May 22, 2015

LOCALLY LOOP ALGEBRAS AND LOCALLY AFFINE LIE ALGEBRAS

JUN MORITA AND YOJI YOSHII

ABSTRACT. In this study, we investigate a new class of Lie algebras, i.e., tame locally extended affine Lie algebras of nullity 1, which are an infinite-rank analog of affine Lie algebras. This type of algebra is called a locally affine Lie algebra. A certain ideal of a locally affine Lie algebra, called a core, is a universal extension of a local version of a loop algebra, which is called a locally loop algebra. We classify locally loop algebras and locally affine Lie algebras.

Throughout this study, F is a field of characteristic 0. All of the algebras are assumed to be unital, except the Lie algebras. The tensor products are over F.

1. INTRODUCTION

Historically, root systems have played very important roles in Lie theory and many other areas. To obtain a root system, we usually need a certain ad-diagonalizable subalgebra $\mathscr{H}$ of a Lie algebra $\mathscr{L}$ over F. Then, we have a decomposition:

$$\mathscr{L} = \bigoplus_{\xi\in\mathscr{H}^*} \mathscr{L}_\xi,$$

where $\mathscr{H}^*$ is the dual space of $\mathscr{H}$ and $\mathscr{L}_\xi = \{x \in \mathscr{L} \mid [h,x] = \xi(h)x \text{ for all } h \in \mathscr{H}\}$. An element $\xi \in \mathscr{H}^*$ is called a **root** if $\mathscr{L}_\xi \neq 0$, and the **set of roots** is defined by

$$R = \{\xi \in \mathscr{H}^* \mid \mathscr{L}_\xi \neq 0\}.$$

The subspace $\mathscr{L}_\xi$ is called the **root space** of ξ and the direct sum above is called the **root space decomposition** of $\mathscr{L}$ associated with $\mathscr{H}$. In many cases, a root has its own length, which may come from a symmetric invariant bilinear form $\mathscr{B}$ on $\mathscr{L}$. Therefore, it is natural to consider a triplet $(\mathscr{L},\mathscr{H},\mathscr{B})$ in general.

Such a triplet $(\mathscr{L},\mathscr{H},\mathscr{B})$ is called a **locally extended affine Lie algebra** if conditions (A1) – (A4) are satisfied, as follows:

(A1) $\mathscr{H}$ is ad-diagonalizable and self-centralizing, i.e.,

$$\mathscr{L} = \bigoplus_{\xi\in\mathscr{H}^*} \mathscr{L}_\xi \quad \text{and} \quad \mathscr{H} = \mathscr{L}_0,$$

(A2) $\mathscr{B}$ is nondegenerate,

(A3) $\mathrm{ad}\,x \in \mathrm{End}_F\,\mathscr{L}$ is locally nilpotent for all $\xi \in R^\times$ and all $x \in \mathscr{L}_\xi$, where $R^\times = \{\xi \in R \mid \mathscr{B}(t_\xi,t_\xi) \neq 0\}$ and t_ξ is an element of $\mathscr{H}$ such that $\xi(h) = \mathscr{B}(t_\xi,h)$ for all $h \in \mathscr{H}$.

(A4) $R^\times$ is irreducible, i.e., there is no nontrivial partition, $R^\times = R_1 \cup R_2$, of $R^\times$ such that $\mathscr{B}(t_{\xi_1},t_{\xi_2}) = 0$ for all $\xi_1 \in R_1$, $\xi_2 \in R_2$.

2000 *Mathematics Subject Classification.* Primary: 17B65, 17B67 ; Secondary: 17B70.

Jun Morita gratefully acknowledges partial financial support from a Grant-in-aid for Science Research (MONKASHO KAKENHI) in Japan.

A locally extended affine Lie algebra is abbreviated as a **LEALA**. We note that the class of LEALAs contains:
(1) all finite dimensional split simple Lie algebras,
(2) all affine Lie algebras,
(3) all locally finite split simple Lie algebras,
(4) all extended affine Lie algebras, and
(5) all Heisenberg Lie algebras (as null systems).
We can refer to them all as LEALAs and study them uniformly. The class of LEALAs is assumed to be the best (or the widest) class of $(\mathscr{L},\mathscr{H},\mathscr{B})$ in the following sense.

If we fix $\xi \in R^\times$ and select $a \in F^\times$ such that $a\mathscr{B}(t_\xi,t_\xi) \in \mathbb{Q}_{>0}$, as well as defining $(\cdot,\cdot)$ as a symmetric bilinear form on $V = \sum_{\xi\in R} \mathbb{Q}\xi \subset \mathfrak{h}^*$ by

$$\left(\sum_{\xi\in R} p_\xi \xi\ ,\ \sum_{\eta\in R} q_\eta \eta\right) = a\mathscr{B}\left(\sum_{\xi\in R} p_\xi t_\xi\ ,\ \sum_{\eta\in R} q_\eta t_\eta\right)$$

for $p_\xi, q_\eta \in \mathbb{Q}$, then we find that the properties of the form $(\cdot,\cdot)$ are actually $\mathbb{Q}$-valued and that

$$(\cdot,\cdot) \text{ is positive semi-definite, while } (\xi,v)=0 \text{ for all } \xi\in R^0 \text{ and all } v\in V,$$

where $R^0 = \{\xi \in R \mid (\xi,\xi)=0\}$. This property is usually called the **Kac Conjecture**, which was proved for EALAs (defined below) in [AABGP] and for LEALAs in [MY].

Note that $R^\times = \{\xi \in R \mid (\xi,\xi)\neq 0\}$. An element of $R^\times$ is called an **anisotropic root**, and an element of R^0 is called an **isotropic root**. We can also refer to $R^\times$ as the **root system**. Note that $R^\times$ is a finite root system and $R^0=\{0\}$ when $\mathscr{L}$ is a finite-dimensional split simple Lie algebra (cf. [Bo]), while $R^\times$ is an affine root system and $R^0 = \mathbb{Z}\xi_0$ for some $\xi_0 \in \mathscr{H}^*$ when $\mathscr{L}$ is an affine Lie algebra (cf. [Ma]). A LEALA is called an **extended affine Lie algebra** (**EALA**), when $\dim \mathscr{H} < \infty$.

The **nullity** of a LEALA is defined as the rank of the additive group generated by R^0. In particular, we only use the term 'nullity' when the additive group is free (see Remark 4.6). The **core** $\mathscr{L}_c$ of a LEALA $\mathscr{L}$ is defined as the subalgebra of $\mathscr{L}$ generated by $\mathscr{L}_\xi$ for all $\xi \in R^\times$. In fact, $\mathscr{L}_c$ is an ideal of $\mathscr{L}$, which is obtained by the Kac Conjecture. If the centralizer $C_{\mathscr{L}}(\mathscr{L}_c)$ of $\mathscr{L}_c$ in $\mathscr{L}$ is contained in $\mathscr{L}_c$, this LEALA $\mathscr{L}$ is referred to as **tame**. The core $\mathscr{L}_c$ modulo of its center $Z(\mathscr{L}_c)$, i.e., the quotient Lie algebra $\mathscr{L}_c/Z(\mathscr{L}_c)$, is called the **centerless core** of $\mathscr{L}$.

Previously, we classified LEALAs of nullity 0 in [MY]. The second simplest class comprises LEALAs of nullity 1. The main aim of the present study is to classify the class of tame LEALAs of nullity 1, which we call a **locally affine Lie algebra** (**LALA**).

The centerless core

$$L := \mathscr{L}_c/Z(\mathscr{L}_c)$$

of a LALA $\mathscr{L}$ is a local version of a loop algebra, which we call a **locally loop algebra**. In fact, we show that a locally loop algebra is a directed union of loop algebras. We also show that the core $\mathscr{L}_c$ of a LALA $\mathscr{L}$ is a universal covering of a locally loop algebra L. (These classifications were also shown by Neeb [N2, Cor. 3.13] in a different manner.) Thus, we may say that a LALA is also a local analog of an affine Lie algebra. However, a LALA $\mathscr{L}$ has a more complex structure in a complement of the core $\mathscr{L}_c$, such as $\mathscr{L} = \mathscr{L}_c \oplus D$. Note that for an affine Lie algebra, the complement D is simply a 1-dimensional space spanned by the degree derivation. However, for a LALA, the corresponding complement D is rather large in general. Due to tameness, D can be embedded into $\mathrm{Der}_F\, \mathscr{L}_c$. Then, $d \in$

$\mathrm{Der}_F \mathscr{L}_c$ induces a derivation of $L = \mathscr{L}_c/Z(\mathscr{L}_c)$ since $d(Z(\mathscr{L}_c)) \subset Z(\mathscr{L}_c)$, and we see that $d(x) \in Z(\mathscr{L}_c)$ for $x \in \mathscr{L}_c$ implies that $d = 0$ in $\mathrm{Der}_F \mathscr{L}_c$ since $\mathscr{L}_c = [\mathscr{L}_c, \mathscr{L}_c]$. Therefore, we can find $D \subset \mathrm{Der}_F \mathscr{L}_c \subset \mathrm{Der}_F L$, but also $D \subset \mathrm{Oder}_F \mathscr{L}_c \subset \mathrm{Oder}_F L$, where $\mathrm{Oder}_F(\cdot) = \mathrm{Der}_F(\cdot)/\mathrm{ad}(\cdot)$, which comprises the outer derivations. There is a unique maximal choice of D in $\mathrm{Oder}_F L$, such as D^{max} in this case, and there are many minimal choices of D in $\mathrm{Oder}_F L$, such as $D(p)$ with a specific diagonal matrix p. Thus, a homogeneous space D such that $D(p) \subset D \subset D^{max}$ leads to our classification. Thus, the classification of LALAs is obtained by saying that any homogeneous subalgebra $\mathscr{L}$ of $\mathscr{L}^{max} := \mathscr{L}_c \oplus D^{max}$ that satisfies

$$\mathscr{L}(p) := \mathscr{L}_c \oplus D(p) \subset \mathscr{L} \subset \mathscr{L}^{max}$$

is a LALA, and that any LALA can be obtained in this manner. We roughly explain the LALAs of type $\mathrm{A}^{(1)}_{\mathfrak{I}}$ and $\mathrm{C}^{(2)}_{\mathfrak{I}}$ to obtain a better understanding.

Let $\mathfrak{I}$ be an index set. We suppose that $\mathfrak{I}$ is any index set, i.e., $\mathfrak{I}$ can be finite or infinite. Let $M_{\mathfrak{I}}(F) = \{(a_{ij})_{i,j\in\mathfrak{I}} \mid a_{ij} \in F\}$ be the vector space of matrices of size $\mathfrak{I}$, and let $T_{\mathfrak{I}} = T_{\mathfrak{I}}(F) = \{(a_{ij}) \in M_{\mathfrak{I}}(F) \mid a_{ij} = 0 \text{ for } i \neq j\}$ be the subspace of $M_{\mathfrak{I}}(F)$, which comprises all diagonal matrices.

[$\mathrm{A}^{(1)}_{\mathfrak{I}}$] First, we explain the untwisted type $\mathrm{A}^{(1)}_{\mathfrak{I}}$. Let $\mathrm{sl}_{\mathfrak{I}}(F)$ be the subspace of $M_{\mathfrak{I}}(F)$ that comprises trace 0 matrices with only finitely many nonzero entries. Let $F[t^{\pm 1}]$ be the algebra of Laurent polynomials, and let $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}])$ be the Lie algebra $\mathrm{sl}_{\mathfrak{I}}(F) \otimes F[t^{\pm 1}]$. For example, if $\mathfrak{I} = \mathbb{N}$ (the natural numbers), then we see that

$$\mathrm{sl}_{\mathbb{N}}(F[t^{\pm}]) = \bigcup_{n=2}^{\infty} \mathrm{sl}_n(F[t^{\pm}]) = \bigcup_n \left(\begin{array}{c|c} \mathrm{sl}_n(F[t^{\pm}]) & O \\ \hline O & O \end{array}\right).$$

We refer to $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}])$ as a locally loop algebra of type $\mathrm{A}^{(1)}_{\mathfrak{I}}$, which is simply an infinite-rank analog of a loop algebra $\mathrm{sl}_{\ell+1}(F[t^{\pm 1}])$ of type $\mathrm{A}^{(1)}_{\ell}$. We use the following conventions.

$$\begin{cases} \mathrm{sl}_{\mathfrak{I}} \text{ has type } \mathrm{A}_{\mathfrak{I}} & \text{if } \mathfrak{I} \text{ is an infinite index set,} \\ \mathrm{sl}_{\mathfrak{I}} = \mathrm{sl}_{\ell+1} \text{ has type } \mathrm{A}_{\ell} & \text{if } \mathfrak{I} \text{ is a finite index set that comprises } \ell+1 \text{ elements.} \end{cases}$$

As in the case of $\mathrm{sl}_{\ell+1}(F[t^{\pm 1}])$, a universal covering $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}]) \oplus Fc$ of $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}])$ exists, where Fc is the 1-dimensional center. Then we can construct the Lie algebra

$$\mathscr{L}^{ms} := \mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}]) \oplus Fc \oplus Fd^{(0)}, \tag{1}$$

where $d^{(0)} = t\dfrac{d}{dt}$ is the degree derivation. This $\mathscr{L}^{ms}$ is the simplest example of a LALA, which is called a **minimal standard LALA** of type $\mathrm{A}^{(1)}_{\mathfrak{I}}$. In contrast to the affine Lie algebra case, there are more examples of type $\mathrm{A}^{(1)}_{\mathfrak{I}}$, which are obtained by adding diagonal derivations of $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}])$, and we explain these as follows. First, note that

$$\mathrm{sl}_{\mathfrak{I}}(F) + T_{\mathfrak{I}}$$

is a Lie algebra with center $F\iota$, where $\iota = \iota_{\mathfrak{I}} = (\delta_{ij})_{i,j\in\mathfrak{I}}$ is in $T_{\mathfrak{I}}$. Let

$$\mathscr{A}_{\mathfrak{I}} := \big(\mathrm{sl}_{\mathfrak{I}}(F) + T_{\mathfrak{I}}\big)/F\iota \tag{2}$$

be the quotient Lie algebra. We identify the subalgebra

$$\overline{\mathrm{sl}_{\mathfrak{I}}(F)} = \big(\mathrm{sl}_{\mathfrak{I}}(F) + F\iota\big)/F\iota$$

of $\mathscr{A}_{\mathfrak{I}}$ with $\mathrm{sl}_{\mathfrak{I}}(F)$. Consider the Lie algebra $\mathscr{A}_{\mathfrak{I}} \otimes F[t^{\pm 1}]$. We construct the Lie algebra

$$\hat{\mathscr{A}}_{\mathfrak{I}} := \mathscr{A}_{\mathfrak{I}} \otimes F[t^{\pm 1}] \oplus Fc \oplus Fd^{(0)} \tag{3}$$

as described in (1), which contains $\mathscr{L}^{ms}$.

Theorem 1.1. *$\mathscr{L}^{max} := \hat{\mathscr{A}}_{\mathfrak{J}}$ is a* **maximal LALA** *of type $A_{\mathfrak{J}}^{(1)}$, i.e., any LALA of type $A_{\mathfrak{J}}^{(1)}$ is a subalgebra of $\hat{\mathscr{A}}_{\mathfrak{J}}$. In addition, any LALA of type $A_{\mathfrak{J}}^{(1)}$ contains a LALA*

$$\mathscr{L}(p) := \mathrm{sl}_{\mathfrak{J}}(F[t^{\pm 1}]) \oplus Fc \oplus F(p + d^{(0)}) \tag{4}$$

for some $p \in T_{\mathfrak{J}}$.

This $\mathscr{L}(p)$ is called a **minimal LALA determined by** p. In general, we note that $\mathscr{L}(p)$ may be isomorphic to $\mathscr{L}^{ms} = \mathscr{L}(0)$, but not always isomorphic to $\mathscr{L}^{ms}$ (see Example 9.4). Note that a LALA $\mathscr{L}$ of type $\mathrm{A}_{\mathfrak{J}}^{(1)}$ has a decomposition $\mathscr{L} = \mathscr{L}_c \oplus D$ for a homogeneous complement D that satisfies $D(p) = F(p + d^{(0)}) \subset D \subset D^{max}$ and $\mathscr{L}_c \oplus D^{max} = \hat{\mathscr{A}}_{\mathfrak{J}}$.

[$\mathrm{C}_{\mathfrak{J}}^{(2)}$] Next, we explain the twisted type $\mathrm{C}_{\mathfrak{J}}^{(2)}$. Let $s = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}$ be the matrix of size $2\mathfrak{J}$, where $\iota = \iota_{\mathfrak{J}}$, as described above. Define an automorphism σ of period 2 on $\mathrm{sl}_{2\mathfrak{J}}(F) + T_{2\mathfrak{J}}$ by

$$\sigma(x) = sx^T s$$

for $x \in \mathrm{sl}_{2\mathfrak{J}}(F) + T_{2\mathfrak{J}}$, where x^T is the transpose of x. Let $\mathrm{sp}_{2\mathfrak{J}}(F)$ be the fixed subalgebra of $\mathrm{sl}_{2\mathfrak{J}}(F)$ by σ, which is of type $\mathrm{C}_{\mathfrak{J}}$. Let $\mathfrak{s}$ be the (-1)-eigenspace of σ such that

$$\mathrm{sl}_{2\mathfrak{J}}(F) = \mathrm{sp}_{2\mathfrak{J}}(F) \oplus \mathfrak{s}.$$

Moreover, let T^+ be the 1-eigenspace and T^- is the (-1)-eigenspace of $T_{2\mathfrak{J}}$ relative to σ. Note that $T_{2\mathfrak{J}} = T^+ \oplus T^-$, and thus

$$\mathrm{sl}_{2\mathfrak{J}}(F) + T_{2\mathfrak{J}} = \big(\mathrm{sp}_{2\mathfrak{J}}(F) + T^+\big) \oplus \big(\mathfrak{s} + T^-\big).$$

In addition, note that $F\iota_{2\mathfrak{J}}$ is σ-invariant and $F\iota_{2\mathfrak{J}} \subset T^-$. Let

$$\mathscr{A}_{2\mathfrak{J}} := \big(\mathrm{sl}_{2\mathfrak{J}}(F) + T_{2\mathfrak{J}}\big)/F\iota_{2\mathfrak{J}},$$

as described in (2). We have the induced automorphism on $\mathscr{A}_{2\mathfrak{J}}$, which is also denoted by σ for simplicity. Thus, we obtain the fixed algebra

$$\mathscr{A}_{2\mathfrak{J}}^{\sigma} = \big(\mathrm{sp}_{2\mathfrak{J}}(F) + T^+\big),$$

where we again omit the bars. Let

$$\hat{\mathscr{A}}_{2\mathfrak{J}} := \mathscr{A}_{2\mathfrak{J}} \otimes F[t^{\pm 1}] \oplus Fc \oplus Fd^{(0)},$$

as in (3). We extend σ to $\hat{\mathscr{A}}_{2\mathfrak{J}}$ as

$$\hat{\sigma}(x \otimes t^k) := (-1)^k \sigma(x) \otimes t^k,$$

and identically on $Fc \oplus Fd^{(0)}$. Then, we obtain the fixed algebra

$$\hat{\mathscr{A}}_{2\mathfrak{J}}^{\hat{\sigma}} = \big((\mathrm{sp}_{2\mathfrak{J}}(F) + T^+) \otimes F[t^{\pm 2}]\big) \oplus \big((\mathfrak{s} + T^-) \otimes tF[t^{\pm 2}]\big) \oplus Fc \oplus Fd^{(0)}. \tag{5}$$

Note that $\hat{\mathscr{A}}_{2\mathfrak{J}}^{\hat{\sigma}}$ contains the subalgebra

$$\mathscr{L}^{ms} := \big(\mathrm{sp}_{2\mathfrak{J}}(F) \otimes F[t^{\pm 2}]\big) \oplus \big(\mathfrak{s} \otimes tF[t^{\pm 2}]\big) \oplus Fc \oplus Fd^{(0)},$$

which is called a **minimal standard twisted LALA** of type $\mathrm{C}_{\mathfrak{J}}^{(2)}$. As in the case of type $\mathrm{A}_{\mathfrak{J}}^{(1)}$, we have

Theorem 1.2. *$\mathscr{L}^{max} := \hat{\mathscr{A}}^{\hat{\sigma}}_{2\mathfrak{I}}$ is a* **maximal twisted LALA** *of type $C^{(2)}_{\mathfrak{I}}$, i.e., any LALA of type $C^{(2)}_{\mathfrak{I}}$ is a subalgebra of $\hat{\mathscr{A}}^{\hat{\sigma}}_{2\mathfrak{I}}$. Moreover, any LALA of type $C^{(2)}_{\mathfrak{I}}$ contains a* **minimal twisted LALA**

$$\mathscr{L}(p) := \big(sp_{2\mathfrak{I}}(F)\otimes F[t^{\pm 2}]\big)\oplus\big(\mathfrak{s}\otimes tF[t^{\pm 2}]\big)\oplus Fc\oplus F(p+d^{(0)})$$

for some $p\in T^{+}$.

We must emphasize that the usual twisting process works for the locally loop algebra $\mathrm{sl}_{2\mathfrak{I}}(F)\otimes F[t^{\pm 1}]$ but also for the bigger algebra $\big(\mathrm{sl}_{2\mathfrak{I}}(F)+T_{2\mathfrak{I}}\big)\otimes F[t^{\pm 1}]$. Note that a LALA $\mathscr{L}$ of type $\mathrm{C}^{(2)}_{\mathfrak{I}}$ has a decomposition $\mathscr{L}=\mathscr{L}_c\oplus D$ for a homogeneous complement D that satisfies $D(p)=F(p+d^{(0)})\subset D\subset D^{max}$ and $\mathscr{L}_c\oplus D^{max}=\hat{\mathscr{A}}^{\hat{\sigma}}_{2\mathfrak{I}}$.

Next, we explain how the classification of LALAs is conducted. First, we classify the cores of the LALAs. We show that the core of a LALA is a locally Lie 1-torus and that the core is a universal covering of a locally loop algebra. We also show that there is a one to one correspondence between reduced root systems extended by $\mathbb{Z}$ (which are classified in [Y3, Cor.15], as the class of reduced locally affine root systems) and the cores of LALAs.

The second step of the classification process involves determining a complement D of the core $\mathscr{L}_c$ of a LALA $\mathscr{L}=\mathscr{L}_c\oplus D$. As explained above, we can obtain $D\subset \mathrm{Der}_F\,\mathscr{L}_c\subset \mathrm{Der}_F\,L$ or $D\subset \mathrm{Oder}_F\,\mathscr{L}_c\subset \mathrm{Oder}_F\,L$, where $L=\mathscr{L}_c/Z(\mathscr{L}_c)$ is the centerless core (which is a locally loop algebra).

Now, we need some information about $\mathrm{Der}_F\,L$. Derivations of this type of algebra were studied in [BM], [B], and [NY]. However, the derivations of a locally loop algebra are new. We can use some results from [A1] for the untwisted case since L is a tensor product algebra (see Remark 7.4). However, we need to determine the twisted case. Thus, we propose a new method. Clearly, we need to use the classification of $\mathrm{Der}_F\,\mathfrak{g}$ for a locally finite split simple Lie algebra $\mathfrak{g}$, as described by Neeb in [N1]. Fortunately, we do not need all of the information about $\mathrm{Der}_F\,L$ to classify D. In fact, we only need to know the diagonal derivations of degree m. To explain this, we note that L has double grading, i.e.,

$$L=\bigoplus_{\alpha\in\Delta\cup\{0\}}\ \bigoplus_{k\in\mathbb{Z}} L^k_\alpha,$$

where Δ is a locally finite irreducible root system. The diagonal derivations of degree m denote the space

$$(\mathrm{Der}_F\,L)^m_0 := \{d\in \mathrm{Der}_F\,L \mid d(L^k_\alpha)\subset L^{k+m}_\alpha \text{ for all } \alpha\in\Delta \text{ and } k\in\mathbb{Z}\}.$$

It is crucial to determine the case where $m=0$, i.e., $(\mathrm{Der}_F\,L)^0_0$. Next, $(\mathrm{Der}_F\,L)^m_0$ can be determined easily for the untwisted case. However, for the twisted case, $(\mathrm{Der}_F\,L)^m_0$ is still difficult when m is odd. Finally, using some new techniques (see Lemma 8.8 and Lemma 8.9), the classification of $(\mathrm{Der}_F\,L)^m_0$ is completed in Theorem 8.10.

If we take D as a homogeneous complement of the graded algebra $\mathscr{L}_c$, then D has $\mathbb{Z}$-grading, e.g.,

$$D=\bigoplus_{m\in\mathbb{Z}} D^m,$$

and each D^m can be identified with a subspace of the known space $(\mathrm{Der}_F\,L)^m_0$. Finally, we classify the Lie brackets on D and the concrete brackets are described in Example 6.3.

The remainder of this paper is organized as follows. In Section 2, we define a locally Lie G-torus and we consider a locally Lie 1-torus as a special case. In Section 3, we introduce a locally loop algebra, which is a centerless locally Lie 1-torus. We classify locally Lie 1-tori in general. We prove that a centerless locally Lie 1-torus is uniquely determined by a root system extended by $\mathbb{Z}$, and that a locally Lie 1-torus is a locally loop algebra or a universal covering of a locally loop algebra. In Section 4, we recall the definition of a LEALA and we prove some general properties of a LEALA. In Section 5, we summarize and prove several properties related to LEALAs of nullity 0. In Section 6, we define a LALA. We show that the core of a LALA is a universal covering of a locally loop algebra and we then construct many examples of LALAs. In Sections 7 and 8, we classify untwisted LALAs and twisted LALAs. Finally, we provide our main theorem.

Theorem 1.3. *The examples in Example 6.3 comprise all LALAs.*

In Section 9, we discuss standard LALAs.

The authors thank Karl-Hermann Neeb and Erhard Neher for helpful discussions and suggestions regarding this study.

2. LOCALLY LIE G-TORI

To classify $\mathscr{L}_c$ and $L = \mathscr{L}_c/Z(\mathscr{L}_c)$, we need to study localy Lie G-tori, which are very useful. Let Δ be a locally finite irreducible root system (see [LN1]), and we denote the Cartan integer

$$\frac{2(\mu,\nu)}{(\nu,\nu)}$$

by $\langle\mu,\nu\rangle$ for $\mu,\nu\in\Delta$, while we also let $\langle 0,\nu\rangle := 0$ for all $\nu\in\Delta$. Recall that Δ is called **reduced** if $2\alpha\notin\Delta$ for all $\alpha\in\Delta$. We define the subset

$$\Delta^{\mathrm{red}} := \{\alpha\in\Delta \mid \frac{1}{2}\alpha\notin\Delta\}$$

of Δ, which is a reduced locally finite irreducible root system. Note that $\Delta=\Delta^{\mathrm{red}}$ if Δ is reduced. To simplify the description later, we partition the locally finite irreducible root system Δ according to length. The roots of Δ of minimal length are called **short**. The roots of Δ, which are two times a short root of Δ, are called **extra long**. Finally, the roots of Δ, which are neither short nor extra long, are called **long**. We denote the subsets of the short, long, and extra long roots of Δ by Δ_{sh}, Δ_{lg}, and Δ_{ex}, respectively. Thus,

$$\Delta = \Delta_{\mathrm{sh}}\sqcup\Delta_{\mathrm{lg}}\sqcup\Delta_{\mathrm{ex}}.$$

Clearly, the last two terms in this union may be empty. Indeed,

$$\Delta_{\mathrm{lg}}=\emptyset \quad\Longleftrightarrow\quad \Delta \text{ is a simply laced type or type } \mathrm{BC}_1,$$

and

$$\Delta_{\mathrm{ex}}=\emptyset \quad\Longleftrightarrow\quad \Delta=\Delta^{\mathrm{red}}.$$

Let $G=(G,+,0)$ be an arbitrary abelian group. In general, for a subset S of G, the subgroup generated by S is denoted by $\langle S\rangle$.

Definition 2.1. A Lie algebra $\mathscr{L}$ is called a **locally Lie G-torus of type Δ** if:

(LT1) $\mathscr{L}$ has a decomposition into subspaces

$$\mathscr{L} = \bigoplus_{\mu\in\Delta\cup\{0\},\ g\in G}\mathscr{L}_\mu^g$$

such that $[\mathscr{L}_\mu^g,\mathscr{L}_\nu^h]\subset\mathscr{L}_{\mu+\nu}^{g+h}$ for $\mu,\nu,\mu+\nu\in\Delta\cup\{0\}$ and $g,h\in G$;

(LT2) For every $g \in G$, $\mathscr{L}_0^g = \sum_{\mu\in\Delta,\, h\in G} [\mathscr{L}_\mu^h, \mathscr{L}_{-\mu}^{g-h}]$;

(LT3) For each nonzero $x \in \mathscr{L}_\mu^g$ $(\mu \in \Delta, g \in G)$, an element $y \in \mathscr{L}_{-\mu}^{-g}$ exists such that $t := [x,y] \in \mathscr{L}_0^0$ satisfies $[t,z] = \langle \nu, \mu\rangle z$ for all $z \in \mathscr{L}_\nu^h$ $(\nu \in \Delta\cup\{0\}, h \in G)$;

(LT4) $\dim \mathscr{L}_\mu^g \leq 1$ for $\mu \in \Delta$ and $g \in G$, and $\dim \mathscr{L}_\mu^0 = 1$ if $\mu \in \Delta^{\mathrm{red}}$;

(LT5) $\langle \mathrm{supp}\mathscr{L}\rangle = G$, where $\mathrm{supp}\mathscr{L} = \{g \in G \mid \mathscr{L}_\mu^g \neq 0 \text{ for some } \mu \in \Delta\cup\{0\}\}$.

If Δ is finite, we omit the term 'locally' and simply call it a **Lie G-torus**. Furthermore, if $G \cong \mathbb{Z}^n$, then $\mathscr{L}$ is called a **locally Lie n-torus**, or simply a **locally Lie torus**. We refer to the rank of Δ as the **rank** of $\mathscr{L}$.

Remark 2.2. (i) Condition (LT5) is simply for convenience but if it fails to hold, we may replace G by the subgroup generated by $\mathrm{supp}\mathscr{L}$.

(ii) It follows from (LT1) and (LT3) that $\mathscr{L}$ admits a grading by the root lattice $\langle\Delta\rangle$. Let

$$\mathscr{L}_\lambda := \bigoplus_{g\in G} \mathscr{L}_\lambda^g \tag{6}$$

for $\lambda \in \langle\Delta\rangle$, where $\mathscr{L}_\lambda^g = 0$ if $\lambda \notin \Delta\cup\{0\}$. Then, $\mathscr{L} = \oplus_{\lambda\in\langle\Delta\rangle} \mathscr{L}_\lambda$ and $[\mathscr{L}_\lambda, \mathscr{L}_\mu] \subset \mathscr{L}_{\lambda+\mu}$.

(iii) $\mathscr{L}$ is also graded by the group G, i.e., if

$$\mathscr{L}^g := \bigoplus_{\mu\in\Delta\cup\{0\}} \mathscr{L}_\mu^g, \tag{7}$$

then $\mathscr{L} = \oplus_{g\in G} \mathscr{L}^g$ and $[\mathscr{L}^g, \mathscr{L}^h] \subset \mathscr{L}^{g+h}$. In addition, $\mathrm{supp}\mathscr{L} = \{g \in G \mid \mathscr{L}^g \neq 0\}$.

