A(3), \natural)$ as in (3) of Theorem 5.2. Then L is a centreless Lie torus of type BC_1 and nullity 6. Moveover, Z is isomorphic to the algebra of Laurent polynomials in 6 variables and $\widetilde{L}$ is a finite dimensional nonsplit central simple Lie algebra of absolute type E_8 over the rational function field $\widetilde{Z}$ in 6 variables. This Lie algebra $\widetilde{L}$ is not new. For example Lie algebras constructed using the Kantor construction from tensor products of octonion algebras over fields have been studied in [A3]. Moveover, if the base field is extended further to the algebraic closure of $\widetilde{Z}$, one obtains the model of the split simple E_8 described in [K2]. However, the (infinite dimensional) centreless Lie torus L and the corresponding tame EALA's (see §2) are new.

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