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Abstract. A new class of infinite-dimensional Lie algebras, called extended affine Lie algebras, has recently been introduced by physicists. In their description, the classification of so-called tori is an important step. The class of associative tori is known as quantum tori. Alternative tori have also been classified. In this announcement, we will describe the classification of Jordan tori. An important tool in our work is Zelmanov's structure theorem for prime Jordan algebras. Jordan tori can be used to coodinatize extended affine Lie algebras of type G_2 and A_1, while alternative tori can be used for type F_4 and A_2.

Let F be a field of characteristic $\neq 2$ and T a (not necessarily associative) unital algebra over F. To say that T is *graded by an abelian group* A means $T = \bigoplus_{\alpha\in A} T_\alpha$ (direct sum of F-spaces) and $T_\alpha T_\beta \subset T_{\alpha+\beta}$ for all $\alpha, \beta \in A$. We define the centre $Z(T)$ of T as $Z(T) = \{x \in T : [x, y] = (x, y, z) = (y, x, z) = (y, z, x) = 0 \text{ for all } y, z \in T\}$ where $[x, y] = xy - yx$ and $(x, y, z) = (xy)z - x(yz)$.

Definition 1. A unital algebra $T = \bigoplus_{\alpha\in\mathbb{Z}^n} T_\alpha$ graded by $\mathbb{Z}^n$ is called an *n-torus* or simply a *torus* if $\dim_F T_\alpha = 1$ and $T_\alpha T_\beta = T_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}^n$.

As a basic property of tori, we have:

Lemma 1. *A torus has no zero-divisors.*

In particular, the centre $Z = Z(T)$ of a torus T has no zero-divisors, and is therefore an integral domain. Let $\overline{Z}$ be the field of fractions of Z. We define $\overline{T} = \overline{Z} \otimes_Z T$ and call it the *central closure of* T. Then T embeds into $\overline{T}$ via $x \mapsto 1 \otimes x$. We identify T as a subalgebra of $\overline{T}$.

Definition 2. A torus is called an *associative torus* if it is an associative algebra, an *alternative torus* if it is an alternative algebra, and a *Jordan torus* if it is a Jordan algebra.

Example 1. (1) Let $E = F\langle T_1^{\pm1}, \dots, T_n^{\pm1}\rangle$ be the associative algebra of Laurent polynomials in non-commuting variables $T_1, \dots, T_n$ over F, let $\boldsymbol{q} = (q_{ij})$ be an $n \times n$ matrix such that

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$q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$, and let $I_{\boldsymbol{q}}$ be the ideal of E generated by $\{T_jT_i - q_{ij}T_iT_j : 1 \leq i, j \leq n\}$. Then the *quantum torus associated to* $\boldsymbol{q}$ is defined as $F_{\boldsymbol{q}} = E/I_{\boldsymbol{q}}$. One can show that $F_{\boldsymbol{q}}$ is a torus and that every associative torus is isomorphic to some $F_{\boldsymbol{q}}$. In particular, an associative torus which is commutative is isomorphic to $F_{\mathbf{1}}$ where

$$\mathbf{1} = \mathbf{1}_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \text{ (all } q_{ij} = 1).$$

This is nothing but the algebra of Laurent polynomials in n-variable, $F[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. One can also check that $F_{\boldsymbol{q}}^+$ is a Jordan torus if and only if

$$(*) \qquad \prod_{i,j} q_{ij}^{\alpha_i \beta_j} \neq -1 \text{ for all } (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n.$$

(2) Let

$$\boldsymbol{j} = \boldsymbol{j}_n = \begin{pmatrix} 1 & -1 & 1 & \cdots & 1 \\ -1 & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} (q_{12} = -1,\ q_{21} = -1 \text{ and } q_{ij} = 1 \text{ for the other } i, j).$$

Then one can show that the quantum torus $F_{\boldsymbol{j}}$ is a quaternion algebra over its centre $Z = Z(F_{\boldsymbol{j}}) = F[T_1^{\pm 2}, T_2^{\pm 2}, T_3^{\pm 1}, \dots, T_n^{\pm 1}]$, which we call the *quaternion torus*. The Cayley-Dickson doubling process yields an octonion algebra $\mathbb{OT} = (F_{\boldsymbol{j}}, T_3)$ over Z, taking $T_3 \in Z$ as the structure constant. One can show that the F-algebra $\mathbb{OT}$ is an alternative torus, which we call the *octonion torus* (Table 1). This was called the alternative torus in [BGKN].