(iv) From (LT3), for $\mu \in \Delta^{\mathrm{red}}$, we can see that the elements $e_\mu \in \mathscr{L}_\mu^0$, $f_\mu \in \mathscr{L}_{-\mu}^0$, and $\mu^\vee = \mu_{\mathscr{L}}^\vee := [e_\mu, f_\mu]$ exist such that $[\mu^\vee, z] = \langle \nu, \mu\rangle z$ for all $z \in \mathscr{L}_\nu^h$, $\nu \in \Delta$ and $h \in G$. Thus, the elements $e_\mu, f_\mu, \mu^\vee$ determine a canonical basis for a copy of the Lie algebra $\mathrm{sl}_2(F)$. (Note that $\mu^\vee$ is a unique element in $[\mathscr{L}_\mu^0, \mathscr{L}_{-\mu}^0]$ that satisfies the property.) The subalgebra $\mathfrak{g}$ of $\mathscr{L}$ generated by the subspaces $\mathscr{L}_\mu^0$ for $\mu \in \Delta^{\mathrm{red}}$ is a locally finite split simple Lie algebra with the split Cartan subalgebra

$$\mathfrak{h} := \sum_{\mu\in\Delta^{\mathrm{red}}} [\mathscr{L}_\mu^0, \mathscr{L}_{-\mu}^0],$$

and $\mu^\vee$ are the coroots in $\mathfrak{h}$. (We can show this in the same manner as the proof of [MY, Prop.8.3], or see [St, Sec.III]). Note that if Δ is finite, then $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra. Furthermore, Δ^{red} may be replaced by Δ in the definition of $\mathfrak{g}$ and $\mathfrak{h}$ since it can be shown in the same manner described by [Y1, Thm.5.1] that $\mathscr{L}_{2\nu}^0 = 0$ for all $\nu \in \Delta^{\mathrm{red}}$. We say that the pair $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{g}, \mathfrak{h})_{\mathscr{L}}$ is the **grading pair** of $\mathscr{L}$.

(v) A locally Lie G-torus is perfect, and thus it has a universal covering.

(vi) Let $\mathscr{L}$ be a locally Lie G-torus and Z is its center. Then, we can see that $Z \subset \mathscr{L}_0$. In addition, $\mathscr{L}/Z$ is a locally Lie G-torus with a trivial center. In general, a Lie algebra with a trivial center is called **centerless**.

We define the root systems of locally Lie G-tori. Let $\mathscr{L} = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{g\in G} \mathscr{L}_\mu^g$ be a locally Lie G-torus. For each $\mu \in \Delta$, let

$$S_\mu := \{g \in G \mid \mathscr{L}_\mu^g \neq 0\},$$

and we refer to

$$\tilde{\Delta} := \{S_\mu\}_{\mu\in\Delta}$$

as the **root system** of $\mathscr{L}$ (which is called an extension datum in [LN2]). This system fits into the system introduced in [Y1]. Let us state the precise definition. A family of subsets

S_μ of G indexed by Δ, such as $\{S_\mu\}_{\mu\in\Delta}$, is called a **root system extended by** G if

$$\langle \cup_{\mu\in\Delta} S_\mu\rangle = G, \tag{8}$$

$$S_\nu - \langle \nu,\mu\rangle S_\mu \subset S_{\nu-\langle\nu,\mu\rangle\mu} \quad \text{for all } \mu,\nu\in\Delta, \text{ and} \tag{9}$$

$$0\in S_\mu \quad \text{for all } \mu\in\Delta^{\mathrm{red}}. \tag{10}$$

Moreover, $\{S_\mu\}_{\mu\in\Delta}$ is called **reduced** if

$$S_{2\mu}\cap 2S_\mu=\emptyset \quad \text{for all } 2\mu,\mu\in\Delta. \tag{11}$$

In the same manner described in [Y1, Thm 5.1], we can show that the root system $\tilde{\Delta}$ of $\mathscr{L}$ is a reduced root system extended by G, i.e., $\tilde{\Delta}$ satisfies (8), (9), (10), and (11). Moreover,

$$S_\mu = S_\nu \quad \text{if } \mu \text{ and } \nu \text{ are the same length, and}$$

$$S_\nu\subset S_\mu \quad \text{for all } \nu\in\Delta \text{ if } \mu \text{ is a short root.} \tag{12}$$

Finally, if we let

$$S_0 := \{g\in G \mid \mathscr{L}_0^g \neq 0\}, \tag{13}$$

then we obtain

$$S_0 = S_\mu + S_\mu \tag{14}$$

for a short root μ.

Lemma 2.3. *A locally Lie G-torus $\mathscr{L}$ of type Δ is a directed union of Lie G-tori. In particular, $\mathscr{L}=\bigcup_{\Delta'}\mathscr{L}_{\Delta'}$, where Δ' is a finite irreducible full subsystem of Δ that contains a short root and $\mathscr{L}_{\Delta'}$ is the subalgebra of $\mathscr{L}$ generated by $\mathscr{L}_\alpha$ for all $\alpha\in\Delta'$.*

Furthermore, if G is torsion-free, then a locally Lie G-torus $\mathscr{L}$ of type Δ is a directed union of Lie n-tori, where n runs over a certain subset of $\mathbb{N}$. In particular, $\mathscr{L}=\bigcup_{\Delta',G'}\mathscr{L}_{\Delta'}^{G'}$, where G' is a finitely generated subgroup of G and $\mathscr{L}_{\Delta'}^{G'}$ is the subalgebra of $\mathscr{L}$ generated by $\mathscr{L}_\alpha^g$ for all $\alpha\in\Delta'$ and $g\in G'$.

Proof. Since $S=S_\mu$ generates G for a short root μ by (12), then it is easy to check that $\mathscr{L}_{\Delta'}$ is a Lie G-torus. Hence, the statement is true since Δ is a directed union of finite irreducible full subsystems that contain a short root (see [LN2, 3.15 (b) and the proof]). The second statement follows from the fact that G is a directed union of finitely generated subgroups, and the fact that a finitely generated torsion-free abelian group is free. □

Remark 2.4. Let Δ and G be as given in Lemma 2.3. For a locally finite irreducible full subsystem Δ' of Δ, and for a subgroup G' of G, we put $\mathscr{M}=\mathscr{L}_{\Delta'}^{G'}$, which can be defined as given in Lemma 2.3. Then,

$$\mathscr{M}=\bigoplus_{\mu'\in\Delta'\cup\{0\},\ g'\in G'}\mathscr{M}_{\mu'}^{g'},$$

where

$$\mathscr{M}_{\mu'}^{g'}=\mathscr{M}\cap\mathscr{L}_{\mu'}^{g'} \quad (\mu'\in\Delta'\cup\{0\},\ g'\in G').$$

In fact, we obtain

$$\mathscr{M}_{\mu'}^{g'}=\mathscr{L}_{\mu'}^{g'} \quad (\mu'\in\Delta',\ g'\in G'),$$

and since $\mathscr{M}$ is generated by $\mathscr{L}_\mu^g$ for all $\mu\in\Delta'$ and $g\in G'$, then for $g'\in G'$, we have

$$\mathscr{M}_0^{g'}=\sum_{\mu'\in\Delta',\ h'\in G'}[\mathscr{L}_{\mu'}^{h'},\mathscr{L}_{-\mu'}^{g'-h'}]=\sum_{\mu'\in\Delta',\ h'\in G'}[\mathscr{M}_{\mu'}^{h'},\mathscr{M}_{-\mu'}^{g'-h'}].$$

Thus, we can check conditions (LT1) – (LT5) for $\mathscr{M}$, which implies that $\mathscr{M}$ is a locally Lie G'-torus.

3. LOCALLY LOOP ALGEBRAS

For any index set $\mathfrak{I}$, in the introduction, we defined

$$M_{\mathfrak{I}}(F) = \left\{ (a_{ij})_{i,j\in\mathfrak{I}} \,\middle|\, a_{ij} \in F \right\} \approx \mathrm{Map}(\mathfrak{I}\times\mathfrak{I}, F),$$

as the set of all matrices of size $\mathfrak{I}$, which is naturally a vector space over F. Let $\mathrm{gl}_{\mathfrak{I}}(F)$ be the subspace of $M_{\mathfrak{I}}(F)$ that comprises matrices with only a finite number of nonzero entries. Then, $\mathrm{gl}_{\mathfrak{I}}(F)$ is an associative algebra and a Lie algebra with the usual commutator bracket. Furthermore, we can define the trace of a matrix in $\mathrm{gl}_{\mathfrak{I}}(F)$, and the subalgebra of $\mathrm{gl}_{\mathfrak{I}}(F)$ that comprises trace 0 matrices is denoted by $\mathrm{sl}_{\mathfrak{I}}(F)$, as follows.

$$\mathrm{sl}_{\mathfrak{I}}(F) = \{x \in \mathrm{gl}_{\mathfrak{I}}(F) \mid \mathrm{tr}(x) = 0\}$$

We note that $M_{\mathfrak{I}}(F)$ is not an algebra if $\mathfrak{I}$ is infinite, but

$$M^{\mathrm{fin}}_{\mathfrak{I}}(F) := \{x \in M_{\mathfrak{I}}(F) \mid \text{each row and column of } x \text{ have only finitely many nonzero entries}\}$$

is an associative algebra with the identity matrix $\iota = \iota_{\mathfrak{I}}$, and a Lie algebra with the commutator bracket. In fact, this gives the Lie algebra of derivations of $\mathrm{sl}_{\mathfrak{I}}(F)$, as described by Neeb [N1]. In particular, we have

$$[M^{\mathrm{fin}}_{\mathfrak{I}}(F), \mathrm{sl}_{\mathfrak{I}}(F)] \subset \mathrm{sl}_{\mathfrak{I}}(F) \quad \text{and} \quad \mathrm{Der}_F(\mathrm{sl}_{\mathfrak{I}}(F)) \simeq \mathrm{ad}(M^{\mathrm{fin}}_{\mathfrak{I}}(F)).$$

As a result, we note that there are 14 types of locally loop algebras, i.e., we obtain:

$$\mathrm{A}^{(1)}_{\mathfrak{I}},\ \mathrm{B}^{(1)}_{\mathfrak{I}},\ \mathrm{C}^{(1)}_{\mathfrak{I}},\ \mathrm{D}^{(1)}_{\mathfrak{I}},\ \mathrm{B}^{(2)}_{\mathfrak{I}},\ \mathrm{C}^{(2)}_{\mathfrak{I}},\ \mathrm{BC}^{(2)}_{\mathfrak{I}},\ \mathrm{E}^{(1)}_{6},\ \mathrm{E}^{(1)}_{7},\ \mathrm{E}^{(1)}_{8},\ \mathrm{F}^{(1)}_{4},\ \mathrm{G}^{(1)}_{2},\ \mathrm{F}^{(2)}_{4},\ \mathrm{G}^{(3)}_{2},$$

where we mainly assume that $\mathfrak{I}$ is infinite since we already know the affine Lie algebras.

The locally finite split simple Lie algebra of type $\mathrm{X}_{\mathfrak{I}}$ is defined as a subalgebra of $\mathrm{sl}_{\mathfrak{I}}(F)$, $\mathrm{sl}_{2\mathfrak{I}+1}(F)$ or $\mathrm{sl}_{2\mathfrak{I}}(F)$ as follows:
Type $\mathrm{A}_{\mathfrak{I}}$: $\mathrm{sl}_{\mathfrak{I}}(F)$;
Type $\mathrm{B}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}+1}(F) = \{x \in \mathrm{sl}_{2\mathfrak{I}+1}(F) \mid sx = -x^T s\}$;
Type $\mathrm{C}_{\mathfrak{I}}$: $\mathrm{sp}_{2\mathfrak{I}}(F) = \{x \in \mathrm{sl}_{2\mathfrak{I}}(F) \mid sx = -x^T s\}$;
Type $\mathrm{D}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}}(F) = \{x \in \mathrm{sl}_{2\mathfrak{I}}(F) \mid sx = -x^T s\}$,
where $\mathfrak{I}$ is assumed to be infinite, x^T is the transpose of x, and

$$s = \begin{pmatrix} 0 & \iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } \mathrm{B}_{\mathfrak{I}},\ s = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix} \text{ for } \mathrm{C}_{\mathfrak{I}},\ \text{ or } \ s = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix} \text{ for } \mathrm{D}_{\mathfrak{I}}. \tag{15}$$

Note that $s \in M^{\mathrm{fin}}_{2\mathfrak{I}+1}(F)$ for $\mathrm{B}_{\mathfrak{I}}$ and $s \in M^{\mathrm{fin}}_{2\mathfrak{I}}(F)$ for $\mathrm{C}_{\mathfrak{I}}$ or $\mathrm{D}_{\mathfrak{I}}$, and that $s^2 = \iota_{2\mathfrak{I}+1}$ for $\mathrm{B}_{\mathfrak{I}}$, $s^2 = -\iota_{2\mathfrak{I}}$ for $\mathrm{C}_{\mathfrak{I}}$ and $s^2 = \iota_{2\mathfrak{I}}$ for $\mathrm{D}_{\mathfrak{I}}$. In addition, $\mathrm{B}_{\mathfrak{I}}$, $\mathrm{C}_{\mathfrak{I}}$, or $\mathrm{D}_{\mathfrak{I}}$ is the fixed algebra of $\mathrm{sl}_{2\mathfrak{I}+1}(F)$ or $\mathrm{sl}_{2\mathfrak{I}}(F)$ by an automorphism σ, which are defined as

$$\sigma(x) = -sx^T s \ \text{ for } \mathrm{B}_{\mathfrak{I}} \text{ or } \mathrm{D}_{\mathfrak{I}}, \text{ and } \sigma(x) = sx^T s \ \text{ for } \mathrm{C}_{\mathfrak{I}}. \tag{16}$$

In [NS], Neeb and Stumme showed that these algebras comprise all of the infinite-dimensional locally finite split simple Lie algebras. In addition, they are considered to be locally Lie 0-tori (in the case where $G = \{0\}$). Moreover, since locally finite split simple Lie algebras are centrally closed (see [NS]), we have the equality {infinite-dimensional locally Lie 0-tori} ={infinite-dimensional locally finite split simple Lie algebras}. We note that Lie 0-tori are exact finite-dimensional split simple Lie algebras. In the present study, we are interested in the class of locally Lie 1-tori.

Let $F[t^{\pm 1}]$ be the algebra of Laurent polynomials over F. We call one of the following four Lie algebras an **untwisted locally loop algebra**:

(1) Type $\mathrm{A}_{\mathfrak{I}}^{(1)}$: $\mathrm{sl}_{\mathfrak{I}}(F)\otimes F[t^{\pm 1}]$;

(2) Type $\mathrm{B}_{\mathfrak{I}}^{(1)}$: $\mathrm{o}_{2\mathfrak{I}+1}(F)\otimes F[t^{\pm 1}]$;

(3) Type $\mathrm{C}_{\mathfrak{I}}^{(1)}$: $\mathrm{sp}_{2\mathfrak{I}}(F)\otimes F[t^{\pm 1}]$;

(4) Type $\mathrm{D}_{\mathfrak{I}}^{(1)}$: $\mathrm{o}_{2\mathfrak{I}}(F)\otimes F[t^{\pm 1}]$.

(In addition, it is called an untwisted loop algebras if $\mathfrak{I}$ is finite.) Each of the following three Lie algebras is called a **twisted locally loop algebra**:

(5) Type $\mathrm{B}_{\mathfrak{I}}^{(2)}$: $(\mathrm{o}_{2\mathfrak{I}+1}(F)\otimes F[t^{\pm 2}]\oplus(\mathfrak{s}\otimes tF[t^{\pm 2}])$,

where $\mathfrak{s}=F^{(2\mathfrak{I}+1)}$ is the natural $\mathrm{o}_{2\mathfrak{I}+1}(F)$-module;

(6) Type $\mathrm{C}_{\mathfrak{I}}^{(2)}$: $(\mathrm{sp}_{2\mathfrak{I}}(F)\otimes F[t^{\pm 2}])\oplus(\mathfrak{s}\otimes tF[t^{\pm 2}])$,

where $\mathfrak{s}=\{x\in \mathrm{sl}_{2\mathfrak{I}}(F)\mid sx=x^Ts\}$;

(7) Type $\mathrm{BC}_{\mathfrak{I}}^{(2)}$: $(\mathrm{o}_{2\mathfrak{I}+1}(F)\otimes F[t^{\pm 2}])\oplus(\mathfrak{s}\otimes tF[t^{\pm 2}])$,

where $\mathfrak{s}=\{x\in \mathrm{sl}_{2\mathfrak{I}+1}(F)\mid sx=x^Ts\}$. (In addition, it is called a twisted loop algebra if $\mathfrak{I}$ is finite.) Note that $\mathrm{sl}_{2\mathfrak{I}}(F)=\mathrm{sp}_{2\mathfrak{I}}(F)\oplus\mathfrak{s}$ for $\mathrm{C}_{\mathfrak{I}}^{(2)}$ and $\mathrm{sl}_{2\mathfrak{I}+1}(F)=\mathrm{o}_{2\mathfrak{I}+1}(F)\oplus\mathfrak{s}$ for $\mathrm{BC}_{\mathfrak{I}}^{(2)}$.

The Lie bracket of each untwisted type is natural, i.e., $[x\otimes t^m, y\otimes t^n]=[x,y]\otimes t^{m+n}$. The Lie bracket of type $\mathrm{C}_{\mathfrak{I}}^{(2)}$ or $\mathrm{BC}_{\mathfrak{I}}^{(2)}$ is also natural, and we have

$$[\mathrm{sp}_{2\mathfrak{I}}(F),\mathfrak{s}]\subset\mathfrak{s}\quad\text{and}\quad[\mathfrak{s},\mathfrak{s}]\subset\mathrm{sp}_{2\mathfrak{I}}(F)\quad\text{for } \mathrm{C}_{\mathfrak{I}}^{(2)},$$
$$[\mathrm{o}_{2\mathfrak{I}+1}(F),\mathfrak{s}]\subset\mathfrak{s}\quad\text{and}\quad[\mathfrak{s},\mathfrak{s}]\subset\mathrm{o}_{2\mathfrak{I}+1}(F)\quad\text{for } \mathrm{BC}_{\mathfrak{I}}^{(2)}.$$

Note that $\mathrm{C}_{\mathfrak{I}}^{(2)}$ or $\mathrm{BC}_{\mathfrak{I}}^{(2)}$ is the fixed subalgebra of $\mathrm{sl}_{2\mathfrak{I}}(F)\otimes F[t^{\pm 1}]$ or $\mathrm{sl}_{2\mathfrak{I}+1}(F)\otimes F[t^{\pm 1}]$ by the automorphism $\hat{\sigma}$, which is defined as

$$\hat{\sigma}(x\otimes t^m):=(-1)^m\sigma(x)\otimes t^m \tag{17}$$

(see (16)). This construction is called a **twisting construction** by an automorphism σ.

For $\mathrm{B}_{\mathfrak{I}}^{(2)}$, we have $\mathrm{o}_{2\mathfrak{I}+1}(F)\mathfrak{s}\subset\mathfrak{s}$, and thus we define the bracket of $\mathrm{o}_{2\mathfrak{I}+1}(F)$ and $\mathfrak{s}$ by the natural action, i.e., $[x,v]=xv=-[v,x]$ for $x\in\mathrm{o}_{2\mathfrak{I}+1}(F)$ and $v\in\mathfrak{s}$. We define a bracket on $\mathfrak{s}$ such that $[\mathfrak{s},\mathfrak{s}]\subset\mathrm{o}_{2\mathfrak{I}+1}(F)$ as follows. First, let $(\cdot,\cdot)$ be the bilinear form on $\mathfrak{s}$ determined by s. Then, there is a natural identification

$$\mathrm{o}_{2\mathfrak{I}+1}(F)=D_{\mathfrak{s},\mathfrak{s}}:=\mathrm{span}_F\{D_{v,v'}\mid v,v'\in\mathfrak{s}\},$$

where $D_{v,v'}\in\mathrm{End}(\mathfrak{s})$ is defined by $D_{v,v'}(v'')=(v',v'')v-(v,v'')v'$ for $v''\in\mathfrak{s}$. Thus, we define $[v,v']:=D_{v,v'}$. Note that $[v',v]=-[v,v']$. It is easy to check that the bracket

$$\begin{aligned}&[x\otimes t^{2m}+v\otimes t^{2m'+1},x'\otimes t^{2n}+v'\otimes t^{2n'+1}]\\=&[x,x']\otimes t^{2(m+n)}+D_{v,v'}\otimes t^{2(m'+n'+1)}+xv'\otimes t^{2(m+n')+1}-x'v\otimes t^{2(m'+n)+1}\end{aligned}$$

defines a Lie bracket for $m,m',n,n'\in\mathbb{Z}$.

There is a twisting construction for $\mathrm{B}_{\mathfrak{I}}^{(2)}$ (see [N2]), which we discuss in Section 7, but we also consider that the simple description of $\mathrm{B}_{\mathfrak{I}}^{(2)}$ is important for developing the theory of locally Lie tori.

Remark 3.1. We often omit the term 'untwisted' or 'twisted' and we simply refer to a locally loop algebra. In addition, a locally loop algebra can be simply called a loop algebra in more general theory. For example, $A\otimes F[t^{\pm 1}]$ for any algebra A is called a loop algebra of A. However, we use the term 'locally' in this study to distinguish the familiar loop algebras in Kac-Moody theory.

We can easily check that

all locally loop algebras are centerless locally Lie 1-tori.

For example, let Δ be the root system of type $\mathrm{BC}_{\mathfrak{I}}$, and we put $\mathfrak{g} = \mathfrak{o}_{2\mathfrak{I}+1}(F)$ and $\mathfrak{s} \subset \mathrm{sl}_{2\mathfrak{I}+1}(F)$, as defined above. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ that comprises diagonal matrices. Then, $\mathfrak{h}$ decomposes $\mathfrak{g}$ into the root spaces, such as $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta^{\mathrm{red}}} \mathfrak{g}_\mu$, and $\mathfrak{s}$ into the weight spaces, such as $\mathfrak{s} = \bigoplus_{\mu\in\Delta\cup\{0\}} \mathfrak{s}_\mu$, where Δ^{red} is of type $\mathrm{B}_{\mathfrak{I}}$. Therefore, the twisted locally loop algebra $\mathscr{L} := (\mathfrak{g}\otimes F[t^{\pm 2}]) \oplus (\mathfrak{s}\otimes tF[t^{\pm 2}])$ of type $\mathrm{BC}_{\mathfrak{I}}^{(2)}$ is decomposed into

$$\bigoplus_{m\in\mathbb{Z}} \left((\mathfrak{h}\otimes Ft^{2m}) \oplus \bigoplus_{\mu\in\Delta^{\mathrm{red}}} (\mathfrak{g}_\mu \otimes Ft^{2m}) \oplus \bigoplus_{\mu\in\Delta\cup\{0\}} (\mathfrak{s}_\mu \otimes Ft^{2m+1}) \right).$$

This gives a natural double grading by the groups $\langle\Delta\rangle$ and $\mathbb{Z}$, and we can check the axioms of a locally Lie torus. In addition, the center is contained in $\mathscr{L}_0 = \mathfrak{h}\otimes F[t^{\pm 2}]$, and thus $\mathscr{L}$ is a centerless locally Lie 1-torus. The grading subalgebra is equal to $\mathfrak{g} = \mathfrak{o}_{2\mathfrak{I}+1}(F)$. We refer to the $\mathfrak{g}$-module $\mathfrak{s}$ as the **grading module**.

The following lemma was proved for the base field $\mathbb{C}$ in [ABGP], but it also works for our base field F. We use the notation

$$\tilde{\Delta} := \{S_\mu\}_{\mu\in\Delta}$$

(defined in Section 2) for the case where $\langle \cup_{\mu\in\Delta} S_\mu \rangle = \mathbb{Z}$ (the root system Δ extended by $\mathbb{Z}$).

Lemma 3.2. *Let Δ be a* **finite** *irreducible root system. Let $\mathscr{L} = \oplus_{\mu\in\Delta\cup\{0\},\, m\in\mathbb{Z}} \mathscr{L}_\mu^m$ and $\mathscr{M} = \oplus_{\mu\in\Delta\cup\{0\},\, m\in\mathbb{Z}} \mathscr{M}_\mu^m$ be centerless Lie 1-tori, which have the same root system $\tilde{\Delta}$ extended by $\mathbb{Z}$. Then, an isomorphism $\varphi : \mathscr{L} \longrightarrow \mathscr{M}$ exists such that*

$$\varphi(\mu_{\mathscr{L}}^\vee) = \mu_{\mathscr{M}}^\vee \quad \text{and} \quad \varphi(\mathscr{L}_\mu^m) = \mathscr{M}_\mu^m \quad \text{for all } \mu\in\Delta \text{ and } m\in\mathbb{Z}. \tag{18}$$

Remark 3.3. If $\mathscr{L}$ is a loop algebra, then $\tilde{\Delta}$ determines $\mathscr{L}$, i.e., there is a one to one correspondence between loop algebras and root systems extended by $\mathbb{Z}$ (see [Y3]). In particular, $\tilde{\Delta}$ determines whether the loop algebra is untwisted or twisted.

Proof. Let $0 \neq e_{\pm\mu} \in \mathscr{L}_{\pm\mu}^0$ and $\mu^\vee$ be an sl_2-triple for $\mu\in\Pi$, where Π is a root base of Δ and let $0 \neq x_{\pm\nu} \in \mathscr{L}_{\pm\nu}^{\mp 1}$ and $\nu^\vee$ be an sl_2-triple, where $\nu\in\Delta$ is the highest long (or short) root relative to Π (depending on the type $\tilde{\Delta}$). Then, the set

$$\{e_{\pm\mu},\ \mu^\vee,\ x_{\pm\nu},\ \nu^\vee \mid \mu\in\Pi\}$$

satisfies the Serre relations. Hence, using the Gabber-Kac Theorem (e.g., see [MP, Thm 4, p.381]), a homomorphism ψ exists from the derived affine Lie algebra A (which is a 1-dimensional central extension of a loop algebra), which is determined by Δ and ν (or $\tilde{\Delta}$) into $\mathscr{L}$. Let

$$A = \bigoplus_{\mu\in\Delta\cup\{0\},\, m\in\mathbb{Z}} A_\mu^m$$

be the loop realization of A (which could be twisted) viewed as a Lie 1-torus such that $\psi(A_{\pm\mu}^0) = Fe_{\pm\mu}$ and $\psi(A_{\pm\nu}^{\mp 1}) = Fx_{\pm\nu}$. Then, ψ is graded relative to the $\mathbb{Z}$-grading but also to the double grading $\langle\Delta\rangle\times\mathbb{Z}$. Note that a centerless Lie torus is $\mathbb{Z}$-graded simple (see [Y1, Lem.4.4]). Thus, the nontrivial $\mathbb{Z}$-graded ideal of A is exactly the 1-dimensional center Fc, and the image of ψ contains $\cup_{\mu\in\Delta}\mathscr{L}_\mu = \cup_{\mu\in\Delta}\oplus_{m\in\mathbb{Z}}\mathscr{L}_\mu^m$. Therefore, ψ is onto since $\mathscr{L}$ is generated by $\cup_{\mu\in\Delta}\mathscr{L}_\mu$. Thus, the induced graded isomorphism from the loop algebra A/Fc onto $\mathscr{L}$ exists. Similarly, we obtain a graded isomorphism from the loop algebra A/Fc onto $\mathscr{M}$. Based on these isomorphisms, we obtain the graded isomorphism φ described above. □

Thus, a centerless Lie 1-torus is isomorphic to a loop algebra, and a Lie 1-torus with nontrivial center is isomorphic to a derived affine Lie algebra, which has a 1-dimensional center.

For a Lie 1-torus $\mathscr{L} = \oplus_{\mu\in\Delta\cup\{0\},\, m\in\mathbb{Z}}\mathscr{L}_\mu^m$, we have

$$\dim\mathscr{L}_\mu^m \neq 0 \text{ (so } \dim\mathscr{L}_\mu^m = 1\text{) for all } \mu\in\Delta_{sh} \text{ and } m\in\mathbb{Z} \text{ (so } S_\mu = \mathbb{Z}\text{), and} \tag{19}$$

$$\text{the center of } \mathscr{L} \text{ is equal to } [\mathscr{L}_0^m, \mathscr{L}_0^{-m}] \text{ for any } 0\neq m\in\mathbb{Z}. \tag{20}$$

This can be seen easily from the loop realization. Furthermore, we have

$$\dim \sum_{m\in\mathbb{Z}} [\mathscr{L}_\mu^m, \mathscr{L}_{-\mu}^{-m}] = \begin{cases} 1 & \text{if } \mathscr{L} \text{ is loop} \\ 2 & \text{if } \mathscr{L} \text{ is derived affine} \end{cases} \tag{21}$$

since

$$\sum_{m\in\mathbb{Z}} [\mathscr{L}_\mu^m, \mathscr{L}_{-\mu}^{-m}] = \begin{cases} F\mu^\vee & \text{if } \mathscr{L} \text{ is loop} \\ F\mu^\vee + Fc & \text{if } \mathscr{L} \text{ is derived affine} \end{cases}$$

for $\mu\in\Delta$ and a nontrivial central element c.

Lemma 3.4. *The center of a locally Lie 1-torus is at most 1-dimensional. In particular, for a locally Lie 1-torus* $\mathscr{L} = \oplus_{\mu\in\Delta\cup\{0\},\, m\in\mathbb{Z}}\mathscr{L}_\mu^m$,

$$\mathscr{L} \text{ has a 1-dimensional center} \iff \mathscr{L} \text{ is a directed union of derived affine Lie algebras,}$$

and

$$\mathscr{L} \text{ is centerless} \iff \mathscr{L} \text{ is a directed union of loop algebras}$$

in the following sense:

$$\mathscr{L} = \bigcup_{\Delta'\subset\Delta} \mathscr{L}_{\Delta'},$$

where Δ' is a finite irreducible full subsystem of Δ and $\mathscr{L}_{\Delta'}$ is the homogeneous subalgebra of $\mathscr{L}$ generated by $\mathscr{L}_\mu$ for $\mu\in\Delta'$, and $\mathscr{L}_{\Delta'}$ is a derived affine Lie algebra if the center of $\mathscr{L}$ is 1-dimensional and a loop algebra if $\mathscr{L}$ is centerless.

In particular, the properties (19)*,* (20)*, and* (21) *given above hold in a locally Lie* 1*-torus.*

Proof. Most of the statements follow from Lemma 2.3. In fact, Lie 1-tori are either derived affine Lie algebras or loop algebras, and thus $\mathscr{L}$ is a directed union of derived affine Lie algebras or loop algebras. Considering the loop realization of a derived affine Lie algebra, we find (19).

Suppose that C is a 2-dimensional subalgebra contained in the center. Then, a derived affine Lie algebra or a loop algebra exists that contains C. However, this is impossible because their centers have to be 1-dimensional or zero.

Now, we need to show that derived affine Lie algebras and loop algebras cannot appear simultaneously. If this is case, e.g., $\mathscr{L}'$ is a derived affine subalgebra and $\mathscr{L}''$ is a loop subalgebra, then a derived affine or a loop algebra exists that contains both $\mathscr{L}'$ and $\mathscr{L}''$ as graded subalgebras. Suppose that $\mathscr{L}'$ and $\mathscr{L}''$ are contained in $\mathscr{L}'''$ for a loop algebra $\mathscr{L}'''$. However, this is impossible because of property (20) above. Thus, suppose that $\mathscr{L}'$ and

$\mathscr{L}''$ are contained in $\mathscr{L}'''$ for a derived affine Lie algebra $\mathscr{L}'''$. Then, this is also impossible because of property (21) above. Thus, a locally Lie 1-torus is either a directed union of derived affine Lie algebras, such as $\mathscr{L}_{da}$, or a directed union of loop algebras, such as $\mathscr{L}_{lo}$. It is now clear that the center of $\mathscr{L}_{lo}$ is zero. To show the 1-dimensionality of the center of $\mathscr{L}_{da}$, let C' ($\neq 0$) be a finite dimensional central subspace of a derived affine subalgebra of $\mathscr{L}_{da}$. For any $\mu \in \Delta$ and $m \in \mathbb{Z}$, a derived affine subalgebra M exists that contains $\mathscr{L}_\mu^m$ and C'. Considering the loop realization of M, we find that C' is the 1-dimensional center of M and, in particular, C' is the 1-dimensional center of $\mathscr{L}_{da}$.

