LOCALLY EXTENDED AFFINE LIE ALGEBRAS

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Dedicated to Stephen Berman on the occasion of his 60th birthday

Abstract. We propose a new simplified definition of extended affine Lie algebras (EALAs for short), and also discuss a general version of extended affine Lie algebras, called locally extended affine Lie algebras (local EALAs for short). We prove a conjecture by V. Kac for local EALAs. It turns out that the root system of a local EALA becomes a locally finite version of an extended affine root system. Several examples of new EALAs and local EALAs are introduced, and finally we classify local EALAs of nullity 0 and show the connection to locally finite split simple Lie algebras.

Introduction

Extended affine Lie algebras, or EALAs for short, were first introduced by Høegh-Krohn and Torresani in 1990 [HT] (under the name of irreducible quasi-simple Lie algebras), as a generalization of finite-dimensional simple Lie algebras and affine Kac-Moody Lie algebras over the complex numbers $\mathbb{C}$. EALAs were systematically studied by Allison, Azam, Berman, Gao and Pianzola in [A-P]. They proved the so-called Kac conjecture, which implies that the root systems of EALAs are examples of extended affine root systems which were previously introduced by Saito [S] in 1985.

A natural question about the definition of original EALAs is the necessity of working over $\mathbb{C}$. A recent announcement by Neher [N2] has fixed this problem, taking as our base field an

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arbitrary field F of characteristic 0, and he has reported broad results about EALAs over F.

Our initial purpose was to give a more general (and simpler) definition of EALAs over F and prove the Kac conjecture. Roughly speaking, the conjecture says that a naturally induced symmetric bilinear form on the $\mathbb{Q}$-span of the roots is positive semidefinite. It turns out that the conjecture is true, and the proof can be somewhat simplified with an argument in [AKY] (and of course with the original methods in [A-P]). Thus one can define the *nullity* of an EALA as the $\mathbb{Q}$-dimension of the radical of the form, as in the original theory. However, in this note we do not need any assumptions for the set of roots R, except for the irreducibility of the set of anisotropic roots. The original theory assumed R to be discrete, but there is no such concept in our setting because our base field is more general. Also, an isotropic root can be isolated (e.g. all the roots can be isotropic in our setting). On the other hand, Neher assumed that the additive group generated by isotropic roots has finite rank [N2]. As a result, his EALAs are the tame EALAs of finite null rank in our sense.

We will give examples of an EALA of finite nullity but not having finite rank, and an EALA of infinite nullity. The latter algebra is constructed in a way analogous to the construction of affine Kac-Moody Lie algebras from an infinite-loop algebra

$$\mathfrak{g} \otimes_F F[t_i^{\pm 1}]_{i \in \mathbb{N}},$$

where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra over F and $F[t_i^{\pm 1}]_{i \in \mathbb{N}}$ is the ring of Laurent polynomials in infinitely many variables. (One needs $[F : \mathbb{Q}] = \infty$ to make this loop algebra an EALA.) So it seems reasonable to call the algebra an *EALA of nullity* ∞ (see §5).

We have also noticed that the finite dimensionality of our Cartan subalgebra is not used much for the theory. So we exclude this assumption, and instead an axiom, (A3) in §1, is added. This roughly says that R is small enough to be captured by our nondegenerate form. We call this new algebra a *locally extended affine Lie algebra* or a *local EALA* (or *LEALA*) for short. We will prove the Kac conjecture for local EALAs. An interesting phenomenon is that the roots of an EALA consist of a finite irreducible root system and isotropic roots, while the roots of a local EALA consist of a locally finite irreducible root system and isotropic roots. Thus a generalization of Saito's extended affine root systems naturally comes up.

The so-called affine Lie algebras of infinite rank (see [K, §7.11]) or locally finite split simple Lie algebras ([Stu] or [NS]) are our local EALAs of nullity 0. (We expect that local EALAs will be an interesting topic in the context of locally finite algebras and the recent work on locally finite root systems by Loos and Neher [LN].)

Also, in our axioms of a local EALA (or an EALA), even if there is no anisotropic root, the Kac conjecture still holds. Namely the form becomes identically zero. We call such an algebra a *null system*. Heisenberg Lie algebras with derivations are such examples (Example 7.1). Also, we construct an interesting null system of nullity ∞ from a generalized Witt algebra and its dual module (Example 7.3). This null system of finite null rank coincides with a subalgebra (null part) of an EALA of maximal type constructed in [BGK].