(3) Let

$$\boldsymbol{\omega} = \boldsymbol{\omega}_n = \begin{pmatrix} 1 & \omega & 1 & \cdots & 1 \\ \omega^2 & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} (q_{12} = \omega,\ q_{21} = \omega^2 \text{ and } q_{ij} = 1 \text{ for the other } i, j),$$

where $\omega \in F$ is a primitive 3rd root of unity. One can show that the central closure $\overline{F_{\boldsymbol{\omega}}}$ of the quantum torus $F_{\boldsymbol{\omega}}$ is a central (associative) division algebra of degree 3 over the field $\overline{Z} = F(T_1^3, T_2^3, T_3, \dots, T_n)$. Thus we can construct an Albert algebra $(\overline{F}_{\boldsymbol{\omega}}, T_3)$ over $\overline{Z}$ by Tits' 1st construction [Ja], taking $T_3 \in Z \subset \overline{Z}$ as the structure constant. Let $\mathbb{AT}$ be the Jordan F-subalgebra of $(\overline{F}_{\boldsymbol{\omega}}, T_3) = \overline{F}_{\boldsymbol{\omega}} \bigoplus \overline{F}_{\boldsymbol{\omega}} \bigoplus \overline{F}_{\boldsymbol{\omega}}$ generated by $T_1^{\pm 1}, T_2^{\pm 1}, (0,1,0)^{\pm 1}, T_4^{\pm 1}, \dots, T_n^{\pm 1}$ (Note $(0,1,0)^3 = T_3$). Then one can check that $\mathbb{AT} = F_{\boldsymbol{\omega}} \bigoplus F_{\boldsymbol{\omega}} \bigoplus F_{\boldsymbol{\omega}}$ is a Jordan torus (over

F) which we call the *Albert torus* (Table 2). This example appears in [AABGP] where it is called the Jordan torus. It was also found independently by the author.

Theorem 1. *An alternative torus is isomorphic to either a quantum torus or the octonion torus.*

Remark 1. This result was proven for certain base fields (e.g. F is algebraically closed.) in [BGKN]. It is in fact true for any field F of characteristic $\neq 2$.

For the classification of Jordan tori we first prove:

Lemma 2. *A Jordan torus is strongly prime.*

Because of Lemma 2, we can apply Zelmanov's structure theorem of prime Jordan algebras [M-Z]. Hence, a Jordan torus is of Clifford type, hermitian type or exceptional type.

Theorem 2. *Let J be a Jordan torus over F, and assume that $a \in F$ implies $\sqrt{a} \in F$ and that F contains a primitive 3rd root of unity. Then*

(i) *J cannot be of Clifford type,*

(ii) *J is of hermitian type if and only if $J \cong F_{\boldsymbol{q}}^{+}$ with $(*)$* (Example 1 (1)),

(iii) *J is of exceptional type if and only if $J \cong \mathbb{AT}$.*

Remark 2. The first assumption for F is used for the proof of (ii), while the second is used for the proof of (iii).

Corollary 1. *Let J be as above. Then J is special if and only if $J \cong F_{\boldsymbol{q}}^{+}$ with $(*)$, and J is exceptional if and only if $J \cong \mathbb{AT}$.*

We define the *degree* of a torus T as the degree of the generic minimal polynomial of the central closure $\overline{T}$ over $\overline{Z}$ [Ja]. Then we can show:

Lemma 3. (i) *A quantum torus of degree 2 is isomorphic to the quaternion torus $F_{\boldsymbol{j}}$*

(ii) *A quantum torus of degree 3 is isomorphic to $F_{\boldsymbol{\omega}}$.*

Remark 3. (1) For (ii), we assume that $\omega \in F$. Otherwise, there does not exist such a torus.

(2) $F_{\boldsymbol{q}} \cong F_{\boldsymbol{q}'}$ does not imply $\boldsymbol{q} = \boldsymbol{q}'$ after renumbering the rows and columns of q', if necessary. For example, one can check that $F_{\boldsymbol{q}} \cong F_{\boldsymbol{q}'}$ for

$$\boldsymbol{q} = \boldsymbol{j}_3, \boldsymbol{q}' = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \text{ or } \boldsymbol{q} = \boldsymbol{\omega}_3, \boldsymbol{q}' = \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega^2 & 1 \end{pmatrix}.$$

Finally, we can prove:

Corollary 2. *Let A be an alternative torus and J a Jordan torus with the same assumption for F as in Theorem 1. Then:*

(i) *A is of degree 2 if and only if A is isomorphic to either the quaternion torus or the octonion torus.*

(ii) *J is of degree 3 if and only if J is isomorphic to either $F_{\boldsymbol{\omega}}^{+}$ or the Albert torus.*

Remark 4. The central closures $\overline{F_{\boldsymbol{j}}}$, $\overline{F_{\boldsymbol{\omega}}^{+}}$, $\overline{\mathbb{OT}} = (\overline{F}_{\boldsymbol{j}}, T_3)$, and $\overline{\mathbb{AT}} = (\overline{F}_{\boldsymbol{\omega}}, T_3)$ are all division algebras.