Finally, let $\mathscr{L}$ be a locally Lie 1-torus. Then, (21) is clear. To show (20), let $Z := [\mathscr{L}_0^k, \mathscr{L}_0^{-k}]$ for $0 \neq k \in \mathbb{Z}$. For any $z \in Z$, $\mu \in \Delta$, and $m \in \mathbb{Z}$, a derived affine subalgebra or a loop subalgebra exists that contains z and $\mathscr{L}_\mu^m$, and z is in the center of the subalgebra (by (20) for a Lie 1-torus as given above). Hence $[z, \mathscr{L}_\mu^m] = 0$ for all $\mu \in \Delta$ and $m \in \mathbb{Z}$. Therefore, Z is contained in the center of $\mathscr{L}$. Thus, $Z = 0$ or $\dim Z = 1$. If $Z = 0$, then a loop subalgebra exists, and thus $\mathscr{L} = \mathscr{L}_{lo}$. Hence, $Z = 0$ is the center of $\mathscr{L}$. If $\dim Z = 1$, then Z is the center of $\mathscr{L}$ since the center of $\mathscr{L}$ is at most 1-dimensional. □

For any two elements $x \otimes t^m$ and $y \otimes t^n$, in each locally loop algebra $\mathscr{L}$, we define the new bracket on a 1-dimensional central extension

$$\tilde{\mathscr{L}} := \mathscr{L} \oplus Fc$$

by

$$[x \otimes t^m, y \otimes t^n] := [x, y] \otimes t^{m+n} + m(x, y)\delta_{m+n,0}c, \tag{22}$$

where (x, y) is the trace form $\operatorname{tr}(xy)$, or for type $B_3^{(2)}$, the direct sum of the trace form and the bilinear form on $\mathfrak{s}$ is determined by the symmetric matrix s given above. Indeed, this gives a central extension since $\mathscr{L}$ is a directed union of loop algebras and $\tilde{\mathscr{L}}$ is a derived LALA, i.e., a 1-dimensional central extension of a loop algebra.

Lemma 3.5. *A universal covering of a locally loop algebra is given by* (22).

Proof. Suppose that $\hat{\mathscr{L}}$ is a universal covering of a locally loop algebra $\mathscr{L}$. We know that $\dim_F Z(\hat{\mathscr{L}}) \geq 1$ since $\tilde{\mathscr{L}}$ is a covering. Therefore, if $\dim Z(\hat{\mathscr{L}}) > 1$, then a covering $\mathscr{L} \oplus Fc_1 \oplus Fc_2$ of $\mathscr{L}$ exists. Let $x_1, y_1, \ldots, x_m, y_m, u_1, v_1, \ldots, u_n, v_n \in \mathscr{L}$ be such that $\sum_{i=1}^m [x_i, y_i] = c_1$ and $\sum_{i=1}^n [u_i, v_i] = c_2$. Let $\mathscr{L}'$ be a loop subalgebra of $\mathscr{L}$ that contains x_i, y_i, u_j, v_j for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then, $\mathscr{L}' \oplus Fc_1 \oplus Fc_2$ is perfect, and thus this is a covering of $\mathscr{L}'$. However, a universal covering of a loop algebra has a 1-dimensional center, which is a contradiction. Hence, $\dim Z(\hat{\mathscr{L}}) = 1$. However, it is then clear that $\hat{\mathscr{L}} \cong \tilde{\mathscr{L}}$ since the unique morphism from $\hat{\mathscr{L}}$ onto $\tilde{\mathscr{L}}$ has to be one to one. □

Remark 3.6. By Lemma 3.4, a locally Lie 1-torus has at most a 1-dimensional center. Thus, if we show that $\hat{\mathscr{L}}$ is a locally Lie 1-torus, then we also obtain a proof of Lemma 3.5. In fact, Neher showed that a universal covering of a locally Lie torus is a locally Lie torus in general (see [Ne3] and [NeS]).

Now, we classify locally Lie 1-tori. The method we use is derived from [NS]. In particular, we show that there is only one locally Lie 1-torus for each reduced root system extended by $\mathbb{Z}$. The root systems extended by $\mathbb{Z}$ were classified in [Y3, Cor.15] as the class of locally affine root systems (more general results are given in [LN2]). The following is a list of all the reduced root systems extended by $\mathbb{Z}$ of infinite rank:

$$\begin{aligned}
&\mathrm{A}_{\mathfrak{J}}\times\mathbb{Z},\quad \mathrm{B}_{\mathfrak{J}}\times\mathbb{Z},\quad \mathrm{C}_{\mathfrak{J}}\times\mathbb{Z},\quad \mathrm{D}_{\mathfrak{J}}\times\mathbb{Z},\\
&\big((\mathrm{B}_{\mathfrak{J}})_{\mathrm{sh}}\times\mathbb{Z}\big)\sqcup\big((\mathrm{B}_{\mathfrak{J}})_{\mathrm{lg}}\times 2\mathbb{Z}\big),\quad \big((\mathrm{C}_{\mathfrak{J}})_{\mathrm{sh}}\times\mathbb{Z}\big)\sqcup\big((\mathrm{C}_{\mathfrak{J}})_{\mathrm{lg}}\times 2\mathbb{Z}\big),\\
&\Big(\big((\mathrm{BC}_{\mathfrak{J}})_{\mathrm{sh}}\sqcup(\mathrm{BC}_{\mathfrak{J}})_{\mathrm{lg}}\big)\times\mathbb{Z}\Big)\sqcup\big((\mathrm{BC}_{\mathfrak{J}})_{\mathrm{ex}}\times(2\mathbb{Z}+1)\big),
\end{aligned}$$

where we write $\sqcup_{\mu\in\Delta}(\mu\times S_\mu)$ for $\{S_\mu\}_{\mu\in\Delta}$, and for a subset Δ' of Δ, if all S_μ's for $\mu\in\Delta'$ are the same set S, we write $\Delta'\times S$ instead of $\sqcup_{\mu\in\Delta'}(\mu\times S_\mu)$. Furthermore, we simply use a type instead of writing Δ, e.g., $\mathrm{A}_{\mathfrak{J}}$ for Δ of type $\mathrm{A}_{\mathfrak{J}}$.

We can see that these seven systems are the exact root systems of the locally loop algebras introduced above, and thus we label each system by

$$\mathrm{A}_{\mathfrak{J}}^{(1)},\quad \mathrm{B}_{\mathfrak{J}}^{(1)},\quad \mathrm{C}_{\mathfrak{J}}^{(1)},\quad \mathrm{D}_{\mathfrak{J}}^{(1)},\quad \mathrm{B}_{\mathfrak{J}}^{(2)},\quad \mathrm{C}_{\mathfrak{J}}^{(2)},\quad \mathrm{BC}_{\mathfrak{J}}^{(2)}.$$

We also use the label for the root system as the **type** of a locally Lie 1-torus. We efer to the first four types as **untwisted** and the last three types as **twisted**. Note that

$$\text{all } S_\mu\text{'s are } \mathbb{Z} \text{ for the untwisted type, and in general,}$$

$$S_\mu=\mathbb{Z} \text{ for a short root } \mu. \tag{23}$$

First, we provide the following lemma when Δ **is finite**. Suppose that $\Pi\subset\Delta$ is an integral base, i.e., $\Delta\subset\langle\Pi\rangle$, and Π is linearly independent in the vector space that defines Δ, where $\langle\Pi\rangle$ is the additive subgroup generated by Π, i.e., $\langle\Pi\rangle$ is the $\mathbb{Z}$-span of Π. Note that

$$\Pi\subset\Delta^{\mathrm{red}}. \tag{24}$$

Lemma 3.7. *Let $\mathscr{L}=\oplus_{\mu\in\Delta\cup\{0\},m\in\mathbb{Z}}\mathscr{L}_\mu^m$ and $\mathscr{M}=\oplus_{\mu\in\Delta\cup\{0\},m\in\mathbb{Z}}\mathscr{M}_\mu^m$ be centerless Lie 1-tori of the same type $\tilde{\Delta}$, where Δ* **is finite**. *Let Π be an integral base of Δ that contains a fixed short root $\nu\in\Delta$ if $\tilde{\Delta}$ is of the untwisted type or a fixed short root $\nu\in\Delta$ if $\tilde{\Delta}$ is of the twisted type. Let $0\neq x_\mu\in\mathscr{L}_\mu^0$ and $0\neq y_\mu\in\mathscr{M}_\mu^0$ for each $\mu\in\Pi$ (see* (24)*). Furthermore, let $0\neq x\in\mathscr{L}_\nu^1$ and $0\neq y\in\mathscr{M}_\nu^1$ (see* (23)*).*

Then, a unique isomorphism ψ from $\mathscr{L}$ onto $\mathscr{M}$ exists such that $\psi(x)=y$, $\psi(\mu_{\mathscr{L}}^\vee)=\mu_{\mathscr{M}}^\vee$ and $\psi(x_\mu)=y_\mu$ for all $\mu\in\Pi$.

Proof. By (18), an isomorphism $\varphi:\mathscr{L}\longrightarrow\mathscr{M}$ exists such that $\varphi(\mu_{\mathscr{L}}^\vee)=\mu_{\mathscr{M}}^\vee$ and $\varphi(\mathscr{L}_\mu^m)=\mathscr{M}_\mu^m$ for all $\mu\in\Delta$ and $m\in\mathbb{Z}$. Hence, we have $y=a\varphi(x)$ and $y_\mu=a_\mu\varphi(x_\mu)=$ for some a and $a_\mu\in F^\times$. Let $f:\langle\Pi\rangle_{\mathbb{Z}}\times\mathbb{Z}\longrightarrow F^\times$ be the group homomorphism of the abelian groups defined by $f(\mu,0)=a_\mu$ and $f(0,1)=a$. Let D_f be the diagonal linear automorphism on $\mathscr{M}$ defined by $D_f(y)=f(\mu,m)y$ for $y\in\mathscr{M}_\mu^m$. Then, D_f is an automorhism of Lie algebras. Indeed, $D_f([y,y'])=f(\mu+\mu',m+m')[y,y']=f((\mu,m)+(\mu',m'))[y,y']=f(\mu,m)f(\mu',m')[y,y']=[f(\mu,m)y,f(\mu',m')y']=[D_f(y),D_f(y')]$ for $y\in\mathscr{M}_\mu^m$ and $y'\in\mathscr{M}_{\mu'}^{m'}$. Hence, $\psi:=D_f^{-1}\circ\varphi$ is the required isomorphism.

For the uniqueness, we first note that this isomorphism is unique on $\mathscr{L}_{-\nu}^{-1}$ and $\mathscr{L}_{-\mu}^0$ for all $\mu\in\Pi$ since $[\mathscr{L}_\nu^1,\mathscr{L}_{-\nu}^{-1}]=F\nu^\vee$ (since $\mathscr{L}$ is centerless) and $[\mathscr{L}_\mu^0,\mathscr{L}_{-\mu}^0]=F\mu^\vee$. Thus, it is sufficient to show that $\mathscr{L}$ is generated by $\mathscr{L}_\nu^1$, $\mathscr{L}_{-\nu}^{-1}$, and $\mathscr{L}_{\pm\mu}^0$ for all $\mu\in\Pi$. However, by a standard argument (or see [St, Prop.9.9]), $\mathscr{L}^0$ (= the finite-dimensional split simple Lie algebra $\mathfrak{g}$) is generated by $\mathscr{L}_{\pm\mu}^0$ for all $\mu\in\Pi$. Then, we can choose a root base of Δ such that ν is the negative highest long root if $\tilde{\Delta}$ is of the untwisted type or the negative highest short root if $\tilde{\Delta}$ is of the twisted type. Using the loop realization of $\mathscr{L}$, it is clear that $\mathscr{L}$ is generated by $\mathscr{L}^0 = \mathfrak{g}$ and $\mathscr{L}^{\pm 1}_{\pm\nu}$. □

Now, we can prove that there is a one to one correspondence between the class of centerless locally Lie 1-tori and the class of reduced root systems extended by $\mathbb{Z}$, and that locally loop algebras exhaust all of the centerless locally Lie 1-tori. Note that this method works for any cardinality of Δ.

Theorem 3.8. *Let $\mathscr{L} = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{m\in\mathbb{Z}} \mathscr{L}^m_\mu$ be a locally Lie 1-torus of type $\tilde{\Delta}$. If $\mathscr{L}$ is centerless, then $\mathscr{L}$ is graded isomorphic to the locally loop algebra of type $\tilde{\Delta}$, and if $\mathscr{L}$ has a nontrivial center, then $\mathscr{L}$ is graded isomorphic to a universal covering of the locally loop algebra of type $\tilde{\Delta}$ given by* (22)*.*

Proof. First, it should be noted that we already know this theorem for Lie 1-tori, i.e., the case where Δ is finite. In addition, it is sufficient to show the case where $\mathscr{L}$ is centerless (see Lemma 3.5), and thus we assume that $\mathscr{L}$ is centerless. Let $\mathscr{M} = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{m\in\mathbb{Z}} \mathscr{M}^m_\mu$ be a locally loop algebra of type $\tilde{\Delta}$. Furthermore, let $\tilde{\Delta} = \{S_\mu\}_{\mu\in\Delta}$.

Fix a long root ν if $\tilde{\Delta}$ is of the untwisted type, or a short root ν if $\tilde{\Delta}$ is of the twisted type, and let $0 \neq x \in \mathscr{L}^1_\nu$ and $0 \neq e_\nu \otimes t \in \mathscr{M}^1_\nu$ (see (23)). Let Π be an integral base of Δ that contains ν. Let $0 \neq x_\mu \in \mathscr{L}^0_\mu$ and $0 \neq e_\mu \otimes 1 \in \mathscr{M}^0_\mu$ for each $\mu \in \Pi$ (see (24)). Then, we claim that the map $\psi : \mu^\vee_{\mathscr{L}} \mapsto \mu^\vee_{\mathscr{M}}$ and $x_\mu \mapsto e_\mu \otimes 1$ for all $\mu \in \Pi$, and $x \mapsto e_\nu \otimes t$ extends to an isomorphism from $\mathscr{L}$ onto $\mathscr{M}$. Indeed, if we let $\Gamma \subset \Pi$ be a finite irreducible subset that contains ν, then Γ is an integral base of the irreducible root system $\Delta_\Gamma := \Delta \cap \langle\Gamma\rangle$.

Let $\tilde{\Delta}_\Gamma = \{S_\mu\}_{\mu\in\Delta_\Gamma}$ be the root system extended by $\mathbb{Z}$. Let $\mathscr{L}_\Gamma$ be the subalgebra determined by Δ_Γ, i.e., the subalgebra of $\mathscr{L}$ generated by $\mathscr{L}^m_\mu$ for all $\mu \in \Delta_\Gamma$ and $m \in \mathbb{Z}$, which is a centerless Lie 1-torus of type $\tilde{\Delta}_\Gamma$ (see Lemma 3.4). Similarly, let $\mathscr{M}_\Gamma$ be the subalgebra of $\mathscr{M}$ determined by Δ_Γ. Then, by Lemma 3.7, a unique graded isomorphism ψ_Γ from $\mathscr{L}_\Gamma$ onto $\mathscr{M}_\Gamma$ exists such that $\psi_\Gamma(x_\mu) = e_\mu \otimes 1$ for all $\mu \in \Gamma$ and $x \mapsto e_\nu \otimes t$.

Suppose that $\Gamma_1, \Gamma_2 \subset \Pi$ are finite irreducible subsets that contain ν such that $\mathscr{L}_{\Gamma_1} \subset \mathscr{L}_{\Gamma_2}$. Then, the uniqueness of the isomorphisms ψ_{Γ_1} and ψ_{Γ_2} implies that they agree on $\mathscr{L}_{\Gamma_1}$. Since $\mathscr{L}$ is the directed union of the subalgebras $\mathscr{L}_\Gamma$ ($\Gamma \subset \Pi$ is a finite irreducible subset), we can define an isomorphism $\psi : \mathscr{L} \longrightarrow \mathscr{M}$ by $\psi(x) = \psi_\Gamma(x)$ for $x \in \mathscr{L}_\Gamma$, which has the required properties. □

Note that in (22), we defined the Lie bracket of a universal covering of a locally loop algebra using a symmetric bilinear form $(\cdot,\cdot)$ on a locally loop algebra. In particular, we can write $(\cdot,\cdot) = \mathrm{tr}(\cdot,\cdot) \otimes \varepsilon(\cdot,\cdot)$, where $\varepsilon(t^m, t^n) = \delta_{m+n,0}$. In fact, it is easy to check that this form is invariant, graded (as a form of a Lie torus defined in [Y2]), and nondegenerate. We simply refer to a **form** for a symmetric invariant graded bilinear form on a Lie G-torus. We use the following lemma later.

Lemma 3.9. *A nonzero form on a locally Lie* 1*-torus exists. In addition, this form is unique up to a nonzero scalar. In particular, a form of a locally loop algebra is equal to $c(\cdot,\cdot)$ for some $c \in F$, where $(\cdot,\cdot)$ is used in* (22)*.*

Proof. Only the uniqueness part is not clear (since we already use a form in (22)). However, this form is unique up to a scalar for a Lie 1-torus (e.g., see [Y2]). Thus, the uniqueness follows from a local argument since a locally Lie 1-torus is a directed union of Lie 1-tori. □

4. LALAs

Let us recall LEALAs in [MY]. A subalgebra $\mathscr{H}$ of a Lie algebra $\mathscr{L}$ is called ad-diagonalizable if

$$\mathscr{L} = \bigoplus_{\xi \in \mathscr{H}^*} \mathscr{L}_\xi,$$

where $\mathscr{H}^*$ is the dual space of $\mathscr{H}$ and

$$\mathscr{L}_\xi = \{x \in \mathscr{L} \mid [h,x] = \xi(h)x \text{ for all } h \in \mathscr{H}\}.$$

This decomposition is called the **root space decomposition** (of $\mathscr{L}$ with respect to an ad-diagonalizable subalgebra $\mathscr{H}$). Note that an ad-diagonalizable subalgebra $\mathscr{H}$ is automatically abelian. To confirm this, we need the well-known fact that every submodule of a weight module is also a weight module. We can use a common trick to obtain the proof, e.g., as given in [MP, Prop.2.1], but they assumed that $\mathscr{H}$ is abelian. To ensure that this assumption is unnecessary, we prove it here. First we show that:

Claim 4.1. *$\mathscr{H} = \oplus_{\xi \in \mathscr{H}^*} \mathscr{H}_\xi$, where $\mathscr{H}_\xi = \mathscr{L}_\xi \cap \mathscr{H}$.*

Proof. Suppose that $\mathscr{H} \neq \oplus_{\xi \in \mathscr{H}^*} \mathscr{H}_\xi$. Then, $x \in \mathscr{H}$ exists such that x can be written as $x = x_1 + \cdots + x_n$ with $n > 1$, which satisfies $x_i \in \mathscr{L}_{\xi_i} \setminus \mathscr{H}$ for all i. Take $x \in \mathscr{H}$ among all of these elements such that n is minimal, and choose $h \in \mathscr{H}$ such that $\xi_1(h) \neq \xi_2(h)$. Then, $x' := \operatorname{ad} h(x) - \xi_1(h)x = (\xi_2(h) - \xi_1(h))x_2 + \cdots + (\xi_n(h) - \xi_1(h))x_n \in \mathscr{H}$. This contradicts the minimality of n. Hence, we have $\mathscr{H} = \oplus_{\xi \in \mathscr{H}^*} \mathscr{H}_\xi$. □

Now, suppose that $h \in \mathscr{H}_\xi$ and $h' \in \mathscr{H}_{\xi'}$. Then, $[h,h'] = \xi'(h)h' = -\xi(h')h$. Hence, if h and h' are linearly independent, then $[h,h'] = 0$. Furthermore, we can see that $[h,h'] = 0$ if they are linearly dependent. Thus, $\mathscr{H}$ is always abelian.

In particular, we have

$$\mathscr{H} = \mathscr{H}_0 \subset \mathscr{L}_0 = C_{\mathscr{L}}(\mathscr{H}),$$

where $C_{\mathscr{L}}(\mathscr{H})$ is the centralizer of $\mathscr{H}$ in $\mathscr{L}$.

An element of the set

$$R = \{\xi \in \mathscr{H}^* \mid \mathscr{L}_\xi \neq 0\}$$

is called a **root**. (We do not call this R a root system and we simply call it the **set of roots**.)

Let $\mathscr{L}$ be a Lie algebra, $\mathscr{H}$ is a subalgebra of $\mathscr{L}$, and $\mathscr{B}$ is a symmetric invariant bilinear form of $\mathscr{L}$. A triple $(\mathscr{L}, \mathscr{H}, \mathscr{B})$ (or simply $\mathscr{L}$) is called a **LEALA** if it satisfies the following four axioms (we explain $R^\times$ shortly):

(A1) $\mathscr{H}$ is ad-diagonalizable and self-centralizing, i.e.,

$$\mathscr{L} = \bigoplus_{\xi \in \mathscr{H}^*} \mathscr{L}_\xi \quad \text{and} \quad \mathscr{H} = \mathscr{L}_0;$$

(A2) $\mathscr{B}$ is nondegenerate;

(A3) $\operatorname{ad} x \in \operatorname{End}_F \mathscr{L}$ is locally nilpotent for all $\xi \in R^\times$ and all $x \in \mathscr{L}_\xi$,

(A4) $R^\times$ is irreducible.

Moreover,

(i) If $\mathscr{H}$ is finite-dimensional, then $\mathscr{L}$ is called an **EALA**.

(ii) If $R^\times = \emptyset$, then $(\mathscr{L}, \mathscr{H}, \mathscr{B})$ is called a **null LEALA** (or a **null EALA** if $\mathscr{H}$ is finite-dimensional) or simply a **null system**. Note that if $R^\times = \emptyset$, then the axioms (A3) and (A4) are empty statements.

Now, using (A1) and (A2), we find that $\mathscr{B}_{\mathscr{L}_\xi \times \mathscr{L}_{-\xi}}$ is nondegenerate for all $\xi \in R$. In particular,

$$\mathscr{B}_{\mathscr{H} \times \mathscr{H}} \text{ is nondegenerate.}$$

Lemma 4.2. *For each $\xi \in R$, a unique $t_\xi \in \mathscr{H}$ exists such that $\mathscr{B}(h, t_\xi) = \xi(h)$ for all $h \in \mathscr{H}$.*

Proof. By the nondegeneracy of $\mathscr{B}_{\mathscr{L}_\xi \times \mathscr{L}_{-\xi}}$, $x \in \mathscr{L}_\xi$ and $y \in \mathscr{L}_{-\xi}$ exist such that $\mathscr{B}(x,y) = 1$. Let $t_\xi := [x,y] \in \mathscr{H}$. Then,

$$\mathscr{B}(h, t_\xi) = \mathscr{B}(h, [x,y]) = B([h,x], y) = \xi(h)\mathscr{B}(x,y) = \xi(h)$$

for all $h \in \mathscr{H}$. The uniqueness of t_ξ follows from the nondegeneracy of $\mathscr{B}_{\mathscr{H} \times \mathscr{H}}$. □

Using these t_ξs, we can define an **induced form on the vector space spanned by R over F**, which is simply denoted as $(\cdot,\cdot)$, by

$$(\xi, \eta) := \mathscr{B}(t_\xi, t_\eta)$$

for $\xi, \eta \in R$. Note that the form $(\cdot,\cdot)$ is well defined, which is easily confirmed by:

$$\begin{aligned}
\mathscr{B}(\textstyle\sum_\xi p_\xi t_\xi, \sum_\eta q_\eta t_\eta) &= \textstyle\sum_\xi p_\xi \mathscr{B}(t_\xi, \sum_\eta q_\eta t_\eta) &&= \textstyle\sum_\xi p_\xi \xi(\sum_\eta q_\eta t_\eta) \\
&= \textstyle\sum_\xi p'_\xi \xi(\sum_\eta q_\eta t_\eta) &&= \textstyle\sum_\xi p'_\xi \mathscr{B}(t_\xi, \sum_\eta q_\eta t_\eta) \\
&= \mathscr{B}(\textstyle\sum_\xi p'_\xi t_\xi, \sum_\eta q_\eta t_\eta) &&= \mathscr{B}(\textstyle\sum_\eta q_\eta t_\eta, \sum_\xi p'_\xi t_\xi) \\
&= \textstyle\sum_\eta q_\eta \mathscr{B}(t_\eta, \sum_\xi p'_\xi t_\xi) &&= \textstyle\sum_\eta q_\eta \eta(\sum_\xi p'_\xi t_\xi) \\
&= \textstyle\sum_\eta q'_\eta \eta(\sum_\xi p'_\xi t_\xi) &&= \textstyle\sum_\eta q'_\eta \mathscr{B}(t_\eta, \sum_\xi p'_\xi t_\xi) \\
&= \mathscr{B}(\textstyle\sum_\eta q'_\eta t_\eta, \sum_\xi p'_\xi t_\xi) &&= \mathscr{B}(\textstyle\sum_\xi p'_\xi t_\xi, \sum_\eta q'_\eta t_\eta)
\end{aligned}$$

for $\sum_\xi p_\xi \xi = \sum_\xi p'_\xi \xi$ and $\sum_\eta q_\eta \eta = \sum_\eta q'_\eta \eta$.

Now we call an element of

$$R^\times := \{\xi \in R \mid (\xi,\xi) \neq 0\}$$

an **anisotropic root**. Axiom (A4) means that $R^\times = R_1 \cup R_2$ and $(R_1, R_2) = 0$, which imply that $R_1 = \emptyset$ or $R_2 = \emptyset$.

Remark 4.3. Null systems have not been studied widely. In [AABGP], they assumed that $R^\times \neq \emptyset$ for an EALA. We also assume that $R^\times \neq \emptyset$ throughout this study.

Remark 4.4. We note that there was one more axiom for a LEALA in [MY], but we showed that axiom is unnecessary by Lemma 4.2 above.

We say that a triple $(\mathscr{L}, \mathscr{H}, \mathscr{B})$ is **admissible** if it satisfies (A1) and (A2). A fundamental property of admissible triples is as follows.

Lemma 4.5. *For $\xi \in R$ and all $x \in \mathscr{L}_\xi$ and $y \in \mathscr{L}_{-\xi}$, we have*

$$[x,y] = \mathscr{B}(x,y)t_\xi, \tag{25}$$

where t_ξ is defined in Lemma 4.2.

Proof. Let $h := [x,y] - \mathscr{B}(x,y)t_\xi \in \mathscr{H}$. Then, for all $h' \in \mathscr{H}$, we have

$$\mathscr{B}(h,h') = \mathscr{B}(x,[y,h']) - \mathscr{B}(x,y)\mathscr{B}(t_\xi,h') = \mathscr{B}(x,y)\xi(h') - \mathscr{B}(x,y)\xi(h') = 0.$$

Hence, by the nondegeneracy of $\mathscr{B}_{\mathscr{H} \times \mathscr{H}}$, we obtain $h = 0$. □

We can scale the above form $(\cdot,\cdot)$ by a nonzero scalar such that $(\xi,\eta)\in\mathbb{Q}$ for all $\xi,\eta\in R^\times$ (see [AABGP, p.16] or [MY, §3]). Let V be the $\mathbb{Q}$-span of R, such as

$$V := \operatorname{span}_{\mathbb{Q}} R.$$

We showed the Kac Conjecture in [MY, Thm 3.10], which states that

$$\text{the scaled form } (\cdot,\cdot) \text{ on } V \text{ is positive semidefinite, and } (R^0,V)=0, \tag{26}$$

where

$$R^0 := \{\xi\in R \mid (\xi,\xi)=0\},$$

the set of **isotropic roots** or **null roots**. As a corollary, $(W,R^\times)$ becomes a reduced locally extended affine root system (LEARS), where $W=\operatorname{span}_{\mathbb{Q}} R^\times$ (see [MY, §4] and [Y3]). We simply refer to R as the set of roots, but we refer to $R^\times$ as a **LEARS**. This $R^\times$ satisfies the fundamental properties of classical finite irreducible root systems, locally finite irreducible root systems, and affine root systems in the sense of Macdonald [Ma], or extended affine root systems in the sense of Saito [S]. We do not recall the definition of LEARS because is is not needed in this study. The reader can find the precise definition in [Y3].

The dimension of the radical of V is called the **null dimension** for a LEALA. If the additive subgroup of V generated by R^0 is free, we call the rank the **nullity** of a LEALA. Thus, we only use the term *nullity* when $\langle R^0\rangle$ is a free abelian group.

Remark 4.6. Of course, there is a notion of rank for non-free abelian groups, but to be consistent with the original theory of EALAs, as given in [AABGP] and [Ne2], we assume that $\langle R^0\rangle$ is free for nullity. Thus, if we say that a LEALA $\mathscr{L}$ has nullity, this means that $\langle R^0\rangle$ is a free abelian group. (In [MY], we used the term *null rank* for nullity, and *nullity* for null dimension, but we have changed these terms to maintain consistency with the notion of nullity in [Ne2].)

The **core** of a LEALA $\mathscr{L}$, denoted by $\mathscr{L}_c$, is the subalgebra of $\mathscr{L}$ generated by the root spaces $\mathscr{L}_\alpha$ for all $\alpha\in R^\times$. Then, by the Kac Conjecture (26), $\mathscr{L}_c$ is an ideal of $\mathscr{L}$. If the centralizer of $\mathscr{L}_c$ in $\mathscr{L}$ is contained in $\mathscr{L}_c$, then $\mathscr{L}$ is called **tame**. Note that the core is zero for a null system (since it is generated by an empty set), so a null system is not tame.

Now, as mentioned earlier, $(W,R^\times)$ is a reduced LEARS. Thus, by [Y3], a locally finite irreducible root system Δ and a **reflectable section** W' of W exist such that Δ^{red} is contained in $R^\times\cap W'$. In particular, W' is a complement of $\operatorname{rad}W$, such as $W=W'\oplus\operatorname{rad}W$, where $\operatorname{rad}W$ is the radical of W relative to the defining positive semidefinite form of the LEARS $(W,R^\times)$. Moreover, a family of subsets $\{S_\mu\}_{\mu\in\Delta}$ of $\operatorname{rad}W$ indexed by Δ exists such that

$$R^\times=\bigcup_{\mu\in\Delta}(\mu+S_\mu), \tag{27}$$

and $\{S_\mu\}_{\mu\in\Delta}$ is a reduced root system extended by $G=\langle\cup_{\mu\in\Delta}S_\mu\rangle$, as defined in Section 2. We note that

$$\operatorname{rad}W=(\operatorname{rad}V)\cap W,$$

by the Kac Conjecture (26).

We can give the graded structure of the core $\mathscr{L}_c$ from (27). For each $\mu\in\Delta$ and $g\in G$, if $g\in S_\mu$, where we let

$$(\mathscr{L}_c)_\mu^g := \mathscr{L}_c\cap\mathscr{L}_{\mu+g},$$

and if $g\notin S_\mu$, where we let $(\mathscr{L}_c)_\mu^g := 0$. Then, we can easily show that

$$\mathscr{L}_c=\bigoplus_{\mu\in\Delta\cup\{0\}}\bigoplus_{g\in G}(\mathscr{L}_c)_\mu^g,$$

where $(\mathscr{L}_c)_0^g := \sum_{\mu\in\Delta}\sum_{g=h+k}\,[(\mathscr{L}_c)_\mu^h,(\mathscr{L}_c)_{-\mu}^k]$, and that

$$\mathscr{L}_c \text{ is a locally Lie } G\text{-torus of type } \Delta, \tag{28}$$

or more precisely, of type $\{S_\mu\}_{\mu\in\Delta}$. Furthermore, if we let

$$\mathscr{L}_c^g := \bigoplus_{\mu\in\Delta\cup\{0\}} (\mathscr{L}_c)_\mu^g,$$

then we obtain a G-graded Lie algebra

$$\mathscr{L}_c = \bigoplus_{g\in G} \mathscr{L}_c^g.$$

Next, we note some properties related to R^0 for LEALAs. As mentioned in Section 2, $S_\mu = S_\nu$ if μ and ν are the same length for $\mu,\nu\in\Delta$. If we let

$$S := S_\mu$$

for a short root μ, then S contains all S_ν, as in (12), and S satisfies $0\in S$ and $2S-S\subset S$. In addition,

$$S \text{ spans } \operatorname{rad} W$$

(see [Thm 8, Y2]).

Lemma 4.7. *Let $\mathscr{L}$ be a LEALA. Then, $S+S\subset R^0$, and $S+S=R^0$ if $\mathscr{L}$ is tame.*

Moreover, we have $\operatorname{rad} V = \operatorname{span}_{\mathbb{Q}} R^0$. *In particular, if $\mathscr{L}$ has nullity, then (nullity of $\mathscr{L}$) = (null dimension of $\mathscr{L}$).*

Proof. The first statement follows from (14) in Section 1, but we present this for convenience with respect to the next statement. Let $s,s'\in S$. Then, $\mathscr{L}_{-\mu+s}\neq 0$ and $\mathscr{L}_{\mu+s'}\neq 0$ for $\mu\in\Delta_{sh}$, and $[\mathscr{L}_{-\mu+s},\mathscr{L}_{\mu+s'}]\neq 0$, by sl_2-theory. (Consider the sl_2-subalgebra generated by $\mathscr{L}_{\mu-s}$ and $\mathscr{L}_{-\mu+s}$, and let it act on $\mathscr{L}_{\mu+s'}$.) Therefore, $0\neq[\mathscr{L}_{-\mu+s},\mathscr{L}_{\mu+s'}]\subset\mathscr{L}_{s+s'}$ and hence $s+s'\in R^0$. Thus, $S+S\subset R^0$.

Suppose that $\mathscr{L}$ is tame. Let $\sigma\in R^0$. If $\alpha+\sigma\notin R$ for all $\alpha\in R^\times$, then $\mathscr{L}_\sigma$ centralizes the core, and thus $\mathscr{L}_\sigma$ is in the core. Therefore, $\mathscr{L}_\sigma=\sum_{\mu\in\Delta,\ s+s'=\sigma}[\mathscr{L}_{\mu+s},\mathscr{L}_{-\mu+s'}]$, and thus $\sigma=s+s'$ for some $s,s'\in S_\mu=S_{-\mu}\subset S$. However, $0\neq\mathscr{L}_{\mu+s}=\mathscr{L}_{\mu-s'+\sigma}$ and $0\neq\mathscr{L}_{\mu-s'}$ since $-s'\in S_\mu$. Therefore, $\mu-s'+\sigma\in R$ with $\mu-s'\in R^\times$, which is a contradiction. Thus, $\alpha\in R^\times$ exists such that $\alpha+\sigma\in R$. (This property is called **nonisolated**. Therefore, we have shown that any isotropic root is nonisolated if $\mathscr{L}$ is tame.) Note that $\alpha=\mu+s$ for some $\mu\in\Delta$ and $s\in S$. Hence, $s+\sigma\in S$, so $\sigma\in S-S=S+S$. Thus, $S+S=R^0$.

For the last statement, it is sufficient to show that $\operatorname{rad} V\subset V^0 := \operatorname{span}_{\mathbb{Q}} R^0$ (the other inclusion is clear). Since $V=W+V^0$ (where $W=\operatorname{span}_{\mathbb{Q}} R^\times$), it is sufficient to show that $(\operatorname{rad} V)\cap W=\operatorname{rad} W\subset V^0$. However, this is clear since $\operatorname{rad} W=\operatorname{span}_{\mathbb{Q}} S$, as above. □

Note that if we put

$$R_c^0 := \{\delta\in R^0 \mid \mathscr{L}_\delta\cap\mathscr{L}_c\neq 0\},$$

then (14) in Section 2 means that we always have

$$R_c^0 = S+S. \tag{29}$$

Remark 4.8. (1) In fact, the rank of $\langle R^0\rangle$ as a torsion-free abelian group is always of the null dimension since the null dimension is now simply the $\mathbb{Q}$-dimension of $\operatorname{span}_{\mathbb{Q}} R^0$ by Lemma 4.7.

(2) There are notions of null dimension and nullity for LEARS $(W,R^\times)$, i.e., (null dimension of $R^\times$) := $\dim\operatorname{rad} W$ and (nullity of $R^\times$) := $\operatorname{rank}\langle S\rangle$ if $\langle S\rangle$ is free (see [Y3]).

For example, if $S=\mathbb{Q}$, then the null dimension is 1, and the rank (= the largest cardinality of linearly independent elements over $\mathbb{Z}$) of $\mathbb{Q}$ as a torsion-free group is also 1. However, we do not say that the nullity is 1 when $S=\mathbb{Q}$.

In general, (null dimension of $\mathscr{L}$) $\geq$ (null dimension of $R^\times$). If $\mathscr{L}$ has nullity, then so does $R^\times$ and (nullity of $\mathscr{L}$) $\geq$ (nullity of $R^\times$) since any subgroup of a free abelian group is free (e.g., see [G]). If $\mathscr{L}$ is tame, then (null dimension of $\mathscr{L}$) $=$ (null dimension of $R^\times$), and if $\mathscr{L}$ has nullity, then

(nullity of $\mathscr{L}$) = (null dimension of $\mathscr{L}$) = (nullity of $R^\times$) = (null dimension of $R^\times$)

since $S+S=R^0$.

Now, we present some basic properties of the center of a LEALA.

Proposition 4.9. *Let $(\mathscr{L},\mathscr{H},\mathscr{B})$ be a LEALA over F with the center $Z(\mathscr{L})$, and R^0 is the set of isotropic roots of $\mathscr{L}$. Then:*

(1) We have

$$\sum_{\delta\in R^0} Ft_\delta \subset Z(\mathscr{L}) \subset \mathscr{H},$$

where t_δ is a unique element in $\mathscr{H}$ defined by (25) *in Lemma 4.5.*

(2) Let $\mathscr{L}_c$ be the core of $\mathscr{L}$ and $R^0_c=\{\delta\in R^0 \mid \mathscr{L}_\delta\cap\mathscr{L}_c\neq 0\}$. Then, for $\delta\in R^0_c$, we have $t_\delta\in\mathscr{L}_c$ and

$$\sum_{\delta\in R^0_c} Ft_\delta = Z(\mathscr{L}_c)\cap\mathscr{H} \subset Z(\mathscr{L}).$$

(3) Let $R^\times$ be the set of anisotropic roots of $\mathscr{L}$ (which is a LEARS). Let $m=\dim_{\mathbb{Q}}(\operatorname{rad} W)$ be the null dimension of $R^\times$, i.e., the dimension of the radical of the induced form from $\mathscr{B}$ on $W=\operatorname{span}_{\mathbb{Q}} R^\times$. Then, $m\geq \dim_F\big(Z(\mathscr{L}_c)\cap\mathscr{H}\big)$, and if $m\geq 1$, then $\dim_F\big(Z(\mathscr{L}_c)\cap\mathscr{H}\big)\geq 1$. Hence, $m=1$ implies that $\dim_F\big(Z(\mathscr{L}_c)\cap\mathscr{H}\big)=1$ and $\dim_F Z(\mathscr{L})\geq 1$.

(4) If $\mathscr{L}$ is tame, then $\sum_{\delta\in R^0} Ft_\delta=\sum_{\delta\in R^0_c} Ft_\delta = Z(\mathscr{L}_c)\cap\mathscr{H}=Z(\mathscr{L})$.

Furthermore, let n be the null dimension of $\mathscr{L}$, i.e., $n=\dim_{\mathbb{Q}}\operatorname{span}_{\mathbb{Q}} R^0$. Then, $m=n\geq\dim_F Z(\mathscr{L})$. Moreover, if $n\geq 1$, then $\dim_F Z(\mathscr{L})\geq 1$. Hence, $n=1$ implies that $\dim_F Z(\mathscr{L})=1$.

Proof. (1): Since each δ is an isotropic root, we have $[t_\delta,x]=0$ for any root vector $x\in\mathscr{L}_\xi$. In fact, $[t_\delta,x]=\xi(t_\delta)x=(\xi,\delta)x=0$ since δ is in the radical of the form (see (26)). Hence, $[t_\delta,\mathscr{L}]=0$, i.e., $t_\delta\in Z(\mathscr{L})$. Thus, $\sum_{\delta\in R^0} Ft_\delta\subset Z(\mathscr{L})$. The second inclusion is clear due to the fact that $\mathscr{H}$ is self-centralizing.

(2): For $\delta\in R^0_c$, let $0\neq x\in\mathscr{L}_\delta\cap\mathscr{L}_c$. Then, $t_\delta=[x,y]$ for some $y\in\mathscr{L}_{-\delta}$, and hence $t_\delta\in\mathscr{L}_c$ since $\mathscr{L}_c$ is an ideal. Thus, $\sum_{\delta\in R^0_c} Ft_\delta\subset\mathscr{L}_c\cap\mathscr{H}$, and by (1), we obtain $\sum_{\delta\in R^0_c} Ft_\delta\subset\mathscr{L}_c\cap Z(\mathscr{L})\subset Z(\mathscr{L}_c)$. Therefore, we obtain $\sum_{\delta\in R^0_c} Ft_\delta\subset Z(\mathscr{L}_c)\cap\mathscr{H}$.

For the other inclusion, let $x\in Z(\mathscr{L}_c)\cap\mathscr{H}$. Since

$$\mathscr{L}_c\cap\mathscr{H}=\sum_{\xi\in R^\times}[\mathscr{L}_\xi,\mathscr{L}_{-\xi}]+\sum_{\delta\in R^0_c}[\mathscr{L}_\delta,\mathscr{L}_{-\delta}],$$

we can write

$$x=\sum_{\xi\in R^\times} a_\xi t_\xi+\sum_{\delta\in R^0_c} a_\delta t_\delta,$$

where $a_\xi,a_\delta\in F$. Let $\Delta\subset R^\times$ be a locally finite irreducible root system determined by a reflectable section of $\bar{R}^\times$ and S is a reflection space for a short root in Δ. Then, we know that $R^\times \subset \Delta + S$ and $R_c^0 = S+S$ (see (29)). Thus, we obtain

$$\begin{aligned} x &= \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_{\alpha+\delta'} + \sum_{\delta\in S+S} a_\delta t_\delta \\ &= \sum_{\alpha\in\Delta,\delta'\in S} (a_{\alpha+\delta'} t_\alpha + a_{\alpha+\delta'} t_{\delta'}) + \sum_{\delta\in S+S} a_\delta t_\delta \\ &= \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_\alpha + \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_{\delta'} + \sum_{\delta\in S+S} a_\delta t_\delta, \end{aligned}$$

and hence,

$$y := \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_\alpha \in Z(\mathscr{L}_c).$$

However, $y \in \mathfrak{h} \subset \mathfrak{g}$, and since $\mathfrak{g}$ is a locally finite split **simple** Lie algebra, then y has to be 0. Therefore,

$$x = \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_{\delta'} + \sum_{\delta\in S+S} a_\delta t_\delta \in \sum_{\delta\in R_c^0} F t_\delta,$$

and we obtain $Z(\mathscr{L}_c)\cap\mathscr{H} \subset \sum_{\delta\in R_c^0} F t_\delta$. Hence, $\sum_{\delta\in R_c^0} F t_\delta = Z(\mathscr{L}_c)\cap\mathscr{H}$. The second inclusion follows from (1).

(3): We know that $R^\times \subset \Delta+S$ and $m = \dim_{\mathbb{Q}}(\mathrm{rad}\, W) = \dim_{\mathbb{Q}} \mathrm{span} S$. However, since $R_c^0 = S+S$, we have $m = \dim_{\mathbb{Q}} \mathrm{span} R_c^0$. Using $\mathscr{B}$, we define an injective linear map

$$\varphi_{\mathscr{B}} \,:\, \mathscr{H} \longrightarrow \mathscr{H}^*,$$

where $\varphi_{\mathscr{B}}(h) \in \mathscr{H}^*$ for $h\in\mathscr{H}$ is given by $\varphi_{\mathscr{B}}(h)(h') = \mathscr{B}(h,h')$ for all $h'\in\mathscr{H}$. Note that $\varphi_{\mathscr{B}}(t_\mu) = \mu$ for $\mu\in R$, where $t_\mu\in\mathscr{H}$ satisfies $[x,y] = \mathscr{B}(x,y)t_\mu$ for $x\in\mathscr{L}_\mu$ and $y\in\mathscr{L}_{-\mu}$. Set $\mathscr{H}^\circ = \mathrm{im}\varphi_{\mathscr{B}} \subset \mathscr{H}^*$. If we put

$$t = \varphi_{\mathscr{B}}^{-1} \,:\, \mathscr{H}^\circ \longrightarrow \mathscr{H}$$

and $t_\nu = t(\nu) = \varphi_{\mathscr{B}}^{-1}(\nu) \in \mathscr{H}$ for $\nu\in\mathscr{H}^\circ$, then we find that $t_{\nu+\nu'} = t_\nu + t_{\nu'}$ for all $\nu,\nu'\in\mathscr{H}^\circ$, and $t_{a\nu} = a t_\nu$ for $\nu\in\mathscr{H}^\circ$ and $a\in F$. Since $R\subset\mathscr{H}^\circ$, there is a one to one correspondence

$$\{\delta\in R_c^0\} \leftrightarrow \{t_\delta\}_{\delta\in R_c^0},$$

and, in particular, we can see that $t_{\delta+\delta'} = t_\delta + t_{\delta'}$ for $\delta,\delta'\in R_c^0$ and $t_{a\delta} = a t_\delta$ for $\delta\in R_c^0$ and $a\in F$. Thus, $m = \dim_{\mathbb{Q}} \sum_{\delta\in R_c^0} \mathbb{Q} t_\delta \geq \dim_F \sum_{\delta\in R_c^0} F t_\delta = \dim_F \big(Z(\mathscr{L}_c)\cap\mathscr{H}\big)$. Finally, if $m\geq 1$, then $0\neq\delta\in R_c^0$ exists and thus $t_\delta\neq 0$. Thus, $Ft_\delta\neq 0$, and hence we obtain the last statement.

(4): We have $R^0 = S+S = R_c^0$ since $\mathscr{L}$ is tame (see Lemma 4.7). Hence, $\sum_{\delta\in R^0} F t_\delta = \sum_{\delta\in R_c^0} F t_\delta$. Furthermore, by (2), we already have $\sum_{\delta\in R_c^0} F t_\delta = Z(\mathscr{L}_c)\cap\mathscr{H} \subset Z(\mathscr{L})$. Moreover, for $x\in Z(\mathscr{L})$, we have $x\in Z(\mathscr{L}_c)$ since $\mathscr{L}$ is tame. Hence, $Z(\mathscr{L}_c)\cap\mathscr{H} = Z(\mathscr{L})$. The remaining assertions follow from the fact that $R^0 = R_c^0$ using (3) and Lemma 4.7. □

Remark 4.10. There are examples of a tame LEALA or EALA where the nullity is ∞ but the center is simply 1-dimensional. For example, $\mathscr{L} = \mathrm{sl}_2(\mathbb{C}[t_i^{\pm 1}]_{i\in\mathbb{N}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ is a tame EALA over $\mathbb{C}$ of type A_1, where $d = \sum_{i=1}^\infty a_i d_i$ with degree derivation $d_i = t_i \frac{\partial}{\partial t_i}$, and $\{a_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ is linearly independent over $\mathbb{Q}$. This $\mathscr{L}$ has nullity of ∞ but the center is equal to Fc. Note that the Cartan subalgebra $\mathscr{H}$ of $\mathscr{L}$ is simply 3-dimensional (for details, see [MY, Rem.5.2(2)]).

Lemma 4.11. *Let $(\mathscr{L},\mathscr{H},\mathscr{B})$ be a tame LEALA. Then, we have the natural embedding*

$$\mathscr{L}/Z(\mathscr{L}_c) \hookrightarrow \mathrm{Der}_F\, \mathscr{L}_c \quad \textit{and} \quad \mathscr{L}/Z(\mathscr{L}_c) \hookrightarrow \mathrm{Der}_F\big(\mathscr{L}_c/Z(\mathscr{L})\big).$$

(Note that $\mathrm{Z}(\mathscr{L}) = \mathrm{Z}(\mathscr{L}_c) \cap \mathscr{H}$ *by Proposition 4.9.)*

In particular, if $\mathscr{N}$ *is a complement of the core* $\mathscr{L}_c$*, i.e.,* $\mathscr{L} = \mathscr{L}_c \oplus \mathscr{N}$*, then* $\mathscr{N}$ *can be identified with a subspace of* $\mathrm{Oder}_F\big(\mathscr{L}_c/\mathrm{Z}(\mathscr{L})\big)$*, i.e., the outer derivations.*

Proof. Since $\mathscr{L}_c$ is an ideal of $\mathscr{L}$, we find that the restriction $\mathrm{ad}x\mid_{\mathscr{L}_c}$ for $x \in \mathscr{L}$ is in $\mathrm{Der}_F \mathscr{L}_c$ and that $\mathrm{Z}(\mathscr{L}_c)$ is an ideal of $\mathscr{L}$. Let

$$f\ :\ \mathscr{L} \longrightarrow \mathrm{Der}_F \mathscr{L}_c$$

be the induced map obtained by this restriction. Since $\mathscr{L}$ is tame, we have $\ker f = C_{\mathscr{L}}(\mathscr{L}_c) = \mathrm{Z}(\mathscr{L}_c)$. Hence, we obtain the first embedding. In addition, $\mathrm{ad}x\mid_{\mathscr{L}_c}$ induces a derivation of $\mathscr{L}_c/\mathrm{Z}(\mathscr{L})$ since $\mathrm{Z}(\mathscr{L}) = \mathrm{Z}(\mathscr{L}_c) \cap \mathscr{H} \subset \mathrm{Z}(\mathscr{L}_c)$. Let

$$f' : \mathscr{L} \longrightarrow \mathrm{Der}_F\big(\mathscr{L}_c/\mathrm{Z}(\mathscr{L})\big)$$

be the induced map. Let $x \in \ker f'$. Then, we have $[x, \mathscr{L}_c] \subset \mathrm{Z}(\mathscr{L}) = \mathrm{Z}(\mathscr{L}_c) \cap \mathscr{H}$. For any $w \in \mathscr{L}_c$, since $\mathscr{L}_c$ is perfect, we can write $w = \sum_i [u_i, v_i]$ for some $u_i, v_i \in \mathscr{L}_c$. Then, $[x, y] = \sum_i [[x, u_i], v_i] + \sum_i [u_i, [x, v_i]] = 0$, and thus $[x, \mathscr{L}_c] = 0$. Hence, $\ker f' \subset \mathrm{Z}(\mathscr{L}_c)$. It is clear that $\mathrm{Z}(\mathscr{L}_c) \subset \ker f'$. Thus, $\ker f' = \mathrm{Z}(\mathscr{L}_c)$, and hence we obtain the second embedding.

For the second assertion, suppose that $\mathrm{ad}x$ for $x \in \mathscr{N}$ is inner in $\mathrm{Der}_F\big(\mathscr{L}_c/\mathrm{Z}(\mathscr{L})\big)$, i.e., $\mathrm{ad}x = \mathrm{ad}y$ on $\mathscr{L}_c/\mathrm{Z}(\mathscr{L})$ for some $y \in \mathscr{L}_c$. Then, we have $[x-y, \mathscr{L}_c] \subset \mathrm{Z}(\mathscr{L})$. However, since $\mathscr{L}_c$ is perfect, for $w = \sum_i [u_i, v_i]$ $(u_i, v_i \in \mathscr{L}_c)$, we have $[x-y, w] = \sum_i [[x-y, u_i], v_i] + \sum_i [u_i, [x-y, v_i]] = 0$. Hence, $x - y \in C_{\mathscr{L}}(\mathscr{L}_c) = \mathrm{Z}(\mathscr{L}_c)$ by tameness. In particular, $x - y \in \mathscr{L}_c$, but $x \in \mathscr{L}_c$, which forces x to be 0. Therefore, $\mathrm{ad}x$ is an outer derivation of $\mathscr{L}_c/\mathrm{Z}(\mathscr{L})$. □

Finally, we give some definitions for later use.

Definition 4.12. Let V be a vector space over $\mathbb{Q}$, and G is an additive subgroup of V. Let

$$\mathscr{A} = \bigoplus_{g \in G} \mathscr{A}^g$$

be a G-graded algebra. Define a linear transformation d_i on $\mathscr{A}$ by

$$d_i(a_g) = g_i a_g$$

for $a_g \in \mathscr{A}^g$, where g_i is the i-coordinate of g obtained by a fixed basis of V. Note that d_i depends on a basis of V. Then, d_i is a derivation of $\mathscr{A}$ where we have

$$d_i(a_g a_h) = (g_i + h_i) a_g a_h = g_i a_g a_h + h_i a_g a_h = d_i(a_g) g_h + a_g d_i(a_h)$$

for $a_h \in \mathscr{A}^h$ and $h \in G$. We refer to each d_i as an ***i*-th coordinate-degree derivation**.

If $\dim_F V = 1$, then d_1 is simply called a **degree derivation**.

We define a standard LEALA.

Definition 4.13. If a LEALA $\mathscr{L}$ contains all coordinate-degree derivations that act on the G-graded core, i.e., a locally Lie G-torus, then $\mathscr{L}$ is called **standard**. This concept depends on the G-graded structure of the core, which is not unique. Thus, when we use this term more precisely, we say that $\mathscr{L}$ is standard (or **non-standard**) relative to the locally Lie G-torus.

We define the minimality of a LEALA (see [N2] and Remark 9.2).

Definition 4.14. A LEALA $\mathscr{L}$ is called **minimal** if $\mathscr{L}$ is the only LEALA that contains $\mathscr{L}_c$ and which is contained in $\mathscr{L}$ (equivalently, if there is no LEALA $\mathscr{L}'$ that satisfies $\mathscr{L}_c \subset \mathscr{L}' \subsetneq \mathscr{L}$). Note that if the nullity is positive, then $\mathscr{L}_c$ is never a LEALA. Thus, if $\mathscr{L}$ has positive nullity and $\mathscr{L}_c$ is a hyperplane in $\mathscr{L}$ (i.e., $\dim \mathscr{L}/\mathscr{L}_c = 1$), then $\mathscr{L}$ is minimal.

Example 4.15. Let $\mathscr{L}^{ms} = \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}]) \oplus Fc \oplus Fd^{(0)}$ (as explained in the Introduction), $\mathscr{L}_1 = \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}]) \oplus Fc \oplus F(e_{11} + d^{(0)})$, where e_{11} is the matrix unit of size $\mathbb{N}$ (only $(1,1)$-entry is 1 and all the other entries are 0), and $\mathscr{L}_2 = \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}]) \oplus Fc \oplus F(p + d^{(0)})$, where

$$p = \mathrm{diag}(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots) \quad \text{(a diagonal matrix of size } \mathbb{N}).$$

Then, these three Lie algebras are all minimal LEALAs. (See Definition 6.1. In fact, these are minimal LALAs.) In addition, $\mathscr{L}^{ms}$ is standard, but $\mathscr{L}_1$ and $\mathscr{L}_2$ are not standard.

In Example 9.3, we show that $\mathscr{L}_1$ is isomorphic to $\mathscr{L}^{ms}$. We note that the concept of standard is not an isomorphic invariant because it depends on the grading of the core. In Example 9.4, we also show that $\mathscr{L}_2$ is not isomorphic to $\mathscr{L}^{ms}$.

5. LEALAs of nullity 0

We classified LEALAs of nullity 0 in [MY, Thm 8.7]. Now, we describe the tame LEALAs of nullity 0 in a slightly different manner compared with the description in [MY].

Let $M := M_{\mathfrak{I}}(F)$, $M_{2\mathfrak{I}+1}(F)$ or $M_{2\mathfrak{I}}(F)$ be the space of matrices of an infinite size $\mathfrak{I}$, $2\mathfrak{I}+1$, or $2\mathfrak{I}$, respectively, and $T_{\mathfrak{I}}$, $T_{2\mathfrak{I}+1}$, or $T_{2\mathfrak{I}}$ is the subspace of M that comprises diagonal matrices. Let T' be a complement of $F\iota_{\mathfrak{I}}$ in $T_{\mathfrak{I}}$, where $\iota_{\mathfrak{I}}$ is the identity matrix such that

$$T_{\mathfrak{I}} = T' \oplus F\iota_{\mathfrak{I}}.$$

Then, the following list comprises infinite-dimensional **maximal tame LEALAs of nullity** 0. (The term "maximal" is used in the usual sense, i.e., no tame LEALA contains each listed LEALA of each type.)

- Type $\mathrm{A}_{\mathfrak{I}}$:
$$\mathrm{sl}_{\mathfrak{I}}(F) + T' \text{ with a Cartan subalgebra } T' \tag{30}$$
(Note that T' is the unique modulo $F\iota_{\mathfrak{I}}$. In addition, see Remark 5.9 and Lemma 5.10),
- Type $\mathrm{B}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}+1}(F) + T^+$ with a Cartan subalgebra T^+, where
$$T^+ := \{x \in T_{2\mathfrak{I}+1} \mid sx = -xs\},$$
- Type $\mathrm{C}_{\mathfrak{I}}$: $\mathrm{sp}_{2\mathfrak{I}}(F) + T^+$ with a Cartan subalgebra T^+, where
$$T^+ := \{x \in T_{2\mathfrak{I}} \mid sx = -xs\},$$
- Type $\mathrm{D}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}}(F) + T^+$ with a Cartan subalgebra T^+, where
$$T^+ := \{x \in T_{2\mathfrak{I}} \mid sx = -xs\},$$

  and each matrix s is the same as s defined in (15).

We note that $F\iota_{\mathfrak{I}}$ is the center of $\mathrm{sl}_{\mathfrak{I}}(F) + T_{\mathfrak{I}}$, and that

$$\mathrm{sl}_{\mathfrak{I}}(F) + T' \cong \big(\mathrm{sl}_{\mathfrak{I}}(F) + T_{\mathfrak{I}}\big)/F\iota_{\mathfrak{I}}$$

for any T'. It is sometimes better to embed T' into $T_{\mathfrak{I}}/F\iota_{\mathfrak{I}}$.

As with locally finite split simple Lie algebras, each of type $\mathrm{B}_{\mathfrak{I}}$, $\mathrm{C}_{\mathfrak{I}}$, or $\mathrm{D}_{\mathfrak{I}}$ is the fixed algebra of $\mathrm{sl}_{2\mathfrak{I}+1}(F) + T_{2\mathfrak{I}+1}$ or $\mathrm{sl}_{2\mathfrak{I}}(F) + T_{2\mathfrak{I}}$ by the automorphism σ defined in (16). This is why we write T^+ because this is the eigenspace of eigenvalue 1 of σ. We write the eigenspace of eigenvalue -1 of σ as T^-.

Any subalgebra of a maximal tame LEALA of nullity 0 that contains each locally finite split simple Lie algebra is a tame LEALA of nullity 0. Thus, let $\mathscr{L}$ be a tame LEALA of nullity 0. Then,

- Type $A_{\mathfrak{J}}$: $\mathrm{sl}_{\mathfrak{J}}(F) \subset \mathscr{L} \subset \mathrm{sl}_{\mathfrak{J}}(F) + T'$ with a Cartan subalgebra $\mathscr{L} \cap T'$,
- Type $B_{\mathfrak{J}}$: $\mathrm{o}_{2\mathfrak{J}+1}(F) \subset \mathscr{L} \subset \mathrm{o}_{2\mathfrak{J}+1}(F) + T^+$ with a Cartan subalgebra $\mathscr{L} \cap T^+$,
- Type $C_{\mathfrak{J}}$: $\mathrm{sp}_{2\mathfrak{J}}(F) \subset \mathscr{L} \subset \mathrm{sp}_{2\mathfrak{J}}(F) + T^+$ with a Cartan subalgebra $\mathscr{L} \cap T^+$,
- Type $D_{\mathfrak{J}}$: $\mathrm{o}_{2\mathfrak{J}}(F) \subset \mathscr{L} \subset \mathrm{o}_{2\mathfrak{J}}(F) + T^+$ with a Cartan subalgebra $\mathscr{L} \cap T^+$.

(We describe the defining bilinear form $\mathscr{B}$ shortly.)

We consider $T_{\mathfrak{J}}$ more carefully. Set

$$T_{\mathfrak{J}}^{as} = \{d \in T_{\mathfrak{J}} \mid d \text{ is } \textbf{almost scalar}\} = \{d \in T_{\mathfrak{J}} \mid d - a\iota_{\mathfrak{J}} \in \mathrm{gl}_{\mathfrak{J}}(F) \text{ for some } a \in F\},$$

i.e., d has **only finitely many different diagonal entries from the identity** $\iota_{\mathfrak{J}}$. Clearly, $T_{\mathfrak{J}}^{as}$ is a subspace of $M_{\mathfrak{J}}(F)$.

Lemma 5.1. *Let $\mathfrak{h}$ be the diagonal subalgebra of $\mathrm{sl}_{\mathfrak{J}}(F)$. Then, we have*

$$T_{\mathfrak{J}}^{as} = \mathfrak{h} \oplus F\iota_{\mathfrak{J}} \oplus Fe_{jj}, \tag{31}$$

where e_{jj} is the matrix in $M_{\mathfrak{J}}(F)$ such that the (j,j)-entry is 1 and all the other entries are 0 for any fixed index $j \in \mathfrak{J}$. In particular, we have

$$\mathrm{gl}_{\mathfrak{J}}(F) = \mathrm{sl}_{\mathfrak{J}}(F) \oplus Fe_{jj}$$

for any $j \in \mathfrak{J}$.

Furthermore, let I be any finite subset of $\mathfrak{J}$, and $\iota_I := \sum_{i \in I} e_{ii}$. Then, we have

$$T_{\mathfrak{J}}^{as} = \mathfrak{h}_I \oplus F\iota_I \oplus T_{\mathfrak{J}\setminus I}^{as}, \tag{32}$$

where $\mathfrak{h}_I$ is the subspace of $\mathfrak{h}$ such that all (k,k)-components of $k \in \mathfrak{J} \setminus I$ are 0, and $T_{\mathfrak{J}\setminus I}^{as}$ is the subspace of $T_{\mathfrak{J}}^{as}$ such that all (i,i)-components of $i \in I$ are 0.

Moreover, we have

$$T_{\mathfrak{J}} = \mathfrak{h}_I \oplus F\iota_I \oplus T_{\mathfrak{J}\setminus I}, \tag{33}$$

where $T_{\mathfrak{J}\setminus I}$ is the subspace of $T_{\mathfrak{J}}$ such that all (i,i)-components of $i \in I$ are 0.

Proof. It is clear that $T_{\mathfrak{J}}^{as} \supset \mathfrak{h} \oplus F\iota_{\mathfrak{J}} \oplus Fe_{jj}$. For the other inclusion, let $x \in T_{\mathfrak{J}}^{as}$. Then, $a \in F$ exists such that $y := x - a\iota_{\mathfrak{J}} \in T_{\mathfrak{J}} \cap \mathrm{gl}_{\mathfrak{J}}(F)$. Hence, $y = y - \mathrm{tr}(y)e_{jj} + \mathrm{tr}(y)e_{jj}$ and note that $h := y - \mathrm{tr}(y)e_{jj} \in \mathfrak{h}$. Thus, $x = h + a\iota_{\mathfrak{J}} + \mathrm{tr}(y)e_{jj} \in \mathfrak{h} \oplus F\iota_{\mathfrak{J}} \oplus Fe_{jj}$. This completes the description of (29).

For the second decomposition (30), we have $T_{\mathfrak{J}}^{as} = T_I \oplus T_{\mathfrak{J}\setminus I}^{as}$, where T_I is the subset of $T_{\mathfrak{J}}^{as}$ such that all (k,k)-components of $k \in \mathfrak{J} \setminus I$ are 0. However, it is then easy to see that $T_I = \mathfrak{h}_I \oplus F\iota_I$. The last decomposition (31) is now clear. ☐

We have not mentioned the defining bilinear form $\mathscr{B}$ of a tame LEALA $\mathscr{L}$ of nullity 0. Thus, as described in [MY], let $\mathfrak{g}$ be one of the the locally finite split simple Lie algebra

$$\mathrm{sl}_{\mathfrak{J}}(F),\ \mathrm{o}_{2\mathfrak{J}+1}(F),\ \mathrm{sp}_{2\mathfrak{J}}(F) \quad \text{or} \quad \mathrm{o}_{2\mathfrak{J}}(F),$$

contained in $\mathscr{L}$, as defined above. The restriction $\mathscr{B}_{\mathscr{L}\times\mathfrak{g}}$ of $\mathscr{B}$ to the space $\mathscr{L} \times \mathfrak{g}$ is a nonzero scalar multiple of the trace form, and the remaining part, i.e., the restriction to $\mathfrak{C} \times \mathfrak{C}$, where $\mathfrak{C}$ is a complement of $\mathfrak{g}$, can be any symmetric bilinear form.