Appendix

Table 1. Multiplication table for the octonion torus:

$$\mathbb{OT} = (F_{\boldsymbol{j}}, T_3) = F_{\boldsymbol{j}} \oplus F_{\boldsymbol{j}} = \bigoplus_{\alpha \in \mathbb{Z}^n} F t_\alpha$$

where, for $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$t_\alpha = \begin{cases} (T^\alpha T_3^m, 0) & \text{if } \alpha_3 = 2m \\ (0, T^\alpha T_3^m) & \text{if } \alpha_3 = 2m+1 \quad (m \in \mathbb{Z}) \end{cases}$$

and $T^\alpha = T_1^{\alpha_1} T_2^{\alpha_2} T_4^{\alpha_4} \cdots T_n^{\alpha_n}$. For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, the multiplication in $\mathbb{OT}$ is given by

(I) $\alpha_3 \equiv \beta_3 \equiv 0$ (mod 2) (all $\equiv$ are mod 2 below) :

$$t_\alpha t_\beta = (-1)^{\alpha_2 \beta_1} t_{\alpha+\beta}$$

(II) $\alpha_3 \equiv 0$, $\beta_3 \equiv 1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ (-1)^{\alpha_2 \beta_1 + 1} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(III) $\alpha_3 \equiv 1$, $\beta_3 \equiv 0$:

$$t_\alpha t_\beta = (-1)^{\alpha_1 \beta_2} t_{\alpha+\beta}$$

(IV) $\alpha_3 \equiv 1$, $\beta_3 \equiv 1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ (-1)^{\alpha_1 \beta_2 + 1} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

The structure constants are $\{\pm 1\}$.

Table 2. Multiplication table for the Albert torus:

$$\mathbb{AT} = F_{\boldsymbol{\omega}} \oplus F_{\boldsymbol{\omega}} \oplus F_{\boldsymbol{\omega}} = \langle T_1^{\pm 1}, T_2^{\pm 1}, (0,1,0)^{\pm 1}, T_4^{\pm 1}, \dots, T_n^{\pm 1} \rangle = \bigoplus_{\alpha \in \mathbb{Z}^n} F t_\alpha$$

where, for $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$t_\alpha = \begin{cases} (T^\alpha T_3^m, 0, 0) & \text{if } \alpha_3 = 3m \\ (0, T^\alpha T_3^m, 0) & \text{if } \alpha_3 = 3m+1 \\ (0, 0, T^\alpha T_3^m) & \text{if } \alpha_3 = 3m-1 \ (m \in \mathbb{Z}) \end{cases}$$

and $T^\alpha = T_1^{\alpha_1} T_2^{\alpha_2} T_4^{\alpha_4} \cdots T_n^{\alpha_n}$. For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, the multiplication in $\mathbb{AT}$ is given by

(I) $\alpha_3 \equiv \beta_3 \equiv 0 \pmod 3$ (all $\equiv$ are mod 3 below) :

$$t_\alpha t_\beta = \frac{1}{2}(\omega^{\alpha_1 \beta_2} + \omega^{\alpha_2 \beta_1}) t_{\alpha+\beta}$$

(II) $\alpha_3 \equiv 0$, $\beta_3 \equiv 1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ -\frac{1}{2}\omega^{\alpha_2 \beta_1} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(III) $\alpha_3 \equiv 0$, $\beta_3 \equiv -1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ -\frac{1}{2}\omega^{\alpha_1 \beta_2} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(IV) $\alpha_3 \equiv 1$, $\beta_3 \equiv -1$:

$$t_\alpha t_\beta = \begin{cases} \omega^{\alpha_2 \beta_1} t_{\alpha+\beta} & \text{if } \alpha_1 + \beta_1 \equiv \alpha_2 + \beta_2 \equiv 0 \\ -\frac{1}{2}\omega^{\alpha_2 \beta_1} t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(V) $\alpha_3 \equiv \beta_3 \equiv 1$ or $\alpha_3 \equiv \beta_3 \equiv -1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv \beta_1 \equiv \beta_2 \equiv 0 \\ -\frac{1}{2} t_{\alpha+\beta} & \text{if } [\alpha_1 \equiv \alpha_2 \equiv 0 \text{ and } (\beta_1 \not\equiv 0 \text{ or } \beta_2 \not\equiv 0)] \\ & \quad \text{or } [(\alpha_1 \not\equiv 0 \text{ or } \alpha_2 \not\equiv 0) \text{ and } \beta_1 \equiv \beta_2 \equiv 0] \\ -\frac{1}{4}(\omega^{\alpha_1 \beta_2} + \omega^{\alpha_2 \beta_1}) t_{\alpha+\beta} & \text{if } (\alpha_1 \not\equiv 0 \text{ or } \alpha_2 \not\equiv 0) \text{ and } (\beta_1 \not\equiv 0 \text{ or } \beta_2 \not\equiv 0) \\ & \quad \text{and } \alpha_1 + \beta_1 \equiv \alpha_2 + \beta_2 \equiv 0 \\ \frac{1}{2}(\omega^{\alpha_1 \beta_2} + \omega^{\alpha_2 \beta_1}) t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

Since $\omega^2 + \omega + 1 = 0$, the structure constants are

$$\{1, \omega, \omega^2, -\frac{1}{2}, -\frac{\omega}{2}, -\frac{\omega^2}{2}, \frac{1}{4}, \frac{\omega}{4}, \frac{\omega^2}{4}\}.$$

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