In fact, in [MY], we did not state clearly why the restriction $\mathscr{B}_{\mathscr{L}\times\mathfrak{g}}$ of $\mathscr{B}$ is a nonzero scalar multiple of the trace form. However, this follows from the perfectness of $\mathfrak{g}$ and the invariance of $\mathscr{B}$. We summarize this phenomenon in a slightly more general setup. Let us refer to a symmetric invariant bilinear form simply as a **form** for convenience.

Lemma 5.2. *Let L be a Lie algebra with a form B and let $\mathfrak{g}$ be a perfect ideal of L. If any form of $\mathfrak{g}$ is equal to $B' := B\mid_{\mathfrak{g}\times\mathfrak{g}}$ up to a scalar, then any invariant bilinear form on $L\times\mathfrak{g}$ or on $\mathfrak{g}\times L$ is equal to $B\mid_{L\times\mathfrak{g}}$ or $B\mid_{\mathfrak{g}\times L}$ up to a scalar. In this case, "invariant on $L\times\mathfrak{g}$" means that $B([x,y],z) = B(x,[y,z])$ for $x,y\in L$ and $z\in\mathfrak{g}$.*

Proof. Let E be an invariant bilinear form on $L\times\mathfrak{g}$. For $x\in L$ and $y\in\mathfrak{g}$, since $y=\sum_i[u_i,v_i]$ for some $u_i,v_i\in\mathfrak{g}$, then we have

$$E(x,y) = E(x,\sum_i[u_i,v_i]) = c\sum_i B'([x,u_i],v_i) = c\sum_i B([x,u_i],v_i) = cB(x,\sum_i[u_i,v_i]) = cB(x,y)$$

for some $c\in F$. We can prove the result for $\mathfrak{g}\times L$ in a similar manne. □

Recall that the associative algebra

$$M_{\mathfrak{J}}^{\text{fin}}(F) = \{x\in M_{\mathfrak{J}}(F) \mid \text{each row and column of } x \text{ have only finitely many nonzero entries}\}$$

is a Lie algebra under the commutator. Using the matrix $s\in M_{\mathfrak{J}}^{\text{fin}}(F)$ defined in (15), we can define an automorphism of $M_{\mathfrak{K}}^{\text{fin}}(F)$, where $\mathfrak{K} = 2\mathfrak{J}$ or $2\mathfrak{J}+1$, by the same definition of σ in (16). We also denote the automorphism by σ. Thus, each fixed Lie algebra $M_{\mathfrak{K}}^{\text{fin}}(F)^{\sigma}$ contains a locally finite split simple Lie algebra $\mathfrak{g} := \mathrm{sl}_{\mathfrak{K}}(F)^{\sigma}$.

Lemma 5.3. *Let L be any subalgebra of $M_{\mathfrak{J}}^{\text{fin}}(F)$, and let M be any subalgebra of $\mathrm{gl}_{\mathfrak{J}}(F)$. Then, the trace form tr on $L\times M$ and $M\times L$ is well defined and it is invariant.*

Hence, if L contains $\mathrm{sl}_{\mathfrak{J}}(F)$, then any invariant bilinear form on $L\times\mathrm{sl}_{\mathfrak{J}}(F)$ or on $\mathrm{sl}_{\mathfrak{J}}(F)\times L$ is equal to $c\,\mathrm{tr}$ for some $c\in F$. In particular, $\mathrm{sl}_{\mathfrak{J}}(F)$ is a perfect ideal of L.

Moreover, if L is a subalgebra of $M_{\mathfrak{K}}^{\text{fin}}(F)^{\sigma}$ that contains $\mathfrak{g} = \mathrm{sl}_{\mathfrak{K}}(F)^{\sigma}$, then $\mathfrak{g}$ is a perfect ideal of L, and any invariant bilinear form on $L\times\mathfrak{g}$ or on $\mathfrak{g}\times L$ is equal to $c\,\mathrm{tr}$ for some $c\in F$.

Proof. Since $xy\in\mathrm{gl}_{\mathfrak{J}}(F)$ for $x\in L$ and $y\in M$, then the trace form $\mathrm{tr}(xy)$ is well defined. To show the invariance, i.e., $\mathrm{tr}([A,B]y) = \mathrm{tr}(A[B,y])$ for $A,B\in L$ and $y\in M$, it is sufficient to show this for $y=e_{ij}$ (the matrix unit of (i,j)-component).

Let $A=(a_{mn})$, $B=(b_{mn})$, and $C=(c_{mn})=[A,B]$. Then, $c_{mn}=\sum_k(a_{mk}b_{kn}-b_{mk}a_{kn})$ and $\mathrm{tr}([A,B]y) = \mathrm{tr}((c_{mn})e_{ij}) = c_{ji} = \sum_k(a_{jk}b_{ki}-b_{jk}a_{ki})$ and

$$\mathrm{tr}(A[B,y]) = \mathrm{tr}((a_{mn})\big(\sum_m b_{mi}e_{mj} - \sum_n b_{jn}e_{in}\big)) = \sum_k(a_{jk}b_{ki}-a_{ki}b_{jk}).$$

Therefore, the trace form is invariant. We can prove this for the case where $M\times L$ in a similar manner. We note that $\mathrm{sl}_{\mathfrak{J}}(F)$ or $\mathfrak{g}$ is a perfect ideal of L. By [NS, Lem. II.11], any form on $\mathrm{sl}_{\mathfrak{J}}(F)$ is equal to $c\,\mathrm{tr}$ for some $c\in F^{\times}$. Therefore, the second and last statements follow from Lemma 5.2. □

Remark 5.4. We employ the notation given in Lemma 5.3. We can identify $M_{\mathfrak{J}}^{\text{fin}}(F)$ with the derivation algebra $\mathrm{Der}\big(\mathrm{gl}_{\mathfrak{J}}(F)\big)$, and $M_{\mathfrak{K}}^{\text{fin}}(F)^{\sigma}$ with the derivation algebra $\mathrm{Der}\,\mathfrak{g}$ (see [N1]).

Suppose that $\mathscr{B}$ is a symmetric invariant bilinear form on

$$\mathscr{M}_{\mathfrak{J}} := \mathrm{sl}_{\mathfrak{J}}(F) + T_{\mathfrak{J}}.$$

Then, by Lemma 5.3, the restriction of $\mathscr{B}$ to $\mathscr{M}_{\mathfrak{J}}\times\mathrm{sl}_{\mathfrak{J}}(F)$ or $\mathrm{sl}_{\mathfrak{J}}(F)\times\mathscr{M}_{\mathfrak{J}}$ is equal to $c\,\mathrm{tr}$ for some $c\in F$. We claim that such a form $\mathscr{B}$ does exist. Therefore, we select any complement $\mathfrak{h}^c$ of $\mathfrak{h}$ in $T_{\mathfrak{J}}$, i.e.,

$$T_{\mathfrak{J}} = \mathfrak{h}^c\oplus\mathfrak{h}.$$

Let

$$\psi : \mathfrak{h}^c \times \mathfrak{h}^c \longrightarrow F$$

be an arbitrary symmetric bilinear form. Now, we define a symmetric bilinear form $\mathscr{B}$ on $\mathscr{M}_{\mathfrak{J}}$ as

$$\mathscr{B}(x,y) = \psi(x,y)$$

on $\mathfrak{h}^c$, and $c\,\mathrm{tr}$ on $\mathscr{M}_{\mathfrak{J}} \times \mathrm{sl}_{\mathfrak{J}}(F)$ and $\mathrm{sl}_{\mathfrak{J}}(F) \times \mathscr{M}_{\mathfrak{J}}$. To show that $\mathscr{B}$ is invariant, we prove the following.

Claim 5.5. *Let $x \in T_{\mathfrak{J}} \setminus F\iota_{\mathfrak{J}}$ and $y_k \in \mathrm{sl}_{\mathfrak{J}}(F)$ for $k = 1,2,\dots,r$. Then, a finite subset I of $\mathfrak{J}$, $0 \neq h \in \mathfrak{h}$ and $g \in T_{\mathfrak{J}}$ exist such that $y_k \in \mathrm{sl}_I(F)$ for all k, $h \in \mathfrak{h}_I$,*

$$x = h + g, \quad [x,y_k] = [h,y_k] \quad \textit{and} \quad \mathscr{B}(x,y_k) = \mathscr{B}(h,y_k)$$

for all k. Moreover, if $\mathscr{B}$ is nontrivial, then $y \in \mathrm{sl}_{\mathfrak{J}}(F)$ and $h' \in \mathfrak{h}$ exist such that $[x,y] \neq 0$ and

$$\mathscr{B}(x,h') \neq 0. \tag{34}$$

Proof. Let I be a finite subset of $\mathfrak{J}$ such that $y_k \in \mathrm{sl}_I(F)$ for all k. Moreover, if the $I \times I$-block submatrix of x is a scalar matrix, then we enlarge I until the $I \times I$-block submatrix of x is not a scalar matrix. For I, by (33) in Lemma 5.1, $0 \neq h \in \mathfrak{h}_I$ exists such that $x = h + b\iota_I + x'$ for some $b \in F$ and $x' \in T_{\mathfrak{J}\setminus I}$. Put $g := b\iota_I + x'$. Then, clearly $[g,y_k] = 0$. In addition, we have $\mathscr{B}(g,y_k) = c\,\mathrm{tr}(gy_k) = cb\,\mathrm{tr}(y_k) = 0$ since $\mathrm{tr}(y_k) = 0$.

To show the second statement, it is sufficient to select $y \in \mathrm{sl}_I(F)$ and $h' \in \mathfrak{h}_I$ such that $[h,y] \neq 0$ and $\mathrm{tr}(hh') \neq 0$. □

Claim 5.6. *$\mathscr{B}$ is invariant.*

Proof. It is sufficient to consider the case that involves some elements in $\mathfrak{h}^c$. Since $\mathfrak{h}^c$ is an abelian subalgebra, the case that involves three elements in $\mathfrak{h}^c$ is clear.

For the case that involves one element in $\mathfrak{h}^c$, let $x \in \mathfrak{h}^c$ and $y,z \in \mathrm{sl}_{\mathfrak{J}}(F)$. Then, it is sufficient to show that

$$\mathscr{B}([x,y],z) = \mathscr{B}(x,[y,z]).$$

If $x \in F\iota$, then both sides are clearly 0. Thus, by Claim 5.5, we can change x into h for y and $[y,z]$ such that $\mathscr{B}([x,y],z) = \mathscr{B}([h,y],z)$ and $\mathscr{B}(x,[y,z]) = \mathscr{B}(h,[y,z])$. This follows from the invariance on $\mathrm{sl}_{\mathfrak{J}}(F)$.

The case that involves two elements in $\mathfrak{h}^c$ can be shown in a similar manner. Let $x,y \in \mathfrak{h}^c$ and $z \in \mathrm{sl}_{\mathfrak{J}}(F)$. Then, it is sufficient to show that

$$\mathscr{B}(x,[y,z]) = 0 \quad \text{and} \quad \mathscr{B}([x,z],y) = \mathscr{B}(x,[z,y]).$$

Again, if x or $y \in F\iota$, then both sides of both equations are clearly 0. Thus, by Claim 5.5, the left-hand side of the first equation is equal to $\mathscr{B}(h,[h',z])$ for some $h,h' \in \mathfrak{h}_I$, and this is equal to 0 by the invariance on $\mathrm{sl}_{\mathfrak{J}}(F)$. For the second equation, change x into h for z and $[z,y]$ such that (LHS) $= \mathscr{B}([h,z],y)$ and (RHS) $= \mathscr{B}(h,[z,y])$. However, these are equal according to the case involving one element, as described above. Thus, we have proved that the symmetric bilinear form $\mathscr{B}$ is invariant. □

The radical of $\mathscr{B}$ is contained in $F\iota_{\mathfrak{J}}$ whenever the restriction to $\mathrm{sl}_{\mathfrak{J}}(F)$ is not zero. In fact, this follows from [MY, Lem. 8.5] since the center of $\mathscr{M}_{\mathfrak{J}} = \mathrm{sl}_{\mathfrak{J}}(F) + T_{\mathfrak{J}}$ is equal to $F\iota_{\mathfrak{J}}$. However, for convenience, we show this directly. First, let us mention the graded structure of $\mathscr{M}_{\mathfrak{J}}$.

Let $\mathfrak{g} := \mathrm{sl}_{\mathfrak{J}}(F)$ and let $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\mu \in \mathrm{A}_{\mathfrak{J}} \subset \mathfrak{h}^*} \mathfrak{g}_\mu\right)$ be the root-space decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$. We extend each root $\mu \in \mathfrak{h}^*$ to an element in $T_{\mathfrak{J}}^*$ as follows.

Let $\mathrm{A}_{\mathfrak{I}} = \{\pm(\varepsilon_i - \varepsilon_j) \mid i, j \in \mathfrak{I}\}$, where ε_i is the linear form of $\mathrm{gl}_{\mathfrak{I}}(F)$ determined by $e_{kl} \mapsto \delta_{lk}\delta_{ki}$. Since an element $p \in T_{\mathfrak{I}}$ can be written as $p = \mathrm{diag}(a_{ii})_{i\in\mathfrak{I}}$, we can define $\varepsilon_i(p) = a_{ii}$. In this manner, we can embed $\mathrm{A}_{\mathfrak{I}}$ into $T_{\mathfrak{I}}^*$. Thus, $\mathscr{M} := \mathscr{M}_{\mathfrak{I}}$ has the root-space decomposition

$$\mathscr{M} = \bigoplus_{\mu \in T_{\mathfrak{I}}^*} \mathscr{M}_\mu$$

relative to $T_{\mathfrak{I}}$, where $\mathscr{M}_\mu = \mathfrak{g}_\mu$ for $\mu \neq 0$ and $\mathscr{M}_0 = T_{\mathfrak{I}}$, and $\mathscr{M}_\mu = 0$ if $\mu \notin \mathrm{A}_{\mathfrak{I}}$. This is an $\langle \mathrm{A}_{\mathfrak{I}} \rangle$-graded Lie algebra, and $\mathscr{B}$ is **graded** in the sense that $\mathscr{B}(\mathscr{M}_\xi, \mathscr{M}_\eta) = 0$, unless $\xi + \eta = 0$ for all $\xi, \eta \in \mathrm{A}_{\mathfrak{I}}$. In general, a symmetric invariant bilinear form on a Lie algebra with a root-space decomposition relative to a subalgebra is graded.

In particular, the radical of $\mathscr{B}$ is graded. Thus, we can check the nondegeneracy for each homogeneous element. The elements of degree $\mu \in \mathrm{A}_{\mathfrak{I}}$ cannot be in the radical by Lemma 5.3. For the elements of degree 0, the only candidate is an element in $F\iota_{\mathfrak{I}}$ by (34), which implies that the radical of $\mathscr{B}$) is contained in $F\iota_{\mathfrak{I}}$.

Therefore, we have the following.

Lemma 5.7. *Let $\mathscr{B}$ be nontrivial. Then, the radical of $\mathscr{B}$ is equal to $F\iota_{\mathfrak{I}}$ if $\mathscr{B}(\iota_{\mathfrak{I}}, \iota_{\mathfrak{I}}) = 0$, and $\mathscr{B}$ is nondegenerate if $\mathscr{B}(\iota_{\mathfrak{I}}, \iota_{\mathfrak{I}}) \neq 0$.* □

Thus, for any symmetric bilinear form ψ on $\mathfrak{h}^c$ with the radical $F\iota_{\mathfrak{I}}$, the quotient Lie algebra $\mathscr{M}_{\mathfrak{I}}/F\iota_{\mathfrak{I}}$ with the induced form $\bar{B}$ is a LEALA of type $\mathrm{A}_{\mathfrak{I}}$ of nullity 0. Note that $\mathscr{M}_{\mathfrak{I}}/F\iota_{\mathfrak{I}}$ is isomorphic to $\mathscr{M}'_{\mathfrak{I}} := \mathrm{sl}_{\mathfrak{I}}(F) \oplus \mathfrak{t}$, where $\mathfrak{t}$ is a complement of $\mathfrak{h} \oplus F\iota_{\mathfrak{I}}$ in $T_{\mathfrak{I}}$. Conversely, if ψ' is any symmetric bilinear form on $\mathfrak{t}$, we can define a symmetric nondegenerate invariant form $\mathscr{B}'$ on $\mathscr{M}'_{\mathfrak{I}}$ as described above, and $\mathscr{M}'_{\mathfrak{I}}$ is isomorphic to $\mathscr{M}_{\mathfrak{I}}/F\iota_{\mathfrak{I}}$. By a similar argument, we can say that a LEALA of type $\mathrm{A}_{\mathfrak{I}}$ of nullity 0 is isomorphic to a subalgebra of $\mathscr{M}_{\mathfrak{I}}/F\iota_{\mathfrak{I}}$ that contains $\mathrm{sl}_{\mathfrak{I}}(F) = (\mathrm{sl}_{\mathfrak{I}}(F) + F\iota_{\mathfrak{I}})/F\iota_{\mathfrak{I}}$ with the induced form $\bar{B}$.

Example 5.8. The centerless Lie algebra $\mathrm{gl}_{\mathfrak{I}}(F) = \mathrm{sl}_{\mathfrak{I}}(F) \oplus Fe_{jj}$ is an example of a LEALA of type $\mathrm{A}_{\mathfrak{I}}$ of nullity 0, where e_{jj} is the matrix unit for $j \in \mathfrak{I}$. However, $\mathrm{gl}_n(F) = \mathrm{sl}_n(F) \oplus Fe_{jj}$ has the center $F\iota_n$ if $j \in \{1, 2, \cdots, n\}$, where ι_n is the identity matrix on $\mathrm{gl}_n(F)$, and this is a non-tame EALA of nullity 0.

Suppose that $\mathscr{B}$ is a nondegenerate form on $\mathrm{gl}_{\mathfrak{I}}(F)$. Then, $\mathscr{B}$ is a nonzero scalar multiple $c \in F$ of the trace form, except on $Fe_{jj} \times Fe_{jj}$, by Lemma 5.3. Conversely, we can take any value to $\mathscr{B}(e_{jj}, e_{jj})$ and extend a nondegenerate form to $\mathrm{gl}_{\mathfrak{I}}(F)$.

For the finite case where $\mathrm{gl}_n(F) = \mathrm{sl}_n(F) \oplus Fe_{jj}$, suppose that $\mathfrak{B}$ is a nondegenerate form on $\mathrm{gl}_n(F)$. Since $\mathrm{rad}\,\mathfrak{B}$ is in the center of $\mathrm{gl}_n(F)$, we find that $\mathfrak{B}$ is nondegenerate $\iff \mathfrak{B}(\iota_n, \iota_n) \neq 0$. Moreover, this is equivalent to

$$\mathfrak{B}(e_{jj}, e_{jj}) \neq \frac{n-1}{n} c. \tag{35}$$

In fact, consider the expression $\iota_n = \iota_n - ne_{jj} + ne_{jj}$, where we note that $\mathrm{tr}(\iota_n - ne_{jj}) = 0$. Since $x := \iota_n - ne_{jj} \in \mathrm{sl}_{\mathfrak{I}}(F)$, we have $\mathfrak{B}(\iota_n, \iota_n) =$

$$\begin{aligned}
\mathfrak{B}(x + ne_{jj}, x + ne_{jj}) &= \mathfrak{B}(x, x) + 2n\mathfrak{B}(x, e_{jj}) + n^2\mathfrak{B}(e_{jj}, e_{jj}) \\
&= c\,\mathrm{tr}(x^2) + 2nc\,\mathrm{tr}(xe_{jj}) + n^2\mathfrak{B}(e_{jj}, e_{jj}) \\
&= c\,\mathrm{tr}(\iota_n - 2ne_{jj} + n^2 e_{jj}) + 2nc\,\mathrm{tr}(e_{jj} - ne_{jj}) + n^2\mathfrak{B}(e_{jj}, e_{jj}) \\
&= c(n - 2n + n^2) + 2nc(1 - n) + n^2\mathfrak{B}(e_{jj}, e_{jj}) \\
&= cn - cn^2 + n^2\mathfrak{B}(e_{jj}, e_{jj}).
\end{aligned}$$

Hence, $\mathfrak{B}(\iota_n,\iota_n)=0$ if and only if $n^2\mathfrak{B}(e_{jj},e_{jj})=c(n^2-n)$, and thus (35) holds.

Remark 5.9. In the classification of tame LEALAs of nullity 0 of type $\mathrm{A}_{\mathfrak{I}}$ in [MY], we select $\mathrm{sl}_{\mathfrak{I}}(F)\oplus\mathfrak{t}'$ for a complement $\mathfrak{t}'$ of $T^{as}_{\mathfrak{I}}$ in $T_{\mathfrak{I}}$ as the maximal one. However, a subalgebra of the bigger Lie algebra $\mathrm{sl}_{\mathfrak{I}}(F)\oplus\mathfrak{t}$ defined above is actually a maximal tame LEALA of nullity 0, which is shown essentially by the following lemma.

Lemma 5.10. *Let $p\in T_{\mathfrak{I}}$. Suppose that $[p,\mathrm{sl}_{\mathfrak{I}}(F)]=0$. Then, $p\in F\iota_{\mathfrak{I}}$. In particular, $\mathrm{sl}_{\mathfrak{I}}(F)+T'$ is a tame LEALA of nullity 0 for any complement T' of $F\iota_{\mathfrak{I}}$ in $T_{\mathfrak{I}}$.*

Proof. Let I be any finite subset of $\mathfrak{I}$. Decompose $p=h\oplus s\iota_I\oplus q$ in $T_{\mathfrak{I}}=\mathfrak{h}_I\oplus F\iota_I\oplus T_{\mathfrak{I}\setminus I}$ for some $s\in F$ (see (33)). Since $\mathrm{sl}_I(F)\subset\mathfrak{g}$ and $[\iota_I,\mathrm{sl}_I(F)]=0$, we have $0=[p,\mathrm{sl}_I(F)]=[h,\mathrm{sl}_I(F)]$. Hence, $h=0$, and thus $p=s\iota_I\oplus q$. For a different subset I', we have $p=s'\iota_{I'}\oplus q'$. However, for $I''=I\cup I'$, we have $p=s''\iota_{I''}\oplus q''$. Since $I,I'\subset I''$, we have $s=s'=s''$. Therefore, $p=s\iota_{\mathfrak{I}}$. □

Now, we consider the forms on the other types $\mathrm{B}_{\mathfrak{I}}$, $\mathrm{D}_{\mathfrak{I}}$ and $\mathrm{C}_{\mathfrak{I}}$. Let $\mathscr{B}$ be a symmetric invariant form on

$$\mathscr{M}_{\mathfrak{K}}=\mathrm{sl}_{\mathfrak{K}}(F)+T_{\mathfrak{K}}$$

such that the restriction to $\mathrm{sl}_{\mathfrak{K}}(F)$ is not zero, where $\mathfrak{K}=2\mathfrak{I}$ or $2\mathfrak{I}+1$. Let $\mathscr{M}^{\sigma}_{\mathfrak{K}}$ be the fixed algebra by the automorphism σ defined above with the restricted form $\mathscr{B}^{\sigma}$. Then, $\mathscr{B}^{\sigma}$ is still invariant, and by Lemma 5.3, the restriction to $\mathrm{sl}_{\mathfrak{K}}(F)^{\sigma}$ is equal to $c\,\mathrm{tr}$ for some $c\in F^{\times}$.

Moreover, $\mathscr{B}^{\sigma}$ is nondegenerate. This follows from [MY, Lem. 8.5] since $\mathscr{M}^{\sigma}_{\mathfrak{K}}$ has a trivial center. We can also show this using the following lemma, which is similar to Lemma 5.1. Recall that T^+ denotes the eigenspace of eigenvalue $+1$ of σ, and T^- is the eigenspace of eigenvalue -1 of σ.

Lemma 5.11. *Let I be any finite subset of $\mathfrak{I}$ and fix some index $i_0\in I$. Then, we have*

$$T^+_{2\mathfrak{I}}=\mathfrak{h}^+_{2I}\oplus T^+_{2\mathfrak{I}\setminus 2I}\quad\text{and}\quad T^-_{2\mathfrak{I}}=\mathfrak{h}^-_{2I}\oplus F(e_{i_0i_0}+e_{\mathfrak{I}+i_0,\mathfrak{I}+i_0})\oplus T^-_{2\mathfrak{I}\setminus 2I},$$

where $\mathfrak{h}^+_{2I}$ or $\mathfrak{h}^-_{2I}$ is a subset of $\mathfrak{h}^+$ or $\mathfrak{h}^-$ such that all (k,k) and $(\mathfrak{I}+k,\mathfrak{I}+k)$ components for $k\in\mathfrak{I}\setminus I$ are 0, and $T^+_{2\mathfrak{I}\setminus 2I}$ or $T^-_{2\mathfrak{I}\setminus 2I}$ is a subset of $T^+_{2\mathfrak{I}}$ or $T^-_{2\mathfrak{I}}$ such that all (i,i) and $(\mathfrak{I}+i,\mathfrak{I}+i)$ components for $i\in I$ are 0.

Furthermore, we have

$$T^+_{2\mathfrak{I}+1}=\mathfrak{h}^+_{2I+1}\oplus T^+_{(2\mathfrak{I}+1)\setminus(2I+1)}\quad\text{and}\quad T^-_{2\mathfrak{I}+1}=\mathfrak{h}^-_{2I+1}\oplus Fe_{2\mathfrak{I}+1,2\mathfrak{I}+1}\oplus T^-_{(2\mathfrak{I}+1)\setminus(2I+1)},$$

where $\mathfrak{h}^+_{2I+1}$ or $\mathfrak{h}^-_{2I+1}$ is a subset of $\mathfrak{h}^+$ or $\mathfrak{h}^-$ such that the (k,k) and $(\mathfrak{I}+k,\mathfrak{I}+k)$ components of all $k\in\mathfrak{I}\setminus I$ are 0, and $T^+_{(2\mathfrak{I}+1)\setminus(2I+1)}$ or $T^-_{(2\mathfrak{I}+1)\setminus(2I+1)}$ is a subset of $T^+_{2\mathfrak{I}+1}$ or $T^-_{2\mathfrak{I}+1}$ such that the $(2\mathfrak{I}+1,2\mathfrak{I}+1)$ component and the (i,i) and $(\mathfrak{I}+i,\mathfrak{I}+i)$ components of all $i\in I$ are 0.

Moreover, we have

$$T^-_{2\mathfrak{I}}=\mathfrak{h}^-_{2I}\oplus F\iota_{2I}\oplus T^-_{2\mathfrak{I}\setminus 2I}\quad\text{and}\quad T^-_{2\mathfrak{I}+1}=\mathfrak{h}^-_{2I+1}\oplus F\iota_{2I+1}\oplus T^-_{(2\mathfrak{I}+1)\setminus(2I+1)}.\tag{36}$$

Proof. All of these statements are clear except (36). To show this, we consider the two equalities

$$T^-_{2I}=\mathfrak{h}^-_{2I}\oplus F(e_{i_0i_0}+e_{\mathfrak{I}+i_0,\mathfrak{I}+i_0})\quad\text{and}\quad T^-_{2I+1}=\mathfrak{h}^-_{2I+1}\oplus Fe_{2\mathfrak{I}+1,2\mathfrak{I}+1},$$

where T^-_{2I} is a subset of $T^-_{2\mathfrak{I}}$ such that (i,i) and the $(\mathfrak{I}+i,\mathfrak{I}+i)$ components of all $i\in\mathfrak{I}\setminus I$ are 0, and T^-_{2I+1} is a subset of $T^-_{2\mathfrak{I}+1}$ such that (i,i) and the $(\mathfrak{I}+i,\mathfrak{I}+i)$ components of all

$i \in \mathfrak{I} \setminus I$ are 0. However, as in the proof of Lemma 5.1, for $y \in T^-_{2I}$ or T^-_{2I+1}, it follows from the equation that

$$y = y - \frac{1}{2}\operatorname{tr}(y)(e_{i_0 i_0} + e_{\mathfrak{I}+i_0,\mathfrak{I}+i_0}) + \frac{1}{2}\operatorname{tr}(y)(e_{i_0 i_0} + e_{\mathfrak{I}+i_0,\mathfrak{I}+i_0})$$

or

$$y = y - \operatorname{tr}(y)e_{2\mathfrak{I}+1,2\mathfrak{I}+1} + \operatorname{tr}(y)e_{2\mathfrak{I}+1,2\mathfrak{I}+1}.$$

Hence, (36) holds. □

Corollary 5.12. *Let $x \in T^+$ or $x \in T^- \setminus F\iota$. Then, some $0 \neq h \in \mathfrak{h}^{\pm}$ exists such that $\mathscr{B}(x,h) \neq 0$.*

Proof. By (36) in Lemma 5.11, a finite subset $I \subset \mathfrak{I}$ and $0 \neq h' \in \mathfrak{h}^{\pm}_{2I}$ or $\mathfrak{h}^{\pm}_{2I+1}$ exist such that $x = h' + b\iota_{2I} + x'$ or $x = h' + b\iota_{2I+1} + x'$ for some $b \in F$ and $x' \in T_{2\mathfrak{I}\setminus 2I}$ or $T_{2\mathfrak{I}+1\setminus 2I+1}$ (since $x \notin F\iota$). Since the trace form is nondegenerate on $\mathfrak{h}^{\pm}_{2I}$ or $\mathfrak{h}^{\pm}_{2I+1}$, we can select $h \in \mathfrak{h}^{\pm}_{2I}$ or $\mathfrak{h}^{\pm}_{2I+1}$ such that $\operatorname{tr}(h'h) \neq 0$. Then, we have $\mathscr{B}(x,h) = \operatorname{tr}(h'h) + b\operatorname{tr}(h) + \operatorname{tr}(x'h) = \operatorname{tr}(h'h) \neq 0$ (since $x'h = 0$). □

By Corollary 5.12 related to T^+, we can also see that $\mathscr{B}^{\sigma}$ is nondegenerate. (We use the result related to T^- later.) Moreover, the restriction of $\mathscr{B}^{\sigma}$ to any subalgebra $\mathscr{L}$ of $\mathscr{M}^{\sigma}_{\mathfrak{K}}$ that contains $\mathrm{sl}_{\mathfrak{K}}(F)^{\sigma}$ is still a nondegenerate form.

Conversely, let U be a complement of $\mathfrak{h}^{\sigma}$ in $\mathscr{L} \cap T^{\sigma}$, and φ is an arbitrary symmetric bilinear form on U. Then, we can extend φ to a nondegenerate form on $\mathscr{L}$, using Lemma 5.11 (or embedding $\mathscr{L}$ into $\mathscr{M}_{\mathfrak{K}}$) and Corollary 5.12 again. Consequently, we can say that a LEALA of type $\mathrm{X}_{\mathfrak{I}} \neq \mathrm{A}_{\mathfrak{I}}$ of nullity 0 is isomorphic to a subalgebra of $\mathscr{M}^{\sigma}_{\mathfrak{K}}$ that contains $\mathrm{sl}_{\mathfrak{K}}(F)^{\sigma}$.

6. LALAs

The next interesting objects are LEALAs of null dimension 1. In fact, our aim in this study is to classify tame LEALAs of nullity 1.

Definition 6.1. A tame LEALA of nullity 1 is called a **LALA**.

We know that the core of a LALA is a locally Lie G-torus (see (28)), and since $R^{\times}$ is a LEARS of nullity 1, the core is a locally Lie 1-torus. Moreover, using the notations in Section 4, we have the following.

Lemma 6.2. *Let $\mathscr{L}$ be a LALA. Then:*

(1) The core $\mathscr{L}_c$ is a universal covering of a locally loop algebra.

(2) $Z(\mathscr{L}) = Z(\mathscr{L}_c)$, and a natural embedding $\operatorname{ad}\mathscr{L} \hookrightarrow \operatorname{Der}_F\big(\mathscr{L}_c/Z(\mathscr{L}_c)\big)$ exists.

In particular, if $\mathscr{N}$ is a complement of the core $\mathscr{L}_c$, i.e., $\mathscr{L} = \mathscr{L}_c \oplus \mathscr{N}$, then $\mathscr{N}$ can be identified with a subspace of $\operatorname{Oder}_F\big(\mathscr{L}_c/Z(\mathscr{L}_c)\big)$, i.e., the outer derivations of a locally loop algebra.

Proof. By Proposition 4.9, $\mathscr{L}_c$ has a nontrivial center. Hence, by Theorem 3.8, (1) is true. For (2), we have $Z(\mathscr{L}_c) = Z(\mathscr{L})$ by Proposition 4.9. Since $\operatorname{ad}\mathscr{L} \cong \mathscr{L}/Z(\mathscr{L})$, we obtain the embedding using Lemma 4.11. □

To complete the classification of LALAs, we need to classify a complement of the core. First, we give some examples of LALAs.

Example 6.3. Let $\mathfrak{I}$ be an arbitrary index set. We can construct 14 minimal standard LALAs (see Defintion 4.13 and 4.14) from the 14 locally loop algebras $L(\mathrm{X}_{\mathfrak{I}}^{(i)})$ given in Section 3. Thus,

$$\mathscr{L}^{ms} = \mathscr{L}^{ms}(\mathrm{X}_{\mathfrak{I}}^{(i)}) := L(\mathrm{X}_{\mathfrak{I}}^{(i)}) \oplus Fc \oplus Fd^{(0)}$$

is a LALA of type $\mathrm{X}_{\mathfrak{I}}^{(i)}$, where c is central and $d^{(0)}$ is the degree derivation, i.e.,

$$d^{(0)}(t^m) = mt^m$$

with a Cartan subalgebra

$$\mathscr{H} = \mathfrak{h} \oplus Fc \oplus Fd^{(0)},$$

where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}(\mathrm{X}_{\mathfrak{I}})$, which comprises diagonal matrices if $\mathfrak{I}$ is infinite or any Cartan subalgebra if $\mathfrak{I}$ is finite. In addition, a nondegenerate invariant symmetric bilinear form $\mathscr{B}$ on $\mathscr{L}^{ms}$ is an extension of the form defined in Section 3 for loop algebras using the trace form or the Killing form if $\mathfrak{I}$ is finite, and a nondegenerate symmetric associative bilinear form on $F[t^{\pm 1}]$, and by defining $\mathscr{B}(c, d^{(0)}) = 1$. In particular, we define $\mathscr{B}(d^{(0)}, d^{(0)}) = 0$ as usual, although $\mathscr{B}(d^{(0)}, d^{(0)})$ can be any number in F. These LALAs are minimal LALAs. Note that any standard LALA contains a minimal standard LALA. In addition, we note that if $\mathfrak{I}$ is finite, then LALAs are automatically minimal standard LALAs, which are the affine (Kac-Moody) Lie algebras. Note that a minimal standard LALA $\mathscr{L}^{ms}$ is also denoted by $\mathscr{L}(0)$.

Now, we give examples of bigger (and the biggest) LALAs when $\mathfrak{I}$ is infinite. Note that

$$\mathrm{sl}_{\mathfrak{I}}(F) + T = \mathrm{gl}_{\mathfrak{I}}(F) + T,$$

where $T = T_{\mathfrak{I}}$ is the subspace of all the diagonal matrices in the matrix space $M_{\mathfrak{I}}(F)$ of size $\mathfrak{I}$, which is a Lie algebra with the split center $F\iota$, where ι is the diagonal matrix and its diagonal entries are all 1. Thus, its loop algebra

$$\mathscr{U} = \mathscr{U}_{\mathfrak{I}} := \big(\mathrm{sl}_{\mathfrak{I}}(F) + T\big) \otimes F[t^{\pm 1}] \tag{37}$$

is a Lie algebra with the split center $\iota \otimes F[t^{\pm 1}]$.

Assume that $\mathscr{B}$ is a symmetric invariant bilinear form on $\mathscr{U}$, which is not a zero on $\mathrm{sl}_{\mathfrak{I}}(F)$. Then, by Lemma 3.9 and Lemma 5.2, $\mathscr{B}$ is unique up to a scalar to $\mathrm{tr} \otimes \varepsilon$ on

$$\big(\mathrm{sl}_{\mathfrak{I}}(F) \otimes F[t^{\pm 1}]\big) \times \mathscr{U} \quad \text{and} \quad \mathscr{U} \times \big(\mathrm{sl}_{\mathfrak{I}}(F) \otimes F[t^{\pm 1}]\big), \tag{38}$$

i.e., for $x, y \in \mathscr{U}$, and if x or $y \in \mathrm{sl}_{\mathfrak{I}}(F)$, then

$$\mathscr{B}(x \otimes t^m, y \otimes t^n) = a\,\mathrm{tr}(xy)\delta_{n,-m} \tag{39}$$

for some $a \in F^{\times}$. We claim that such a form $\mathscr{B}$ does exist. As in the case of nullity 0, we select a complement $\mathfrak{h}^c$ of $\mathfrak{h}$ in T, i.e., $T = \mathfrak{h}^c \oplus \mathfrak{h}$. For each $m \in \mathbb{Z}$, let

$$\psi_m : \mathfrak{h}^c \times \mathfrak{h}^c \longrightarrow F$$

be an arbitrary bilinear form. We define a symmetric bilinear form $\mathscr{B}$ on $\mathscr{U}$ as

$$\mathscr{B}(x \otimes t^m, y \otimes t^n) = \psi_m(x, y)\delta_{n,-m}$$

on $\mathfrak{h}^c \otimes F[t^{\pm 1}]$, and (39) on (38). We can prove that $\mathscr{B}$ is invariant in a similar manner to the case of nullity 0 using the following claim (which can also be proved in a similar manner to Claim 5.5).

Claim 6.4. *Let $x \in T \setminus F\imath$ and $y_k \in \mathrm{sl}_{\mathfrak{J}}(F)$ for $k = 1, 2, \dots, r$. Then, a finite subset I of $\mathfrak{J}$, $0 \neq h \in \mathfrak{h}$ exist and $g \in T$ such that $y_k \in \mathrm{sl}_I(F)$ for all k, and $h \in \mathfrak{h}_I$,*

$$x = h + g, \quad [x \otimes t^m, y_k \otimes t^n] = [h \otimes t^m, y_k \otimes t^n] \quad \text{and} \quad \mathscr{B}(x \otimes t^m, y_k \otimes t^n) = \mathscr{B}(h \otimes t^m, y_k \otimes t^n)$$

for all $m, n \in \mathbb{Z}$ and all k. Moreover, $y \in \mathrm{sl}_{\mathfrak{J}}(F)$ and $h' \in \mathfrak{h}$ exist such that

$$[x \otimes t^m, y \otimes t^n] \neq 0 \quad \text{and} \quad \mathscr{B}(x \otimes t^m, h' \otimes t^{-m}) \neq 0. \tag{40}$$

□

Now, we can use a general construction, i.e., a one-dimensional central extension by the 2-cocycle

$$\varphi(u, v) := \mathscr{B}\big(d^{(0)}(u), v\big)$$

for $u, v \in \mathscr{U}$, where $d^{(0)}$ is the degree derivation on $\mathscr{U}$. This is well known (e.g., see [AABGP]), but for convenience, we show that φ is a 2-cocycle in a slightly more general setup. Note that $d^{(0)}$ is a skew derivation relative to $\mathscr{B}$, i.e.,

$$\mathscr{B}\big(d^{(0)}(u), v\big) = -\mathscr{B}\big(u, d^{(0)}(v)\big).$$

More generally, for a $\mathbb{Z}$-graded algebra $A = \bigoplus_{m \in \mathbb{Z}} A_m$ with a symmetric **graded bilinear form** ψ, the degree derivation $d^{(0)}$ is skew relative to ψ. In fact, for $x = \sum_m x_m$ and $y = \sum_m y_m \in A$, we have $\psi\big(d^{(0)}(x), y\big) = \sum_m m\psi(x_m, y) = \sum_m m\psi(x_m, y_{-m}) = \sum_m m\psi(x, y_{-m}) = -\sum_m m\psi(x, y_m) = -\psi\big(x, d^{(0)}(y)\big)$. Hence, $d^{(0)}$ is skew.

In general, on a Lie algebra L with a symmetric invariant bilinear form B, we can define $\varphi(u, v) := B\big(d(u), v\big)$ for any skew derivation d and $u, v \in L$. Then, $\varphi(u, v)$ is a 2-cocycle (which is also well known). In fact, the first condition of the cocycle clearly holds, i.e., $\varphi(u, u) = 0$ for all $u \in L$, since $\varphi(u, u) = B(d(u), u) = -B(u, d(u)) = -B(d(u), u) = -\varphi(u, u)$. For the second condition, we have

$$\begin{aligned}
&\varphi([u, v], w) + \varphi([v, w], u) + \varphi([w, u], v) \\
&= B\big(d([u, v]), w\big) - B\big([v, w], d(u)\big) - B\big([w, u], d(v)\big) \\
&= B\big([d((u), v]), w\big) + B\big([u, d(v)]), w\big) - B\big([v, w], d(u)\big) - B\big([w, u], d(v)\big) \\
&= B\big(d(u), [v, w]\big) - B\big(d(v), [u, w]\big) - B\big([v, w], d(u)\big) - B\big([w, u], d(v)\big) = 0.
\end{aligned}$$

Thus, we obtain a 1-dimensional central extension

$$\tilde{\mathscr{U}} := \mathscr{U} \oplus Fc$$

using the 2-cocyle $\varphi(u, v) = \mathscr{B}\big(d^{(0)}(u), v\big)$ given above. Then,

$$\hat{\mathscr{U}} = \hat{\mathscr{U}}_{\mathfrak{J}} := \tilde{\mathscr{U}} \oplus Fd^{(0)}$$

is naturally a Lie algebra that defines

$$[c, d^{(0)}] = 0,$$

anti-symmetrically. Thus, the center of $\hat{\mathscr{U}}$ is equal to $Fc \oplus F\imath$. We also extend the form $\mathscr{B}$ by

$$\mathscr{B}(c, d^{(0)}) = 1 \quad \text{and} \quad \mathscr{B}(\mathscr{U}, d^{(0)}) = 0,$$

symmetrically (where the value of $\mathscr{B}(d^{(0)}, d^{(0)})$ can be any). Then, we can also check that this extended form is invariant.

Let $\mathfrak{g} := \mathrm{sl}_{\mathfrak{J}}(F)$ and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta_{\mathfrak{J}} \subset \mathfrak{h}^*} \mathfrak{g}_\mu$$

be the root-space decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let

$$\mathscr{H} := T \oplus Fc \oplus Fd^{(0)}.$$

We extend each root $\mu \in \mathfrak{h}^*$ to an element in $\mathscr{H}^*$ as follows. First, we can extend μ to T as in the case of nullity 0. Then, we define $\mu(Fc \oplus Fd^{(0)}) = 0$. We also define $\delta \in \mathscr{H}^*$ as $\delta(T \oplus Fc) = 0$ and $\delta(d^{(0)}) = 1$. Then, $\hat{\mathscr{U}}$ has the root space decomposition

$$\hat{\mathscr{U}} = \bigoplus_{\xi \in \mathscr{H}^*} \hat{\mathscr{U}}_\xi$$

relative to $\mathscr{H}$, where $\hat{\mathscr{U}}_{\mu+m\delta} = \mathfrak{g}_\mu \otimes t^m$ for $\mu \in \mathrm{A}_\mathfrak{I}$, $\hat{\mathscr{U}}_{m\delta} = T \otimes t^m$ for $m \neq 0$ and $\hat{\mathscr{U}}_0 = \mathscr{H}$, and $\hat{\mathscr{U}}_\xi = 0$ if $\xi \notin \mathrm{A}_\mathfrak{I}^{(1)} = (\mathrm{A}_\mathfrak{I} \cup \{0\}) + \mathbb{Z}\delta$. For convenience, we assume that $0 \notin \mathrm{A}_\mathfrak{I}$ but $0 \in \mathrm{A}_\mathfrak{I}^{(1)}$. In the above, $\hat{\mathscr{U}}$ is an $\langle \mathrm{A}_\mathfrak{I}^{(1)} \rangle$-graded Lie algebra, and $\mathscr{B}$ is graded in the sense that $\mathscr{B}(\hat{\mathscr{U}}_\xi, \hat{\mathscr{U}}_\eta) = 0$, unless $\xi + \eta = 0$ for all $\xi, \eta \in \mathrm{A}_\mathfrak{I}^{(1)}$. In particular, the radical of $\mathscr{B}$ is graded.

Claim 6.5. *The radical of $\mathscr{B}$ is contained in $\iota \otimes F[t^{\pm 1}]$.*

Proof. Since the radical of $\mathscr{B}$ is graded, we can check the nondegeneracy for each homogeneous element. It is clear that the elements of degree $\mu + m\delta$ for $\mu \in \mathrm{A}_\mathfrak{I}$ cannot be in the radical. The elements of degree $m\delta$ are also outside of the radical by (40). Hence, the radical should be in $\iota \otimes F[t^{\pm 1}]$. □

Now, it is easy to check that $(\hat{\mathscr{U}}, \mathscr{H}, \mathscr{B})$ is a LEALA of nullity 1 by defining $\psi_0(\iota, \iota) \neq 0$. Since the center of $\hat{\mathscr{U}}$ is equal to $Fc \oplus F\iota$, this is not tame. However, since $\iota \otimes F[t^{\pm 1}]$ is an ideal of $\hat{\mathscr{U}}$, the quotient LEALA

$$\mathscr{L}^{max} := \hat{\mathscr{U}} / \big(\iota \otimes F[t^{\pm 1}]\big)$$

is tame, by defining $\psi_0(\iota, \iota) = 0$. Thus, $\mathscr{L}^{max}$ is a LALA, which is isomorphic to the Lie algebra (3) described in the Introduction. The core $\mathscr{L}_c^{max}$ is equal to $\mathrm{sl}_\mathfrak{I}(F) \otimes F[t^{\pm 1}] \oplus Fc$. (As stated in the Introduction, we omit bars for the quotient Lie algebra.) Moreover, it is easy to check that a 1-dimensional extension of the core, such as

$$\mathscr{L}(p) = \mathscr{L}_c^{max} \oplus F(d^{(0)} + p)$$

for some $p \in T$, is a minimal LALA of type $\mathrm{A}_\mathfrak{I}^{(1)}$ (which is a subalgebra of $\mathscr{L}^{max}$). In addition, we can show that any homogeneous subalgebra of $\mathscr{L}^{max}$ that contains some $\mathscr{L}(p)$ is a LALA. In Section 6, we show that any LALA of type $\mathrm{A}_\mathfrak{I}^{(1)}$ is a homogeneous subalgebra of $\mathscr{L}^{max}$ that contains some $\mathscr{L}(p)$.

We describe the other untwisted LALAs using $\hat{\mathscr{U}}_{2\mathfrak{I}}$ and $\hat{\mathscr{U}}_{2\mathfrak{I}+1}$, and the automorphism σ is again defined in (16). Let

$$T = T^\sigma \oplus T^-$$

be the decomposition of $T = T_{2\mathfrak{I}}$ or $T_{2\mathfrak{I}+1}$, where T^σ is the eigenspace of eigenvalue 1 (the fixed algebra of T by σ) and T^- is the eigenspace of eigenvalue -1. Instead of T^σ, we use T^+ because we consider the fixed algebra by another automorphism τ later. Thus, we have

$$T^\sigma = \{(a_{kk}) \in T_{2\mathfrak{I}+1} \mid a_{ii} = -a_{\mathfrak{I}+i,\mathfrak{I}+i}\ (\forall i \in \mathfrak{I}),\ a_{2\mathfrak{I}+1,2\mathfrak{I}+1} = 0\} \quad \text{and}$$
$$T^- = \{(a_{kk}) \in T_{2\mathfrak{I}+1} \mid a_{ii} = a_{\mathfrak{I}+i,\mathfrak{I}+i}\ (\forall i \in \mathfrak{I})\} \quad \text{for } \mathrm{B}_{\mathfrak{I}}^{(1)},$$
$$T^\sigma = \{(a_{kk}) \in T_{2\mathfrak{I}} \mid a_{ii} = -a_{\mathfrak{I}+i,\mathfrak{I}+i}\ (\forall i \in \mathfrak{I})\} \quad \text{and}$$
$$T^- = \{(a_{kk}) \in T_{2\mathfrak{I}} \mid a_{ii} = a_{\mathfrak{I}+i,\mathfrak{I}+i}\ (\forall i \in \mathfrak{I})\} \quad \text{for } \mathrm{C}_{\mathfrak{I}}^{(1)} \text{ or } \mathrm{D}_{\mathfrak{I}}^{(1)}. \tag{41}$$

Note that $F\iota$ is σ-invariant and $F\iota \subset T^-$. Since $\mathfrak{h}$ is σ-invariant, we have

$$\mathfrak{h} = \mathfrak{h}^\sigma \oplus \mathfrak{h}^-, \quad \mathfrak{h}^\sigma = \mathfrak{h} \cap T^\sigma \subset T^\sigma \quad \text{and} \quad \mathfrak{h}^- = \mathfrak{h} \cap T^- \subset T^-.$$

Let us extend the automorphism on $\hat{\mathscr{U}} = \hat{\mathscr{U}}_{2\mathfrak{I}}$ or $\hat{\mathscr{U}}_{2\mathfrak{I}+1}$ as

$$\hat{\sigma}(x \otimes t^k) := \sigma(x) \otimes t^k, \quad \hat{\sigma}(c) := c \quad \text{and} \quad \hat{\sigma}(d^{(0)}) := d^{(0)}.$$

Then, the fixed algebra $\hat{\mathscr{U}}^{\hat{\sigma}}$ with the restriction of the form $\mathscr{B}$ is a LALA of type $\mathrm{B}_{\mathfrak{I}}^{(1)}$, $\mathrm{C}_{\mathfrak{I}}^{(1)}$ or $\mathrm{D}_{\mathfrak{I}}^{(1)}$, depending on the type of σ. In particular,

$$\hat{\mathscr{U}}^{\hat{\sigma}} = \big((\mathfrak{g} + T^\sigma) \otimes F[t^{\pm 1}]\big) \oplus Fc \oplus Fd^{(0)},$$

where $\mathfrak{g} = \mathrm{sl}_{2\mathfrak{I}+1}(F)^\sigma$ or $\mathrm{sl}_{2\mathfrak{I}}(F)^\sigma$ is a locally finite split simple Lie algebra of each type.

The nondegeneracy of the restricted form $\mathscr{B}$ follows from the next lemma, where the proof is similar to the case in nullity 0.

Lemma 6.6. *Let $0 \neq x \in T^\sigma$ or $x \in T^- \setminus F\iota$. Then, some $h \in \mathfrak{h}^\sigma$ or $h \in \mathfrak{h}^-$ exist such that*

$$\mathscr{B}(x \otimes t^m, h \otimes t^{-m}) \neq 0$$

for all $m \in \mathbb{Z}$. □

As in the case of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$, a 1-dimensional extension of the core $\hat{\mathscr{U}}_c^{\hat{\sigma}}$, such as

$$\mathscr{L}(p) = \hat{\mathscr{U}}_c^{\hat{\sigma}} \oplus F(d^{(0)} + p)$$

for some $p \in T^\sigma$, is a minimal LALA of each type. In addition, we can check that any homogeneous subalgebra of $\mathscr{L}^{max} = \hat{\mathscr{U}}^{\hat{\sigma}}$ that contains some $\mathscr{L}(p)$ is a LALA of each type. In Section 6, we show that any LALA of each type is a homogeneous subalgebra of $\mathscr{L}^{max} = \hat{\mathscr{U}}^{\hat{\sigma}}$ that contains some $\mathscr{L}(p)$.

We now give examples of twisted LALAs. Again, we use the automorphism σ defined in (16) to obtain the type $\mathrm{C}_{\mathfrak{I}}$ or $\mathrm{B}_{\mathfrak{I}}$, and we extend the automorphism on $\hat{\mathscr{U}} = \hat{\mathscr{U}}_{2\mathfrak{I}}$ or $\hat{\mathscr{U}}_{2\mathfrak{I}+1}$ as

$$\hat{\sigma}(x \otimes t^k) := (-1)^k \sigma(x) \otimes t^k, \quad \hat{\sigma}(c) = c \quad \text{and} \quad \hat{\sigma}(d^{(0)}) := d^{(0)}. \tag{42}$$

Then, the fixed algebra $\hat{\mathscr{U}}^{\hat{\sigma}}$ with the restriction of the form $\mathscr{B}$ is a LALA of type $\mathrm{C}_{\mathfrak{I}}^{(2)}$ or $\mathrm{BC}_{\mathfrak{I}}^{(2)}$, depending on the type of σ. In particular,

$$\hat{\mathscr{U}}^{\hat{\sigma}} = \big((\mathfrak{g}^\sigma \oplus T^\sigma) \otimes F[t^{\pm 2}]\big) \oplus \big((\mathfrak{g}^- + T^-) \otimes tF[t^{\pm 2}]\big) \oplus Fc \oplus Fd^{(0)}, \tag{43}$$

where $\mathfrak{g}^\sigma = \mathrm{sl}_{2\mathfrak{I}}(F)^\sigma = \mathrm{sp}_{2\mathfrak{I}}(F)$ or $\mathrm{sl}_{2\mathfrak{I}+1}(F)^\sigma = \mathrm{o}_{2\mathfrak{I}+1}(F)$, and $\mathfrak{g}^-$ is the minus space of $\mathrm{sl}_{2\mathfrak{I}}(F)$ or $\mathrm{sl}_{2\mathfrak{I}+1}(F)$ by σ. Since $\iota \otimes tF[t^{\pm 2}]$ is an ideal of $\hat{\mathscr{U}}$, the quotient LEALA

$$\overline{\hat{\mathscr{U}}^{\hat{\sigma}}} := \hat{\mathscr{U}}^{\hat{\sigma}} / \big(\iota \otimes tF[t^{\pm 2}]\big)$$

is tame, by defining $\psi_0(\iota, \iota) = 0$. Thus, $\overline{\hat{\mathscr{U}}^{\hat{\sigma}}}$ is a LALA. (For the type $\mathrm{C}_{\mathfrak{I}}^{(2)}$, this is isomorphic to the Lie algebra (5) described in the Introduction.)

The nondegeneracy of the restricted form $\mathscr{B}$ follows from Lemma 6.6. As in the untwisted case, a 1-dimensional extension of the core $(\overline{\hat{\mathscr{U}}^{\hat{\sigma}}})_c$, such as

$$\mathscr{L}(p) := (\overline{\hat{\mathscr{U}}^{\hat{\sigma}}})_c \oplus F(d^{(0)} + p)$$

for some $p \in T^{\sigma}$, is a minimal LALA of each type. We can also show that any homogeneous subalgebra of $\mathscr{L}^{max} := \overline{\hat{\mathscr{U}}^{\hat{\sigma}}}$ that contains some $\mathscr{L}(p)$ is a LALA of each type. In Section 7, we show that any LALA of each type is a homogeneous subalgebra of $\mathscr{L}^{max}$ that contains some $\mathscr{L}(p)$.

For the type $\mathrm{B}_{\mathfrak{J}}^{(2)}$, as described by Neeb in [N2, App.1], we introduce an automorphism τ on the untwisted LALA $\mathscr{M} := \hat{\mathscr{U}}_{2\mathfrak{J}+2}^{\hat{\sigma}}$ of type $\mathrm{D}_{\mathfrak{J}+1}^{(1)}$, which is defined by $s = \begin{pmatrix} 0 & \iota_{\mathfrak{J}+1} \\ \iota_{\mathfrak{J}+1} & 0 \end{pmatrix}$. For convenience, let $\mathfrak{J}+1 = \{j \mid j \in \mathfrak{J}\} \cup \{j_0\}$ and

$$2\mathfrak{J}+2 = (\mathfrak{J}+1)+(\mathfrak{J}+1) = \big(\{j \mid j \in \mathfrak{J}\} \cup \{j_0\}\big) \cup \big(\{-j \mid j \in \mathfrak{J}\} \cup \{-j_0\}\big).$$

Let

$$g = \iota_{2\mathfrak{J}} + e_{j_0,-j_0} + e_{-j_0,j_0}$$

be the matrix of exchanging rows or columns, and let τ be an involutive automorphism of $\mathrm{o}_{2\mathfrak{J}+2}(F)$ defined by

$$\tau(x) = gxg.$$

Then, we can see that the fixed algebra $\mathrm{o}_{2\mathfrak{J}+2}(F)^{\tau} = \mathrm{o}_{2\mathfrak{J}+1}(F)$ (which is of type $\mathrm{B}_{\mathfrak{J}}$) and the minus space

$$\mathfrak{s} := \{x \in \mathrm{o}_{2\mathfrak{J}+2}(F) \mid \tau(x) = -x\} \tag{44}$$

by τ is isomorphic to $F^{2\mathfrak{J}+1}$ as a natural $\mathrm{o}_{2\mathfrak{J}+1}(F)$-module with

$$\mathfrak{s}_0 = \mathfrak{s} \cap \mathfrak{h}^{\sigma} = F(e_{j_0 j_0} - e_{-j_0,-j_0}).$$

We can extend τ on $\mathrm{o}_{2\mathfrak{J}+2}(F) + T_{2\mathfrak{J}+2}^{\sigma}$. Then, we have

$$\big(T_{2\mathfrak{J}+2}^{\sigma}\big)^{\tau} = T_{2\mathfrak{J}+1}^{\sigma} (\cong T_{2\mathfrak{J}}^{\sigma}) \quad \text{and} \quad \{x \in T_{2\mathfrak{J}+2}^{\sigma} \mid \tau(x) = -x\} = \mathfrak{s}_0. \tag{45}$$

We can also extend τ on $\mathscr{M}$ in the same manner as (42), i.e.,

$$\hat{\tau}(x \otimes t^k) := (-1)^k \tau(x) \otimes t^k, \quad \hat{\tau}(c) := c \quad \text{and} \quad \hat{\tau}(d^{(0)}) := d^{(0)},$$

and we obtain a LALA $\mathscr{M}^{\hat{\tau}}$ of type $\mathrm{B}_{\mathfrak{J}}^{(2)}$. In particular, we have

$$\mathscr{M}^{\hat{\tau}} = \big((\mathrm{o}_{2\mathfrak{J}+1}(F) + T_{2\mathfrak{J}+1}^{\sigma}) \otimes F[t^{\pm 2}]\big) \oplus \big(\mathfrak{s} \otimes tF[t^{\pm 2}]\big) \oplus Fc \oplus Fd^{(0)}.$$

(The odd degree part of t is the same as that in an affine Lie algebra of type $\mathrm{B}_{\ell}^{(2)} = \mathrm{D}_{\ell+1}^{(2)}$.)

The nondegeneracy of the restricted form $\mathfrak{B}$ follows from Lemma 6.6. As in the above, a 1-dimensional extension of the core $\mathscr{M}_c^{\hat{\tau}}$, such as

$$\mathscr{L}(p) = \mathscr{M}_c^{\hat{\tau}} \oplus F(d^{(0)} + p)$$

for some $p \in T_{2\mathfrak{J}+1}^{\sigma}$, is a minimal LALA of type $\mathrm{B}_{\mathfrak{J}}^{(2)}$. In addition, we can show that any homogeneous subalgebra of $\mathscr{L}^{max} := \mathscr{M}^{\hat{\tau}}$ that contains some $\mathscr{L}(p)$ is a LALA of type $\mathrm{B}_{\mathfrak{J}}^{(2)}$. In Section 7, we show that any LALA of type $\mathrm{B}_{\mathfrak{J}}^{(2)}$ is a homogeneous subalgebra of $\mathscr{L}^{max}$ that contains some $\mathscr{L}(p)$.

7. CLASSIFICATION OF THE UNTWISTED LALAS

Let $\mathscr{L}$ be an untwisted LALA of infinite rank, i.e., the core $\mathscr{L}_c$ is a universal covering of an untwisted locally loop algebra of type $A_{\mathfrak{J}}^{(1)}$, $B_{\mathfrak{J}}^{(1)}$, $C_{\mathfrak{J}}^{(1)}$ or $D_{\mathfrak{J}}^{(1)}$ for an infinite index $\mathfrak{J}$. By selecting a homogeneous complement of the $\mathbb{Z}$-graded core, we can write

$$\mathscr{L} = \mathscr{L}_c \oplus \bigoplus_{m\in\mathbb{Z}} D^m.$$

Note that the complement is assumed to be included in the null space:

$$\bigoplus_{m\in\mathbb{Z}} D^m \subset \bigoplus_{\delta\in R^0} \mathscr{L}_\delta = \bigoplus_{m\in\mathbb{Z}} \mathscr{L}_{m\delta_1} \quad \text{and} \quad D^m \subset \mathscr{L}_{m\delta_1},$$

where δ_1 is a generator of $\langle R^0\rangle_{\mathbb{Z}} = \langle S+S\rangle_{\mathbb{Z}} \cong \mathbb{Z}$ (see (23) and Lemma 4.7).

Let

$$\mathscr{L}_c' := \mathscr{L}_c / Z(\mathscr{L}_c)$$

be the centerless core. Moreover, let $(\mathfrak{g},\mathfrak{h})$ be the grading pair of the Lie 1-torus $\mathscr{L}_c$ such that $\mathfrak{h}$ is the set of diagonal matrices of a locally finite split simple Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha = [\mathscr{L}_c^0, \mathscr{L}_c^0] \subset \mathscr{L}_c = \bigoplus_{m\in\mathbb{Z}} \mathscr{L}_c^m,$$

where

$$\mathscr{L}_c^m = \bigoplus_{\alpha\in\Delta\cup\{0\}} (\mathscr{L}_c)_\alpha^m.$$

We identify the grading pair $(\mathfrak{g},\mathfrak{h})$ of the Lie 1-torus $\mathscr{L}_c'$ and $\mathscr{L}_c$. Moreover, we identify $\mathscr{L}_c'$ with

$$L := \mathfrak{g} \otimes F[t^{\pm 1}].$$

Now, we classify the **diagonal derivations** of an untwisted locally loop algebra L in general. Let

$$(\mathrm{Der}_F L)_0^0 = \{d \in \mathrm{Der}_F L \mid d(\mathfrak{g}_\alpha \otimes t^m) \subset \mathfrak{g}_\alpha \otimes t^m \text{ for all } \alpha \in \Delta \text{ and } m \in \mathbb{Z}\}.$$

We refer to such an element as a **diagonal derivation of degree** 0. Let $d \in (\det_F L)_0^0$. Note that since $\mathfrak{g}_0 = \mathfrak{h} = \sum_{\alpha\in\Delta}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, then we have

$$\begin{aligned} d(\mathfrak{h}\otimes t^m) &= \sum_{\alpha\in\Delta} d([\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]\otimes t^m) = \sum_{\alpha\in\Delta} d([\mathfrak{g}_\alpha\otimes t^m, \mathfrak{g}_{-\alpha}\otimes 1]) \\ &= \sum_{\alpha\in\Delta} \big([d(\mathfrak{g}_\alpha\otimes t^m), \mathfrak{g}_{-\alpha}\otimes 1] + [\mathfrak{g}_\alpha\otimes t^m, d(\mathfrak{g}_{-\alpha}\otimes 1)]\big) \\ &\subset \sum_{\alpha\in\Delta} [\mathfrak{g}_\alpha\otimes t^m, \mathfrak{g}_{-\alpha}\otimes 1] = \mathfrak{h}\otimes t^m. \end{aligned}$$

In addition, we note that $d\mid_{\mathfrak{g}}$ is a diagonal derivation of $\mathfrak{g}$. Hence, by Neeb [N1], we obtain $d\mid_{\mathfrak{g}} = \mathrm{ad}\, p$ for a certain diagonal matrix p of an infinite size. In particular, we have $p \in P$, where

$$P = T_{\mathfrak{J}} \text{ for } A_{\mathfrak{J}}, \text{ and } T_{2\mathfrak{J}}^+ \text{ or } T_{2\mathfrak{J}+1}^+ \text{ for the other types} \tag{46}$$

as defined in Example 6.3. Put

$$d' := d - \mathrm{ad}\, p \in (\mathrm{Der}_F L)_0^0.$$

Then, we have

$$d'(\mathfrak{g}\otimes 1) = 0.$$

In particular, we have $d'(\mathfrak{h}\otimes 1) = 0$. Thus, for $0 \neq x\otimes t \in \mathfrak{g}_\alpha\otimes t$, if

$$d'(x\otimes t) = ax\otimes t \tag{47}$$

for $a \in F$, then

$$d'(y \otimes t^{-1}) = -ay \otimes t^{-1} \tag{48}$$

for all $y \in \mathfrak{g}_{-\alpha}$. In fact, for $y \neq 0$, since $0 \neq [x,y] = h \in \mathfrak{h}$ and $d'(y \otimes t^{-1}) = by \otimes t^{-1}$ for some $b \in F$, then we have

$$\begin{aligned} 0 = d'(h \otimes 1) = d'([x \otimes t, y \otimes t^{-1}]) &= [d'(x \otimes t), y \otimes t^{-1}] + [x \otimes 1, d'(y \otimes t^{-1})] \\ = (a+b)[x \otimes t, y \otimes t^{-1}] &= (a+b)[x,y] \otimes 1 = (a+b)h \otimes 1. \end{aligned}$$

Hence, $b = -a$.

Lemma 7.1. *Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be a locally finite split simple Lie algebra. Then, $\mathscr{U}(\mathfrak{g}).\mathfrak{g}_\beta = \mathfrak{g}$ for any $\beta \in \Delta$, where $\mathscr{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.*

Proof. Since $\mathscr{U}(\mathfrak{g}).\mathfrak{g}_\beta$ is a nonzero ideal of $\mathfrak{g}$, it must be equal to $\mathfrak{g}$ by simplicity. □

By Lemma 7.1, for a fixed $\alpha \in \Delta$, three subspaces

$$\mathfrak{g} \otimes 1, \quad \mathfrak{g}_\alpha \otimes t, \quad \text{and} \quad \mathfrak{g}_{-\alpha} \otimes t^{-1}$$

generate L as a Lie algebra.

Let

$$d'' := d' - ad^{(0)},$$

where $d^{(0)} = t\dfrac{d}{dt}$. Then, we have $d''(\mathfrak{g} \otimes 1) = d'(\mathfrak{g} \otimes 1) = 0$ and using (47),

$$d''(x \otimes t) = d'(x \otimes t) - ax \otimes t = 0$$

for $x \in \mathfrak{g}_\alpha$. Similarly, using (48),

$$d''(y \otimes t^{-1}) = d'(y \otimes t^{-1}) + ay \otimes t^{-1} = 0$$

for $y \in \mathfrak{g}_{-\alpha}$. Thus, we have $d''(L) = 0$ and $d'' = 0$. Hence, we obtain

$$d = \operatorname{ad} p + ad^{(0)}, \quad \text{and} \quad (\operatorname{Der}_F L)_0^0 = \operatorname{ad} P \oplus F d^{(0)}. \tag{49}$$

We define the **shift map** s_m for $m \in \mathbb{Z}$ on $L = \mathfrak{g} \otimes F[t^{\pm 1}]$ by

$$s_m(x \otimes t^k) := x \otimes t^{k+m}$$

for all $k \in \mathbb{Z}$. (Shift maps were discussed in the classification of affine Lie algebras by Moody in [Mo].) Clearly, the shift maps have the property

$$s_m([x,y]) = [s_m(x), y] = [x, s_m(y)]$$

for $x, y \in L$. (In other words, the shift maps are in the centroid of L.) Thus, $s_m \circ d$ is a derivation for any derivation d of L. In fact, for $x, y \in L$,

$$s_m \circ d([x,y]) = s_m([d(x), y] + [x, d(y)]) = [s_m \circ d(x), y] + [x, s_m \circ d(y)].$$

Now, let

$$d \in (\operatorname{Der}_F L)_0^m = \{d \in \operatorname{Der}_F L \mid d(\mathfrak{g}_\alpha \otimes t^k) \subset \mathfrak{g}_\alpha \otimes t^{k+m} \text{ for all } \alpha \in \Delta \text{ and } k \in \mathbb{Z}\}.$$

Then, we have

$$s_{-m} \circ d \in (\operatorname{Der}_F L)_0^0.$$

Hence, by (49), $p = p_d \in P$ and some $a = a_d \in F$ exist such that

$$s_{-m} \circ d = \operatorname{ad} p + ad^{(0)},$$

and thus

$$d = s_m \circ (\operatorname{ad} p + ad^{(0)}).$$

Therefore, we have classified diagonal derivations of the untwisted locally loop algebra. Thus:

Theorem 7.2. *For all $m \in \mathbb{Z}$, we have*

$$(\mathrm{Der}_F L)_0^m = s_m \circ (\mathrm{Der}_F L)_0^0 = s_m \circ (\mathrm{ad}\, P \oplus F d^{(0)}),$$

where P is defined in (46). □

The following property of diagonal derivations is useful later.

Lemma 7.3. *For all $m \in \mathbb{Z}$, let*

$$(\mathrm{Der}'_F L)_0^m := \{d \in (\mathrm{Der}_F L)_0^m \mid s_n \circ d = d \circ s_n \text{ for some } 0 \neq n \in \mathbb{Z}\}$$

and

$$(\mathrm{Der}''_F L)_0^m := \{d \in (\mathrm{Der}_F L)_0^m \mid s_n \circ d = d \circ s_n \text{ for all } n \in \mathbb{Z}\}.$$

Then, we have

$$(\mathrm{Der}'_F L)_0^m = s_m \circ \mathrm{ad}\, P = (\mathrm{Der}''_F L)_0^m.$$

Proof. First, it is clear that

$$(\mathrm{Der}'_F L)_0^m \supset (\mathrm{Der}''_F L)_0^m \supset s_m \circ \mathrm{ad}\, P$$

for all $m \in \mathbb{Z}$. Thus, it is sufficient to show that

$$(\mathrm{Der}'_F L)_0^m \subset s_m \circ \mathrm{ad}\, P. \tag{50}$$

Therefore, let $s_m \circ (\mathrm{ad}\, p + ad^{(0)}) \in (\mathrm{Der}'_F L)_0^m \subset (\mathrm{Der}_F L)_0^m$. Then, for

$$h \otimes t^k \in \mathfrak{h} \otimes t^k \subset L,$$

we have

$$s_n \circ s_m([p + ad^{(0)}, h \otimes t^k]) = s_n(akh \otimes t^{k+m}) = akh \otimes t^{k+m+n}$$

and

$$[s_m \circ (p + ad^{(0)}), h \otimes t^{k+n}] = a(k+n)h \otimes t^{k+n+m}$$

for some $n \neq 0$. Hence, $an = 0$, and we obtain $a = 0$. Therefore, we obtain

$$s_m \circ (\mathrm{ad}\, p + ad^{(0)}) = s_m \circ \mathrm{ad}\, p \in s_m \circ \mathrm{ad}\, P.$$

Thus, we have shown (50). □

Remark 7.4. We can use some results given by Azam related to the derivations of tensor algebras (see [A2, Thm 2.8]). However, their direct application to our tensor algebra $\mathfrak{g} \otimes_F F[t^{\pm 1}]$ yields an isomorphism such that

$$\mathrm{Der}_F(\mathfrak{g} \otimes_F F[t^{\pm 1}]) \cong \mathrm{Der}_F\, \mathfrak{g} \overleftarrow{\otimes}_F F[t^{\pm 1}] \;\oplus\; C(\mathfrak{g}) \overrightarrow{\otimes}_F \mathrm{Der}_F F[t^{\pm 1}],$$

where $C(\mathfrak{g})$ is the centroid of $\mathfrak{g}$ and $\overleftarrow{\otimes}_F$ and $\overrightarrow{\otimes}_F$ are special types of tensor products (since $\mathfrak{g}$ is infinite-dimensional). Thus, we need to perform some more work to obtain our desired form as given above. We only need a special type of subspace, i.e., $(\mathrm{Der}_F L)_0^m$, so we can approach them directly without using Azam's result. In addition, we investigate derivations of twisted locally loop algebras later that are not tensor algebras.

Now, we return to classifying D^m. Let $d \in D^m$. Then, $\operatorname{ad} d \in (\operatorname{Der}_F L)_0^m$ by Lemma 6.2. Hence, by Theorem 7.2, $p = p_d \in P$ (see (46)) and some $a = a_d \in F$ exist such that

$$\operatorname{ad} d = s_m \circ (\operatorname{ad} p + ad^{(0)}).$$

We claim that $a = 0$ for all $m \neq 0$. First, we note that $h,\ h' \in \mathfrak{h}$ exist such that $\operatorname{tr}(hh') \neq 0$. In addition, we have

$$\mathscr{B}(h \otimes t, h' \otimes t^{-1}) = \mathscr{B}(h \otimes t^m, h' \otimes t^{-m}) = c\operatorname{tr}(hh') \neq 0$$

for all $m \in \mathbb{Z}$ and some $0 \neq c \in F$ since $\mathscr{B} = c\operatorname{tr} \otimes \varepsilon$ (see Lemma 3.9 and we note that ε is defined in the previous paragraph). Now, using a pair h and h', we have

$$\begin{aligned}\mathscr{B}([d, h \otimes t], h' \otimes t^{-m-1}) &= a\mathscr{B}(h \otimes t^{m+1}, h' \otimes t^{-m-1})\\ &= a\mathscr{B}(h \otimes t, h' \otimes t^{-1})\end{aligned}$$

while

$$\begin{aligned}\mathscr{B}([d, h \otimes t], h' \otimes t^{-m-1}) &= -\mathscr{B}(h \otimes t, [d, h' \otimes t^{-m-1}])\\ &= a(m+1)\mathscr{B}(h \otimes t, h' \otimes t^{-1}).\end{aligned}$$

Hence, $a = a(m+1)$, i.e., $am = 0$. Thus, $m \neq 0$ implies that $a = 0$.

Moreover, suppose that $a = a_d = 0$ for all $d \in D^0$. Then, $\operatorname{ad} D^0 \subset \operatorname{ad} P$ (see (46)) and for the Cartan subalgebra $\mathscr{H}$ of the original LALA of $\mathscr{L}$, we have $\mathscr{H} = \mathfrak{h} \oplus Fc \oplus D^0$. However, this contradicts the axiom $\mathscr{L}_0 = \mathscr{H}$ since $\left[\mathfrak{h} \otimes F[t^{\pm 1}], \mathscr{H}\right] = 0$. Hence, $p \in P$ exists such that $\operatorname{ad} p + d^{(0)} \in \operatorname{ad} D^0$.

Consequently, we obtain

$$\operatorname{ad} D^m \subset s_m \circ \operatorname{ad} P$$

for $m \neq 0$, and

$$\operatorname{ad} p + d^{(0)} \in \operatorname{ad} D^0 \subset \operatorname{ad} P + Fd^{(0)}$$

for some $p \in P$.

Remark 7.5. In some cases, $d^{(0)} \notin \operatorname{ad} D^0$. Thus, a LALA is not always standard. We can easily construct a non-standard LALA even if $\dim_F D^0 \geq 2$.

Finally, we investigate the bracket on $D := \bigoplus_{m \in \mathbb{Z}} D^m$. Let $D' := \bigoplus_{m \neq 0} D^m$. First, note that $[D', D']$ acts trivially on L since $[\operatorname{ad}(p \otimes t^m), \operatorname{ad}(p' \otimes t^n)] = \operatorname{ad}[p \otimes t^m, p' \otimes t^n] = 0$ in $\operatorname{Der}_F L$. Hence,

$$[D', D'] \subset Fc = Ft_{\delta_1} \subset \mathscr{H},$$

by tameness. In addition, for $d_m \in D^m$ $(m \neq 0)$ and $d_n \in D^n$ $(n \neq 0)$, by the fundamental property (25) of a LEALA (see Lemma 4.5), we have,

$$[d^m, d^n] = \delta_{m,-n}\mathscr{B}(d_m, d_n)t_{m\delta_1} = m\delta_{m,-n}\mathscr{B}(d_m, d_n)t_{\delta_1}.$$

Note that $\mathscr{B}(d_m, d_n)$ can be zero since $h \in \mathfrak{h}$ exists such that $\operatorname{tr}(d_m h) \neq 0$ (and thus $\mathscr{B}(d_m, h) \neq 0$).

Next, since $D^0 \subset \mathscr{H}$, we have $[D^0, D^0] = 0$. Moreover, for $d \in D^0$ such that $\operatorname{ad}_L d = \operatorname{ad}_L p \in D^0$, we have $[d, D^m] = 0$. For the last case, i.e., for $d \in D^0$ such that $\operatorname{ad}_L d = \operatorname{ad}_L p + ad^{(0)} \in \operatorname{ad} D^0$ and $d_m \in D^m$, we have

$$[d, d_m] = [ad^{(0)}, d_m] = amd_m.$$

Now, $\iota \otimes t^m$ centralizes the $\mathscr{L}_c$, and hence we obtain the following identifiication:

$$\mathscr{L} \cong \left(\mathscr{L} + \iota \otimes F[t^{\pm 1}]\right)/\iota \otimes F[t^{\pm 1}].$$

Thus:

Theorem 7.6. *Let $\mathscr{L}$ be an untwisted LALA. Then, $\mathscr{L}$ is isomorphic to that in Example 6.3.* □

8. Classification of the twisted LALAs

As mentioned earlier, each twisted loop algebra M is a subalgebra of an untwisted loop algebra $\tilde{M}$. In particular, we have

$$\begin{aligned} M \text{ has type } \mathrm{B}_{\mathfrak{I}}^{(2)} &\implies \tilde{M} \text{ has type } \mathrm{D}_{\mathfrak{I}+1}^{(1)} \\ M \text{ has type } \mathrm{C}_{\mathfrak{I}}^{(2)} &\implies \tilde{M} \text{ has type } \mathrm{A}_{2\mathfrak{I}}^{(1)} \\ M \text{ has type } \mathrm{BC}_{\mathfrak{I}}^{(2)} &\implies \tilde{M} \text{ has type } \mathrm{A}_{2\mathfrak{I}+1}^{(1)}. \end{aligned}$$

Remark 8.1. In the case where $\mathfrak{I}$ is finite, such as $\mathfrak{I} = \{1,2,...,n\}$, the type $\mathrm{A}_{\mathfrak{I}}$ usually means the Lie algebra $\mathrm{sl}_{\mathfrak{I}+1}(F)$. Therefore, it may be better to write

$$\begin{aligned} L \text{ has type } \mathrm{C}_{\mathfrak{I}}^{(2)} = \mathrm{C}_n^{(2)} &\implies \tilde{L} \text{ has type } \mathrm{A}_{2\mathfrak{I}-1}^{(1)} = \mathrm{A}_{2n-1}^{(1)} \\ L \text{ has type } \mathrm{BC}_{\mathfrak{I}}^{(2)} = \mathrm{BC}_n^{(2)} &\implies \tilde{L} \text{ has type } \mathrm{A}_{2\mathfrak{I}}^{(1)} = \mathrm{A}_{2n}^{(1)} \end{aligned}$$

in order to follow the common notations. However, in this study, we use the type of the Lie algebra $\mathrm{sl}_{\mathfrak{I}}(F)$ as $\mathrm{A}_{\mathfrak{I}}$, instead of $\mathrm{A}_{\mathfrak{I}+1}$ provided that $\mathfrak{I}$ is an infinite set, as mentioned in the Introduction.

First, we provide some basic lemmas for twisted locally loop algebras, as follows:

$$\begin{aligned} &(1) \quad (g^{\sigma} \otimes F[t^{\pm 2}]) \oplus (\mathfrak{g}^{-} \otimes tF[t^{\pm 2}]) \quad \text{for type } \mathrm{C}_{\mathfrak{I}}^{(2)} \text{ or } BC_{\mathfrak{I}}^{(2)}, \quad \text{and} \\ &(2) \quad (o_{2\mathfrak{I}+2}(F)^{\tau} \otimes F[t^{\pm 2}]) \oplus (\mathfrak{s} \otimes tF[t^{\pm 2}]) \quad \text{for type } \mathrm{B}_{\mathfrak{I}}^{(2)} \end{aligned}$$

where in (1) $\mathfrak{g} = \mathrm{sl}_{2\mathfrak{I}+1}(F)$ or $\mathfrak{g} = \mathrm{sl}_{2\mathfrak{I}}(F)$ (see (43) for σ), and in (2) $\mathfrak{s}$ is the minus space of $\mathrm{o}_{2\mathfrak{I}+2}(F)$ by τ, as described in (44). Note that $\mathrm{o}_{2\mathfrak{I}+2}(F)^{\tau} = \mathrm{o}_{2\mathfrak{I}+1}(F)$, which has type $\mathrm{B}_{\mathfrak{I}}$.

Lemma 8.2. (1) *$\mathfrak{g}^{-}$ is an irreducible $\mathfrak{g}^{\sigma}$-module.*
(2) $\mathfrak{s}$ is an irreducible $\mathrm{o}_{2\mathfrak{I}+2}(F)^{\tau}$-module.

Proof. For (1), it is sufficient to show that $w \in \mathscr{U}(\mathfrak{g}^{\sigma})v$ for any $v, w \in \mathfrak{g}^{-}$, where $\mathscr{U}(\mathfrak{g}^{\sigma})$ is the universal enveloping algebra of $\mathfrak{g}^{\sigma}$. However, this is a local property. Thus, a finite-dimensional split simple subalgebra $\mathfrak{f}$ of $\mathfrak{g}$ exists of the same type such that $v, w \in \mathfrak{f}^{-} \subset \mathfrak{g}^{-}$ and $\mathfrak{f}^{\sigma} \subset \mathfrak{g}^{\sigma}$. It is well known that this property holds in the finite-dimensional case (e.g., see [K]). Thus, we are finished. Similarly, (2) holds. □

Lemma 8.3. (1) *Let C be the centralizer of $\mathfrak{g}^{\sigma}$ in $\mathfrak{g}+T$. If $0 \neq x \in C$, then $x \in T^{-} \setminus \mathfrak{g}^{-}$.*
(2) *Let C be the centralizer of $o_{2\mathfrak{I}+2}(F)^{\tau}$ in $\mathrm{sl}_{2\mathfrak{I}+2}(F)^{\sigma} + T_{2\mathfrak{I}+2}^{\sigma} = o_{2\mathfrak{I}+2}(F) + T_{2\mathfrak{I}+2}^{\sigma}$. Then, $C = 0$.*

Proof. For (1), we can write each Lie algebra as

$$\mathfrak{g} + T = (\mathfrak{g}+T)^{\sigma} \oplus (\mathfrak{g}+T)^{-} = (\mathfrak{g}^{\sigma} + T^{\sigma}) \oplus (\mathfrak{g}^{-} + T^{-}).$$

Let

$$x = x_{+} \oplus x_{-} \in (\mathfrak{g}+T)^{\sigma} \oplus (\mathfrak{g}+T)^{-} = (\mathfrak{g}^{\sigma} + T^{\sigma}) \oplus (\mathfrak{g}^{-} + T^{-})$$

be in C. Then, for any $y \in \mathfrak{g}^\sigma$, we have

$$0 = [x,y] = [x_+,y] + [x_-,y].$$

Hence, $[x_+,y] = 0$ and $[x_-,y] = 0$. However, the centralizer $C_{\mathfrak{g}^\sigma + T^\sigma}(\mathfrak{g}^\sigma) = 0$ since $\mathfrak{g}^\sigma + T^\sigma$ is tame, as well as $Z(\mathfrak{g}^\sigma) = 0$ (given that $\mathscr{L} = \mathscr{L}_c + T^\sigma$ is tame and $\mathscr{L}_c = (g^\sigma \otimes F[t^{\pm 2}]) \oplus (\mathfrak{g}^- \otimes tF[t^{\pm 2}]) \oplus Fc$, cf. Section 4). Hence, $x_+ = 0$, and we obtain $x \in (\mathfrak{g}+T)^- = \mathfrak{g}^- + T^-$. If $x \in \mathfrak{g}^-$, then $\mathscr{U}(\mathfrak{g}^\sigma)x$, where $\mathscr{U}(\mathfrak{g}^\sigma)$ is the universal enveloping algebra of $\mathfrak{g}^\sigma$, is a $\mathfrak{g}^\sigma$-submodule of $\mathfrak{g}^-$. However, since $\dim_F \mathfrak{g}^- > 1$ and $\mathfrak{g}^-$ is an irreducible $\mathfrak{g}^\sigma$-module (by Lemma 8.2), then we have $\mathfrak{g}^- \supset \mathscr{U}(\mathfrak{g}^\sigma)x = Fx$, which implies that x has to be 0 in this case. Similarly, (2) holds by property (45). □

Lemma 8.4. (1) *Let $h \in T_{2\mathfrak{I}+1}$ for type $BC_{\mathfrak{I}}^{(2)}$ or $h \in T_{2\mathfrak{I}}$ for type $C_{\mathfrak{I}}^{(2)}$. Suppose that $[h,\mathfrak{g}^\sigma] \subset \mathfrak{g}^-$, then $h \in T^-_{2\mathfrak{I}+1}$ or $h \in T^-_{2\mathfrak{I}}$, respectively.*

(2) *Let $h \in T^\sigma_{2\mathfrak{I}+2}$ for type $B_{\mathfrak{I}}^{(2)}$. Suppose that $[h, o_{2\mathfrak{I}+2}(F)] \subset \mathfrak{s}$, then $h \in \mathfrak{s}_0 = \mathfrak{s} \cap \mathfrak{h}^\sigma$ (for $\mathfrak{s}_0$, see (44) in the last paragraph of Section 5).*

Proof. For (1), let $x \in \mathfrak{g}^\sigma$ and $y = [h,x] \in \mathfrak{g}^-$. Then, $-y = \sigma(y) = [\sigma(h),x]$. Hence, $[h+\sigma(h),x] = 0$ for all $x \in \mathfrak{g}^\sigma$. Therefore, $h+\sigma(h) \in C$ in Lemma 8.3, and thus $h+\sigma(h) \in T^-$. However since $h+\sigma(h) \in T^\sigma$, we obtain $h+\sigma(h) = 0$. Thus, $\sigma(h) = -h$, i.e., $h \in T^-_{2\mathfrak{I}+1}$ or $T^-_{2\mathfrak{I}}$, respectively.

Similarly for (2), we obtain $h + \tau(h) = 0$ by Lemma 8.3. Thus, $h \in \mathfrak{s}_0$. □

Let $\mathscr{L}$ be a twisted LALA of infinite rank, i.e., the core $\mathscr{L}_c$ is a universal covering of a twisted locally loop algebra of type $\mathrm{B}_{\mathfrak{I}}^{(2)}$, $\mathrm{C}_{\mathfrak{I}}^{(2)}$ or $\mathrm{BC}_{\mathfrak{I}}^{(2)}$ for an infinite index $\mathfrak{I}$. As in the untwisted case, by selecting a homogeneous complement of the $\mathbb{Z}$-graded core, we can write

$$\mathscr{L} = \mathscr{L}_c \oplus \bigoplus_{m\in\mathbb{Z}} D^m, \quad \bigoplus_{m\in\mathbb{Z}} D^m \subset \bigoplus_{\delta\in R^0} \mathscr{L}_\delta = \bigoplus_{m\in\mathbb{Z}} \mathscr{L}_{m\delta_1} \quad \text{and} \quad D^m \subset \mathscr{L}_{m\delta_1},$$

where δ_1 is a generator of $\langle R^0\rangle_{\mathbb{Z}}$. Let

$$\mathscr{L}_c' := \mathscr{L}_c / Z(\mathscr{L}_c)$$

be the centerless core and let $(\mathfrak{g},\mathfrak{h})$ be the grading pair of the Lie 1-torus $\mathscr{L}_c$ such that $\mathfrak{h}$ is the set of diagonal matrices of a locally finite split simple Lie algebra $\mathfrak{g}$, as before. According to this terminology, $\mathscr{L}_c' = \mathscr{L}_c/Z(\mathscr{L}_c)$ can be identified with

$$L := \big(\mathfrak{g} \otimes F[t^{\pm 2}]\big) \oplus \big(\mathfrak{s} \otimes tF[t^{\pm 2}]\big).$$

We note that the subalgebras $\mathfrak{g}^+ = \mathfrak{g}^\sigma$ and $\mathfrak{g}^-$ in the previous terminology correspond to $\mathfrak{g}$ and $\mathfrak{s}$ in this new terminology.

Let L be a locally loop algebra of type $X_{\mathfrak{I}}^{(2)}$. Then, L is Δ-graded, where Δ is a locally finite irreducible root system of type $X_{\mathfrak{I}}$. In addition, we can see that $\mathfrak{g}$ is $\Delta^{\mathrm{red}} \cup \{0\}$-graded and $\mathfrak{s}$ is $\Delta' \cup \{0\}$-graded, where Δ^{red} and Δ' are given as follows.

Δ	Δ^{red}	Δ'
$\mathrm{B}_{\mathfrak{I}}$	Δ	$\Delta_{\mathrm{sh}} = (\mathrm{A}_1)^{\times\mathfrak{I}}$
$\mathrm{C}_{\mathfrak{I}}$	Δ	$\Delta_{\mathrm{sh}} = \mathrm{D}_{\mathfrak{I}}$
$\mathrm{BC}_{\mathfrak{I}}$	$\Delta_{\mathrm{sh}} \cup \Delta_{\mathrm{lg}} = \mathrm{B}_{\mathfrak{I}}$	Δ

In this case, our new notation $(\mathrm{A}_1)^{\times\mathfrak{I}}$ denotes the (orthogonal disjoint) union

$$(\mathrm{A}_1)^{\times\mathfrak{I}} = \dot{\sqcup}_{i\in\mathfrak{I}}\Delta_i = \{\pm\varepsilon_i \mid i \in \mathfrak{I}\}$$

of root systems $\Delta_i = \{\pm\varepsilon_i\}$ of type A_1, which satisfy $\Delta_i \perp \Delta_j$ for distinct i, j. In particular, we have

$$\mathfrak{g} = \bigoplus_{\alpha\in\Delta^{\mathrm{red}}\cup\{0\}} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{s} = \bigoplus_{\beta\in\Delta'\cup\{0\}} \mathfrak{s}_\beta.$$

As in the untwisted case, we can classify diagonal derivations of a twisted locally loop algebra L in general.

Let

$$(\mathrm{Der}_F L)_0^0 := \{d \in \mathrm{Der}_F L \mid d(\mathfrak{g}_\alpha\otimes t^{2m}) \subset \mathfrak{g}_\alpha\otimes t^{2m} \text{ and } d(\mathfrak{s}_\beta\otimes t^{2m+1}) \subset \mathfrak{s}_\beta\otimes t^{2m+1} \text{ for all } \alpha\in\Delta^{\mathrm{red}}, \beta\in\Delta' \text{ and } m\in\mathbb{Z}\}$$

and take $d \in (\mathrm{Der}_F L)_0^0$. Then, as before, $d\,|_{\mathfrak{g}}$ is a diagonal derivation of $\mathfrak{g}$, and thus, by Neeb [N1], $d\,|_{\mathfrak{g}} = \mathrm{ad}\, p$ for some $p \in P$, depending on the type of $\mathfrak{g}$ (see (46)). Let

$$d' := d - \mathrm{ad}\, p \in (\mathrm{Der}_F L)_0^0.$$

Then, we have $d'(\mathfrak{g}\otimes 1) = 0$. In particular, we have $d'(\mathfrak{h}\otimes 1) = 0$. Thus, in the same manner as the untwisted case, we can show that for $0 \neq x\otimes t \in \mathfrak{s}_\beta\otimes t$, if

$$d'(x\otimes t) = ax\otimes t \tag{51}$$

for $a \in F$, then

$$d'(y\otimes t^{-1}) = -ay\otimes t^{-1} \tag{52}$$

for all $y \in \mathfrak{s}_{-\beta}$.

Lemma 8.5. *For the above $\mathfrak{s}$, we have $\mathscr{U}(\mathfrak{g}).\mathfrak{s}_\beta = \mathfrak{s}$ for any $\beta\in\Delta'$, where $\mathscr{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.*

Proof. Since $\mathscr{U}(\mathfrak{g}).\mathfrak{s}_\beta$ is a nonzero submodule of $\mathfrak{s}$, then it must be $\mathfrak{s}$ by the irreducibility of $\mathfrak{s}$. □

By Lemma 8.5, for a fixed $\beta\in\Delta'$, the three subspaces

$$\mathfrak{g}\otimes 1, \quad \mathfrak{s}_\beta\otimes t, \quad \text{and} \quad \mathfrak{s}_{-\beta}\otimes t^{-1}$$

generate L as a Lie algebra. As before, let $d'' := d' - ad^{(0)}$. Then, we have $d''(\mathfrak{g}\otimes 1) = d'(\mathfrak{g}\otimes 1) = 0$ and using (51), we obtain $d''(x\otimes t) = d'(x\otimes t) - ax\otimes t = 0$ for $x\in\mathfrak{s}_\beta$. Similarly, using (52), we have $d''(y\otimes t^{-1}) = d'(y\otimes t^{-1}) + ay\otimes t^{-1} = 0$ for $y\in\mathfrak{s}_{-\beta}$. Thus, we have $d''(L) = 0$ and $d'' = 0$. Hence, we obtain

$$d = \mathrm{ad}\, p + ad^{(0)}. \tag{53}$$

Again, we define the shift map s_{2m} for $m\in\mathbb{Z}$ on $L = \big(\mathfrak{g}\otimes F[t^{\pm 2}]\big)\oplus\big(\mathfrak{s}\otimes tF[t^{\pm 2}]\big)$ by

$$s_{2m}(x\otimes t^{2k}) := x\otimes t^{2k+2m} \text{ and } s_{2m}(v\otimes t^{2k+1}) := v\otimes t^{2k+2m+1}$$

for $x\in\mathfrak{g}$ and $v\in\mathfrak{s}$. Let

$$(\mathrm{Der}_F L)_0^{2m} := \{d \in \mathrm{Der}_F L \mid d(\mathfrak{g}_\alpha\otimes t^{2k}) \subset \mathfrak{g}_\alpha\otimes t^{2k+2m} \text{ and } d(\mathfrak{s}_\beta\otimes t^{2k+1}) \subset \mathfrak{s}_\beta\otimes t^{2k+2m+1} \text{ for all } \alpha\in\Delta^{\mathrm{red}}, \beta\in\Delta' \text{ and } k\in\mathbb{Z}\}$$

and we take $d_{2m} \in (\mathrm{Der}_F L)_0^{2m}$. Then, we have $s_{-2m} \circ d_{2m} \in (\mathrm{Der}_F L)_0^0$. Hence, by (53), some $p = p_{d_{2m}} \in P$ and $a = a_{d_{2m}} \in F$ exist such that $s_{-2m} \circ d_{2m} = \mathrm{ad}\, p + ad^{(0)}$, and thus

$$d_{2m} = s_{2m} \circ \mathrm{ad}\, p + at^{2m} d^{(0)} = s_{2m} \circ \mathrm{ad}\, p + at^{2m+1} \frac{d}{dt}.$$

Therefore, as in Theorem 7.2, we have the following.

Lemma 8.6. *For all $m \in \mathbb{Z}$, we have*

$$(\mathrm{Der}_F L)_0^{2m} = s_{2m} \circ (\mathrm{Der}_F L)_0^0 = s_{2m} \circ (\mathrm{ad}\, P \oplus F d^{(0)}),$$

where P is defined in (46). □

Moreover, as in Lemma 7.3, we have the following.

Lemma 8.7. *For all $m \in \mathbb{Z}$, let*

$$(\mathrm{Der}'_F L)_0^{2m} := \{d \in (\mathrm{Der}_F L)_0^{2m} \mid s_{2n} \circ d = d \circ s_{2n} \textit{ for some } 0 \neq n \in \mathbb{Z}\}$$

and

$$(\mathrm{Der}''_F L)_0^{2m} := \{d \in (\mathrm{Der}_F L)_0^{2m} \mid s_{2n} \circ d = d \circ s_{2n} \textit{ for all } n \in \mathbb{Z}\}.$$

Then, we have

$$(\mathrm{Der}'_F L)_0^{2m} = s_{2m} \circ \mathrm{ad}\, P = (\mathrm{Der}''_F L)_0^{2m}.$$

□

Now, we return to the classification of D^m. Let $d_{2m} \in D^{2m}$. Then, $\mathrm{ad}\, d_{2m} = s_{2m} \circ \mathrm{ad}\, p + at^{2m} d^{(0)}$ for some $p \in P$ and $a \in F$ by Lemma 8.6. Then, as in the untwisted case, we can show that $a = 0$ for all $m \neq 0$, using

$$\mathscr{B}([d, h \otimes t], h' \otimes t^{-m-1}) = -\mathscr{B}(h \otimes t^2, [d_{2m}, h' \otimes t^{-2m-2}])$$

for some $h, h' \in \mathfrak{h}$ such that $\mathrm{tr}(h, h') \neq 0$. Furthermore, as in the untwisted case, some $p \in P$ exists such that $\mathrm{ad}\, p + d^{(0)} \in \mathrm{ad}\, D^0$. Thus, the spaces D^m for even m s coincide with those in Example 6.3.

Next, we determine $(\mathrm{Der}_F L)_0^{2m+1}$, where

$$(\mathrm{Der}_F L)_0^{2m+1} := \{d \in \mathrm{Der}_F L \mid d(\mathfrak{g}_\alpha \otimes t^{2k}) \subset \mathfrak{s}_\alpha \otimes t^{2k+2m+1} \text{ and } d(\mathfrak{s}_\beta \otimes t^{2k+1}) \subset \mathfrak{g}_\beta \otimes t^{2k+2m+2} \text{ for all } \alpha \in \Delta^{\mathrm{red}}, \beta \in \Delta' \text{ and } k \in \mathbb{Z}\}.$$

Lemma 8.8. *Let $q \in (\mathrm{Der}_F L)_0^{2m+1}$. Then, q commutes with a shift map s_{2i} for all $i \in \mathbb{Z}$.*

Proof. We note that

$$q(x_\alpha \otimes t^{2k}) = 0 \quad (x_\alpha \in \mathfrak{g}_\alpha,\ \alpha \in \Delta_{lg},\ k \in \mathbb{Z})$$

for $\mathrm{B}_\mathfrak{J}$ or $\mathrm{C}_\mathfrak{J}$, and that

$$q(x_\beta \otimes t^{2k+1}) = 0 \quad (x_\beta \in \mathfrak{s}_\beta,\ \beta \in \Delta_{ex},\ k \in \mathbb{Z})$$

for $\mathrm{BC}_\mathfrak{J}$, since $\mathfrak{s}_\alpha = 0$ and $\mathfrak{g}_\beta = 0$. Therefore, in particular,

$$q \circ s_{2i}(z) = s_{2i} \circ q(z)$$

for $z = x_\alpha \otimes t^{2k}$ in the case of type $\mathrm{B}_\mathfrak{J}$ or $\mathrm{C}_\mathfrak{J}$, and for $z = x_\beta \otimes t^{2k+1}$ in the case of type $\mathrm{BC}_\mathfrak{J}$. For any other given homogeneous element x we can find suitable homogeneous elements y and z such that $x = [y, z]$ in the following sense.

	x	y	z
$B_{\mathfrak{J}}$ or $C_{\mathfrak{J}}$	$x_\alpha \otimes t^{2k}$ $x_\alpha \in \mathfrak{g}_\alpha\ (\alpha \in \Delta_{sh})$	$y_{\alpha'} \otimes t^{2k}$ $y_{\alpha'} \in \mathfrak{g}_{\alpha'}\ (\alpha' \in \Delta_{sh})$	$z_{\alpha''} \otimes 1$ $z_{\alpha''} \in \mathfrak{g}_{\alpha''}\ (\alpha'' \in \Delta_{lg})$
	$x_\beta \otimes t^{2k+1}$ $x_\beta \in \mathfrak{s}_\beta\ (\beta \in \Delta_{sh})$	$y_{\beta'} \otimes t^{2k+1}$ $y_{\beta'} \in \mathfrak{s}_{\beta'}\ (\beta' \in \Delta_{sh})$	$z_{\alpha''} \otimes 1$ $z_{\alpha''} \in \mathfrak{g}_{\alpha''}\ (\alpha'' \in \Delta_{lg})$

The table shown above is for $B_{\mathfrak{J}}$ or $C_{\mathfrak{J}}$. For example, we can understand that for any $x = x_\alpha \otimes t^{2k}$ $(x_\alpha \in \mathfrak{g}_\alpha,\ \alpha \in \Delta_{sh})$, there are $y = y_{\alpha'} \otimes t^{2k}$ $(y_{\alpha'} \in \mathfrak{g}_{\alpha'}\ \alpha' \in \Delta_{sh})$ and $z = z_{\alpha''} \otimes 1$ $(z_{\alpha''} \in \mathfrak{g}_{\alpha''},\ \alpha'' \in \Delta_{lg})$ such that $x = [y,z]$.

Similarly, for $BC_{\mathfrak{J}}$, we obtain the following table.

	x	y	z
$BC_{\mathfrak{J}}$	$x_\alpha \otimes t^{2k}$ $x_\alpha \in \mathfrak{g}_\alpha\ (\alpha \in \Delta_{sh})$	$y_{\beta'} \otimes t^{2k-1}$ $y_{\beta'} \in \mathfrak{s}_{\beta'}\ (\beta' \in \Delta_{sh})$	$z_{\beta''} \otimes t$ $z_{\beta''} \in \mathfrak{s}_{\beta''}\ (\beta'' \in \Delta_{ex})$
	$x_\alpha \otimes t^{2k}$ $x_\alpha \in \mathfrak{g}_\alpha\ (\alpha \in \Delta_{lg})$	$y_{\alpha'} \otimes t^{2k-1}$ $y_{\beta'} \in \mathfrak{s}_{\beta'}\ (\alpha' \in \Delta_{lg})$	$z_{\beta''} \otimes t$ $z_{\alpha''} \in \mathfrak{s}_{\beta''}\ (\beta'' \in \Delta_{ex})$
	$x_\beta \otimes t^{2k+1}$ $x_\beta \in \mathfrak{s}_\beta\ (\beta \in \Delta_{sh})$	$y_{\beta'} \otimes t^{2k}$ $y_{\alpha'} \in \mathfrak{g}_{\alpha'}\ (\alpha' \in \Delta_{sh})$	$z_{\beta''} \otimes t$ $z_{\beta''} \in \mathfrak{s}_{\beta''}\ (\beta'' \in \Delta_{ex})$
	$x_\beta \otimes t^{2k+1}$ $x_\beta \in \mathfrak{s}_\beta\ (\beta \in \Delta_{lg})$	$y_{\alpha'} \otimes t^{2k}$ $y_{\alpha'} \in \mathfrak{g}_{\alpha'}\ (\alpha' \in \Delta_{lg})$	$z_{\beta''} \otimes t$ $z_{\alpha''} \in \mathfrak{s}_{\beta''}\ (\beta'' \in \Delta_{ex})$

In the expression $x = [y,z]$, we note that $q(z) = 0$ is always true for all $B_{\mathfrak{J}}$, $C_{\mathfrak{J}}$ and $BC_{\mathfrak{J}}$ as before, which is the most important fact in this case. Hence, we obtain

$$\begin{aligned} q \circ s_{2i}(x) &= q \circ s_{2i}([y,z]) = q([y, s_{2i}(z)]) \\ &= [q(y), s_{2i}(z)] + [y, q \circ s_{2i}(z)] \\ &= [q(y), s_{2i}(z)] \\ &= s_{2i}([q(y),z]) \\ &= s_{2i}([q(y),z] + [y,q(z)]) \\ &= s_{2i} \circ q([y,z]) \\ &= s_{2i} \circ q(x). \end{aligned}$$

Therefore, $q \circ s_{2i} = s_{2i} \circ q$ on L. □

Lemma 8.9. *Let* $L = \big(\mathfrak{g} \otimes F[t^{\pm 2}]\big) \oplus \big(\mathfrak{s} \otimes tF[t^{\pm 2}]\big)$ *be a twisted loop algebra, which is double graded by* $\Delta \cup \{0\}$ *and* $\mathbb{Z}$ *as above. Let* d *be in* $(\mathrm{Der}_F L)_0^{2m+1}$ *such that* $s_2 \circ d = d \circ s_2$. *Then, a unique derivation* $\tilde{d}$ *on* $\tilde{L}$ *exists such that*

$$\tilde{d}\,|_L = d, \quad \tilde{d}(x \otimes t^{2k+1}) = s_1 \circ d(x \otimes t^{2k}) \quad \text{and} \quad \tilde{d}(v \otimes t^{2k}) = s_{-1} \circ d(v \otimes t^{2k+1})$$

for all $x \in \mathfrak{g}$, $v \in \mathfrak{s}$, *and* $k \in \mathbb{Z}$. *Moreover,*

$$\tilde{d} \in (\mathrm{Der}_F \tilde{L})_0^{2m+1} \quad \text{such that} \quad s_k \circ \tilde{d} = \tilde{d} \circ s_k \quad \text{for all } k \in \mathbb{Z}.$$

Proof. The uniqueness is clear since the image of all the homogeneous elements has been determined. Therefore, it is sufficient to show that $\tilde{d}$ is a derivation. Thus, we need to check the following: For $x, y \in \mathfrak{g}$ and $v, w \in \mathfrak{s}$,

(a) $\tilde{d}([x \otimes t^{2k}, y \otimes t^{2\ell+1}]) = [\tilde{d}(x \otimes t^{2k}), y \otimes t^{2\ell+1}] + [x \otimes t^{2k}, \tilde{d}(y \otimes t^{2\ell+1})]$
(b) $\tilde{d}([x \otimes t^{2k}, v \otimes t^{2\ell}]) = [\tilde{d}(x \otimes t^{2k}), v \otimes t^{2\ell}] + [x \otimes t^{2k}, \tilde{d}(v \otimes t^{2\ell})]$
(c) $\tilde{d}([x \otimes t^{2k+1}, y \otimes t^{2\ell+1}]) = [\tilde{d}(x \otimes t^{2k+1}), y \otimes t^{2\ell+1}] + [x \otimes t^{2k+1}, \tilde{d}(y \otimes t^{2\ell+1})]$

(d) $\tilde{d}([x\otimes t^{2k+1}, v\otimes t^{2\ell+1}]) = [\tilde{d}(x\otimes t^{2k+1}), v\otimes t^{2\ell+1}] + [x\otimes t^{2k+1}, \tilde{d}(v\otimes t^{2\ell+1})]$
(e) $\tilde{d}([x\otimes t^{2k+1}, v\otimes t^{2\ell}]) = [\tilde{d}(x\otimes t^{2k+1}), v\otimes t^{2\ell}] + [x\otimes t^{2k+1}, \tilde{d}(v\otimes t^{2\ell})]$
(f) $\tilde{d}([v\otimes t^{2k+1}, w\otimes t^{2\ell}]) = [\tilde{d}(v\otimes t^{2k+1}), w\otimes t^{2\ell}] + [v\otimes t^{2k+1}, \tilde{d}(w\otimes t^{2\ell})]$
(g) $\tilde{d}([v\otimes t^{2k}, w\otimes t^{2\ell}]) = [\tilde{d}(v\otimes t^{2k}), w\otimes t^{2\ell}] + [v\otimes t^{2k}, \tilde{d}(w\otimes t^{2\ell})]$.

All of these equations involve simple calculations, but we check them to be sure. For (a), we have

$$\begin{aligned}(LHS) &= \tilde{d}([x,y]\otimes t^{2k+2\ell+1}) = s_1\circ d([x,y]\otimes t^{2k+2\ell}) = s_1\circ d([x\otimes t^{2k}, y\otimes t^{2\ell}])\\ &= s_1\big([d(x\otimes t^{2k}), y\otimes t^{2\ell}] + [x\otimes t^{2k}, d(y\otimes t^{2\ell})]\big)\\ &= [d(x\otimes t^{2k}), y\otimes t^{2\ell+1}] + [x\otimes t^{2k}, s_1\circ d(y\otimes t^{2\ell})] = (RHS).\end{aligned}$$

For (b), we have

$$\begin{aligned}(LHS) &= \tilde{d}([x,v]\otimes t^{2k+2\ell}) = s_{-1}\circ d([x,v]\otimes t^{2k+2\ell+1}])\\ &= s_{-1}\big([d(x\otimes t^{2k}), v\otimes t^{2\ell+1}] + [x\otimes t^{2k}, d(v\otimes t^{2\ell+1})]\big)\\ &= [d(x\otimes t^{2k}), v\otimes t^{2\ell}] + [x\otimes t^{2k}, s_{-1}\circ d(v\otimes t^{2\ell+1})] = (RHS).\end{aligned}$$

For (c), we have

$$\begin{aligned}(LHS) &= \tilde{d}([x,y]\otimes t^{2k+2\ell+2}) = d([x,y]\otimes t^{2k+2\ell+2}])\\ &= d([x\otimes t^{2k}, y\otimes t^{2\ell+2}]) = [d(x\otimes t^{2k}), y\otimes t^{2\ell+2}] + [x\otimes t^{2k}, d(y\otimes t^{2\ell+2})]\\ &= s_1\big([d(x\otimes t^{2k}), y\otimes t^{2\ell+1}]\big) + [x\otimes t^{2k}, d\circ s_2(y\otimes t^{2\ell})]\\ &= [s_1\circ d(x\otimes t^{2k}), y\otimes t^{2\ell+1}] + s_2\big([x\otimes t^{2k}, d(y\otimes t^{2\ell})]\big)\\ &\quad \text{(since } s_2 \text{ and } d \text{ commute)}\\ &= [\tilde{d}(x\otimes t^{2k+1}), y\otimes t^{2\ell+1}] + [x\otimes t^{2k+1}, s_1\circ d(y\otimes t^{2\ell})] = (RHS).\end{aligned}$$

For (d), we have

$$\begin{aligned}(LHS) &= \tilde{d}([x,v]\otimes t^{2k+2\ell+2}) = s_{-1}\circ d([x,v]\otimes t^{2\ell+3}])\\ &= s_{-1}\circ d([x\otimes t^{2k}, v\otimes t^{2\ell+3}])\\ &= s_{-1}\big([d(x\otimes t^{2k}), v\otimes t^{2\ell+3}] + [x\otimes t^{2k}, d(v\otimes t^{2\ell+3})]\big)\\ &= s_1\big([d(x\otimes t^{2k}), v\otimes t^{2\ell+1}]\big) + s_{-2}\big([x\otimes t^{2k+1}, d(v\otimes t^{2\ell+3})]\big)\\ &= [s_1\circ d(x\otimes t^{2k}), v\otimes t^{2\ell+1}] + [x\otimes t^{2k+1}, s_{-2}\circ d(v\otimes t^{2\ell+3})]\\ &= (RHS) \quad \text{(since } s_2 \text{ and } d \text{ commute).}\end{aligned}$$

For (e), we have

$$\begin{aligned}(LHS) &= \tilde{d}([x,v]\otimes t^{2k+2\ell+1}) = d([x,v]\otimes t^{2k+2\ell+1}]) = d([x\otimes t^{2k}, v\otimes t^{2\ell+1}])\\ &= [d(x\otimes t^{2k}), v\otimes t^{2\ell+1}] + [x\otimes t^{2k}, d(v\otimes t^{2\ell+1})]\\ &= [s_1\circ d(x\otimes t^{2k}), v\otimes t^{2\ell}] + [x\otimes t^{2k}, \tilde{d}(v\otimes t^{2\ell+1})] = (RHS).\end{aligned}$$

For (f), we have

$$\begin{aligned}(LHS) &= \tilde{d}([v,w]\otimes t^{2k+2\ell+1}]) = s_1\circ d([v,w]\otimes t^{2k+2\ell}])\\ &= s_1\circ d([v\otimes t^{2k-1}, w\otimes t^{2\ell+1}])\\ &= s_1\big([d(v\otimes t^{2k-1}), w\otimes t^{2\ell+1}] + [v\otimes t^{2k-1}, d(w\otimes t^{2\ell+1})]\big)\\ &= s_2[d(v\otimes t^{2k-1}), w\otimes t^{2\ell}]\big) + [v\otimes t^{2k}, d(w\otimes t^{2\ell+1})]\\ &= [d(v\otimes t^{2k+1}), w\otimes t^{2\ell}] + s_{-1}\big([v\otimes t^{2k+1}, d(w\otimes t^{2\ell+1})]\big) = (RHS).\end{aligned}$$

For (g), we have

$$\begin{aligned}(LHS) &= \tilde{d}([v,w]\otimes t^{2k+2\ell}]) = d([v,w]\otimes t^{2k+2\ell}]) = d([v\otimes t^{2k-1}, w\otimes t^{2\ell+1}])\\ &= [d(v\otimes t^{2k-1}), w\otimes t^{2\ell+1}] + [v\otimes t^{2k-1}, d(w\otimes t^{2\ell+1})]\\ &= [s_1\circ d(v\otimes t^{2k-1}), w\otimes t^{2\ell}] + [v\otimes t^{2k}, s_{-1}\circ d(w\otimes t^{2\ell+1})] = (RHS).\end{aligned}$$

For the second assertion, it is clear that $\tilde{d}\in(\mathrm{Der}_F\tilde{L})_0^{2m+1}$. In addition, since d commutes with s_2, then the same is true of $\tilde{d}$. Hence, by Lemma 7.3, $\tilde{d}$ commutes with s_k for all $k\in\mathbb{Z}$. □

Thus, together with Lemma 8.6, we have classified the diagonal derivations of twisted locally loop algebras.

Theorem 8.10. *Let L be a twisted loop algebra. Then, we have $(\mathrm{Der}_F L)_0^0 = \mathrm{ad}\,P\oplus Fd^{(0)}$, where P is defined in* (46)*, and*

$$(\mathrm{Der}_F L)_0^{2m} = s_{2m}\circ(\mathrm{Der}_F L)_0^0 \quad and \quad (\mathrm{Der}_F L)_0^{2m+1} = s_{2m+1}\circ \mathrm{ad}\,T^-$$

for all $m\in\mathbb{Z}$, where $T^- = \mathfrak{s}_0$ for $B_{\mathfrak{J}}^{(2)}$, $T^- = T^-_{2\mathfrak{J}}$ for $C_{\mathfrak{J}}^{(2)}$ or $T^- = T^-_{2\mathfrak{J}+1}$ for $BC_{\mathfrak{J}}^{(2)}$, as defined in Example 6.3.

Proof. By Lemma 8.8, 8.9, and the classification of the untwisted case, if $d\in(\mathrm{Der}_F L)_0^{2m+1}$, then $\tilde{d}\in s_{2m+1}\circ(\mathrm{Der}_F\tilde{L})_0^0$. In addition, by Lemma 8.8 and Lemma 8.7, we obtain $\tilde{d}\in s_{2m+1}\circ\mathrm{ad}\,P$. Thus, $\mathrm{ad}\,p := s_{-2m-1}\circ\tilde{d}\in\mathrm{ad}\,P$, and we have $[p,\mathfrak{g}^+]\subset\mathfrak{g}^-$ according to the terminology used in Lemma 8.4. Hence, by Lemma 8.4, we obtain $p\in T^-$. Therefore, $d\in s_{2m+1}\circ\mathrm{ad}\,T^-$. □

Remark 8.11. If L is a twisted loop algebra of type $\mathrm{B}_{\mathfrak{J}}^{(2)}$, then $(\mathrm{Der}_F L)_0^{2m+1} = s_{2m+1}\circ \mathrm{ad}\,\mathfrak{s}_0 = \mathrm{ad}(\mathfrak{s}_0\otimes t^{2m+1})$. Thus, there is no outer derivation of odd degree.

We return to the classification of twisted LALAs. By Theorem 8.10, if $d\in D^{2m+1}$, then $\mathrm{ad}_L d\in s_{2m+1}\circ\mathrm{ad}_L T^-$. The bracket on $D := \bigoplus_{m\in\mathbb{Z}} D^m$ can be investigated in the same manner as the untwisted case. In particular, for type $\mathrm{BC}_{\mathfrak{J}}^{(2)}$ or $\mathrm{C}_{\mathfrak{J}}^{(2)}$, we use the isomorphism

$$\mathscr{L}\cong\big(\mathscr{L}+\iota\otimes F[t^{\pm1}]\big)/\iota\otimes tF[t^{\pm2}].$$

Thus, D^m for $m\in\mathbb{Z}$ is an exact example for each type described in Example 6.3. Thus, we have completed the classification.

Theorem 8.12. *Let $\mathscr{L}$ be a twisted LALA. Then, $\mathscr{L}$ is isomorphic to that in Example 6.3.* □

Remark 8.13. We can show that any twisted LALA is the fixed algebra of some untwisted LALA. Moreover, for any untwisted LALA $\mathscr{L}$ of type $A_{\mathfrak{J}}^{(1)}$ or $D_{\mathfrak{J}}^{(1)}$, a twisted LALA $\mathscr{L}'$ exists, which is a subalgebra of $\mathscr{L}$ such that $\mathscr{L}'$ is the intersection of $\mathscr{L}$ and the fixed algebra of a maximal untwisted LALA $\mathscr{L}^{max}$ that contains $\mathscr{L}$. Note that a maximal twisted LALA is also unique up to isomorphism, as in the case of a maximal untwisted LALA.

Remark 8.14. By Theorem 7.6 and Theorem 8.12, the LALAs in Example 6.3 comprise all of the algebras. Given this fact, the following statement is clear and it is a useful criterion.

If a diagonal matrix $p \in T$ with a trace that is a nonzero value (e.g., e_{ii} or $e_{ii} + e_{\mathfrak{J}+i,\mathfrak{J}+i}$, etc.) is used in a LALA, then this LALA must be of type $A_{\mathfrak{J}}^{(1)}$, $C_{\mathfrak{J}}^{(2)}$, or $BC_{\mathfrak{J}}^{(2)}$. Moreover, if the type is $C_{\mathfrak{J}}^{(2)}$ or $BC_{\mathfrak{J}}^{(2)}$, then p has to be used in odd degree.

9. Standard LALAs

We prove the following criterion whether a LALA is standard or not.

Lemma 9.1. *Let $(\mathscr{L}, \mathscr{H}, \mathscr{B})$ be a LALA with center Fc and $\mathscr{L}_c$ is its core, which is a locally Lie 1-torus with grading pair $(\mathfrak{g}, \mathfrak{h})$. If $0 \neq d \in \mathscr{L}$ exists such that $[d, \mathfrak{g}] = 0$ and $\mathscr{B}(d, c) \neq 0$, then the action of d on the $\mathbb{Z}$-graded core coincides with a nonzero multiple of a degree derivation relative to $\mathbb{Z}$, and thus $\mathscr{L}$ contains the degree derivation. Hence, $\mathscr{L}$ is standard.*

Proof. Let $d = \sum_{\xi \in R} x_\xi$ for $x_\xi \in \mathscr{L}_\xi$. If $\xi \in R^\times$, then $[\mathfrak{h}, x_\xi] = F x_\xi \subset \mathscr{L}_\xi$, and thus $x_\xi = 0$ since $[d, \mathfrak{g}] = 0$. If $\xi \in R^0 \setminus \{0\}$, then $x_\xi \in T \otimes t^m$ for some $0 \neq m \in \mathbb{Z}$, by Theorem 7.6 and 8.12. However, if $x_\xi \neq 0$, then a root vector $y \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ exists such that $[y, x_\xi] \neq 0$, which is a contradiction. Hence, $x_\xi = 0$. Thus, $d = x_0 \in \mathscr{L}_0 = \mathscr{H}$. Then, by Theorems 7.6 and 8.12,

$$d = p + ad^{(0)} + bc$$

for some $p \in T = T \otimes t^0$ and $a, b \in F$, as well as $a \neq 0$, since $\mathscr{B}(d, c) \neq 0$. Therefore, we have $0 = [d, \mathfrak{g}] = [p, \mathfrak{g}]$. However, unless $\mathscr{L}$ has type $A_{\mathfrak{J}}^{(1)}$, we have $p \in T^\sigma$, and thus p must be zero. If $\mathscr{L}$ has type $A_{\mathfrak{J}}^{(1)}$, then $p \in F\iota$, by Lemma 5.10, and thus p must again be zero (modulo $F\iota$). Thus, we obtain $d = ad^{(0)} + bc$. □

Remark 9.2. In [N2, Def.3.6], Neeb defined a minimal LALA $\mathscr{L}$, which is minimal in the sense described above and that satisfies one more condition:

$$\exists\, d \in \mathscr{H} \text{ such that } W' := \mathrm{span}_{\mathbb{Q}}\{\alpha \in R^\times \mid \alpha(d) = 0\} \text{ is a reflectable section}$$

of $W = \mathrm{span}_{\mathbb{Q}} R^\times$. Thus, $[\mathfrak{g}, d] = 0$. Moreover, if $\delta(d) = 0$, where δ is a generator of $R^0 \cong \mathbb{Z}$, then $\alpha(d) = 0$ for all $\alpha \in R^\times$. Hence, $W' = W$, which is a contradiction. Thus, $\delta(d) \neq 0$. However, d is a nonzero multiple of a degree derivation modulo of the center by Lemma 9.1, and thus a minimal LALA in [N2] is a minimal standard LALA in our sense.

Example 9.3. The minimal LALA $\mathscr{L} = \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}]) \oplus Fc \oplus F(e_{11} + d^{(0)})$ is isomorphic to a minimal standard LALA $\mathscr{L}^{ms} = \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}]) \oplus Fc \oplus Fd^{(0)}$. In fact, let $g = \mathrm{diag}(t, 1, 1, \ldots)$. Then, $g^{-1}Xg$ for $X \in \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}])$ gives an automorphism f of $\mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}])$. Therefore, we can extend f from $\mathscr{L}^{ms}$ onto $\mathscr{L}$ such that $f(c) = c$ and $f(d^{(0)}) = e_{11} + d^{(0)}$. Thus, $\mathscr{L}$ is isomorphic to $\mathscr{L}^{ms}$, as in Lie algebras.

Example 9.4. Let $p = \mathrm{diag}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ and put $d = p + d^{(0)}$. Then, the minimal LALA $\mathscr{L} = \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}]) \oplus Fc \oplus Fd$ is not isomorphic to a minimal standard LALA $\mathscr{L}^{ms}$. In fact, if $\mathscr{L}$ is isomorphic to $\mathscr{L}^{ms}$, then an isomorphism

$$\psi : \mathscr{L}^{ms} \longrightarrow \mathscr{L}$$

exists such that $\psi(d^{(0)}) = x + ad = x + a(d^{(0)} + p)$ for some $x \in \mathscr{L}_c = \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}]) \oplus Fc$ and some nonzero $a \in F$. Then, we have

$$\psi \circ \mathrm{ad}\, d^{(0)} \circ \psi^{-1} = \mathrm{ad}(\psi(d^{(0)})) = \mathrm{ad}(x + ad^{(0)} + ap)$$

in $\mathrm{Der}_F(\mathscr{L})$. Now, we can compare the eigenvalues of the same operators $\psi \circ \mathrm{ad}\, d^{(0)} \circ \psi^{-1}$ and $\mathrm{ad}(x + ad^{(0)} + ap)$. Note that the eigenvalues of $\psi \circ \mathrm{ad}\, d^{(0)} \circ \psi^{-1}$ are all integers. We can select $h = e_{\ell\ell} - e_{\ell+1,\ell+1} \in \mathrm{sl}_{\mathbb{N}}(F[t^{\pm 1}])$ such that

$$[x, h] = 0,$$

by taking $\ell >> 0$, where e_{ij} is a matrix unit. Then,

$$[x + ad^{(0)} + ap, h \otimes t] = a(h \otimes t),$$

which implies that a is a nonzero integer since a is an eigenvalue of $\mathrm{ad}(x + ad^{(0)} + ap)$. We can also choose sufficiently large different integers $m \neq n >> 0$ that satisfy

$$[x, e_{mn}] = 0$$

and

$$\frac{a(n-m)}{mn} \notin \mathbb{Z}. \tag{54}$$

For these integers, m and n, we can see that

$$[x + ad^{(0)} + ap, e_{mn}] = a\left(\frac{1}{m} - \frac{1}{n}\right) e_{mn} = \frac{a(n-m)}{mn} e_{mn}.$$

Since

$$\frac{a(n-m)}{mn}$$

is an eigenvalue of $\mathrm{ad}(x + ad^{(0)} + ap)$, it must be an integer, which contradicts (54). Hence, $\mathscr{L}$ is not isomorphic to $\mathscr{L}^{ms}$.

REFERENCES

[A1] S. Azam, *Extended affine root systems*, J. Lie Theory, no. 2, **12** (2002), 515–527.

[A2] S. Azam, *Derivations of tensor product of algebras*, Commun. Alg., **36**, 3 (2008), 905–927.

[AABGP] B. Allison, S. Azam, S. Berman, Y. Gao, A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. 126 (1997), no. 603.

[ABGP] B. Allison, S. Berman, Y. Gao, A. Pianzola *A characterization of affine Kac-Moody Lie algebras*, Commun. Math. Phys., **185** (1997) 671–688.

[AF] B. Allison, J. Faulkner, *Isotopy for extended affine Lie algebras and Lie tori*, Prog. Math., **288** (2008), 3-43.

[B] G. Benkart, *Derivations and invariant forms of Lie algebras graded by finite root systems*, Can. J. Math. **50** (1998), 225–241.

[BM] G. Benkart, R. Moody, *Derivations, central extensions, and affine Lie algebras*, Algebras Groups Geom. 3 (1986), 456–492.

[Bo] N. Bourbaki, Groupes et algèbres de Lie, Chap. IV, V, VI, Hermann, Paris (1968).

[G] P. Griffith, *Infinite abelian group theory*, Chicago Lectures in Mathematics (1970).

[K] V. Kac *Infinite dimensional Lie algebras*, Cambridge University Press, third edition (1990).

[LN1] O. Loos, E. Neher, *Locally finite root systems*, Memoirs Amer. Math. Soc., **811** vol.171 (2004).

[LN2] O. Loos, E. Neher, *Reflections systems and partial root systems*, Forum Math., Vol 23(2), 349–411 (2011).

[Ma] M. Macdonald, *Affine root systems and Dedekind's η-functions*, J. Invent. Math., **15** (1972), 91–143.

[Mo] R. Moody, *Euclidean Lie algebras*, Can. J. Math., **21** (1969), 1432-1454.

[MP] R. Moody, A. Pianzola, *Lie algebras with triangular decompositions*, Can. Math. Soc. series of monographs and advanced texts, John Wiley (1995).

[MY] J. Morita, Y. Yoshii, *Locally extended affine Lie algebras*, J. Algebra 301 (2006), 59–81.

[N1] K.-H. Neeb, *Derivations of locally simple Lie algebras*, J. Lie Theory 15 (2005), no. 2, 589–594.

[N2] K.-H. Neeb, *Unitary highest weight modules of locally affine Lie algebras*, Quantum affine algebras, extended affine Lie algebras, and their applications,. Contemp. Math., 506, Amer. Math. Soc., Providence, RI (2010), 227–262.

[Ne1] E. Neher, *Lie tori*, C. R. Math. Rep. Acad. Sci. Canada, **26(3)** (2004), 84–89.

[Ne2] E. Neher, *Extended affine Lie algebras*, C. R. Math. Acad. Sci. Soc. R. Can. **26** (2004), no. 3, 90–96.

[Ne3] E. Neher, *Lectures on extended affine Lie algebras*, Fields Inst. publ., to appear.

[NeS] E. Neher, J. Sun, *Universal central extensions of direct limits of Lie superalgebras*, J. Algebra, 368 (2012), 169–181.

[NS] K.-H. Neeb, N. Stumme, *The classification of locally finite split simple Lie algebras*, J. reine angew. Math., **533** (2001), 25–53.

[NY] E. Neher, Y. Yoshii, *Derivations and invariant forms of Jordan and alternative tori*, Trans. Amer. Math. Soc. **355** (2003) 1079–1108.

[S] K. Saito, *Extended affine root systems 1 (Coxeter transformations)*, RIMS., Kyoto Uviv. **21** (1985), 75–179.

[St] N. Stumme, *The structure of locally finite split Lie algebras*, J. Algebra **220** (1999), 664–693.

[Y1] Y. Yoshii, *Root systems extended by an abelian group and their Lie algebras*, J. Lie Theory **14(2)** (2004), 371–394.

[Y2] Y. Yoshii, *Lie tori — a simple characterization of extended affine Lie algebras*, RIMS., Kyoto Uviv. **42** (2006), 739–762.

[Y3] Y. Yoshii, *Locally extended affine root systems*, Proc. on Quantum Affine Algebras, Extended Affine Lie Algebras and Applications, Contemp. Math., **508** (2010), 285–302.

(Jun Morita) INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA, IBARAKI, 305-8571 JAPAN

E-mail address: `morita@math.tsukuba.ac.jp`

(Yoji Yoshii) IWATE UNIVERSITY, UEDA 3-18-33 MORIOKA, IWATE, JAPAN 020-8550

E-mail address: `yoshii@iwate-u.ac.jp`