We classify local EALAs of nullity 0. The EALAs of nullity 0 are exactly the split central extensions of finite-dimensional split simple Lie algebras. However, a new phenomenon comes into play for local EALAs. We first show that the core of a local EALA L of nullity 0 is a locally finite split simple Lie algebra, and the centre of L is always split. So a problem is to classify the centreless local EALAs of nullity 0 (while the centreless EALAs are exactly the finite-dimensional split simple Lie algebras). Locally finite split simple Lie algebras do not exhaust the class. One of the structural difference comes from the fact that infinite-dimensional locally finite split simple Lie algebras have outer derivations. Consequently, a centreless local EALA of nullity 0 is isomorphic to the semidirect product of a locally finite split simple Lie algebra with a certain family of outer derivations.

In the final section, we discuss the relation between indecomposability and tameness of local EALAs.

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§1 Definition of a local EALA

Throughout the paper, let F be a field of characteristic 0. We say that a Lie algebra L has a *root decomposition* with respect to an abelian subalgebra H if

$$L = \bigoplus_{\xi \in H^*} L_\xi,$$

where H^* is the dual space of H and

$$L_\xi = \{x \in L \mid [h, x] = \xi(h)x \text{ for all } h \in H\}.$$

An element of the set

$$R = R(H) = \{\xi \in H^* \mid L_\xi \neq 0\}$$

is called a *root*. We consider a category $\mathcal{L}$ of Lie algebras with such an H and a symmetric invariant bilinear form $\mathcal{B}$. For $(L, H, \mathcal{B}), (L', H', \mathcal{B}') \in \mathcal{L}$, a morphism $\varphi : (L, H, \mathcal{B}) \longrightarrow (L', H', \mathcal{B}')$ is a Lie algebra homomorphism $\varphi : L \longrightarrow L'$ such that $\varphi(H) \subset H'$ and $\mathcal{B}(x, y) = \mathcal{B}'(\varphi(x), \varphi(y))$ for all $x, y \in L$. An isomorphism $\varphi : (L, H, \mathcal{B}) \longrightarrow (L', H', \mathcal{B}')$ is a morphism such that φ is a Lie algebra isomorphism $\varphi : L \longrightarrow L'$ with $\varphi(H) = H'$. If we take $(L_1, H_1, \mathcal{B}_1), (L_2, H_2, \mathcal{B}_2) \in \mathcal{L}$, then we have

$$(L_1, H_1, \mathcal{B}_1) \oplus (L_2, H_2, \mathcal{B}_2) := (L_1 \oplus L_2, H_1 \oplus H_2, \mathcal{B}_1 \perp \mathcal{B}_2) \in \mathcal{L}.$$

If a triple $(L, H, \mathcal{B})$ can never be expressed as $(L_1, H_1, \mathcal{B}_1) \oplus (L_2, H_2, \mathcal{B}_2)$, a direct sum of nontrivial triples, then $(L, H, \mathcal{B})$ is called *indecomposable*.

We note that the Jacobi identity implies that $[L_\xi, L_\eta] \subset L_{\xi+\eta}$, and the invariance of $\mathcal{B}$ implies $\mathcal{B}(L_\xi, L_\eta) = 0$ unless $\xi + \eta = 0$ for $\xi, \eta \in H^*$.

A triple $(L, H, \mathcal{B}) \in \mathcal{L}$ (or simply L) is called *admissible* if it satisfies

(A1) H is self-centralizing, i.e., $L_0 = H$;
(A2) $\mathcal{B}$ is nondegenerate;

Hence L_ξ and $L_{-\xi}$ form a nondegenerate pair relative to the form $\mathcal{B}$, and $\mathcal{B}\,|_{H\times H}$ is nondegenerate. In particular, $-R = R$.

(A3) $R \subset H^*_{\mathcal{B}}$, where $H^*_{\mathcal{B}}$ is the image of the canonical map $H \longrightarrow H^*$ defined by $h \mapsto \mathcal{B}(h, \cdot)$;

Hence there is the induced form on $H^*_{\mathcal{B}}$ from $\mathcal{B}$, simply denoted $(\cdot,\cdot)$. That is, since $\mathcal{B}\,|_{H\times H}$ is nondegenerate, we define $(\xi,\eta) := \mathcal{B}(t_\xi, t_\eta)$ for $\xi,\eta \in H^*_{\mathcal{B}}$, where t_ξ (or t_η similarly) is the unique element so that $\xi(h) = \mathcal{B}(t_\xi, h)$ for all $h \in H$. Then, we have

$$[x, y] = \mathcal{B}(x, y)t_\xi \tag{1.0}$$

for $\xi \in R$, $x \in L_\xi$ and $y \in L_{-\xi}$.

Remark 1.1. (1) Under the assumptions on H, the normalizer and the centralizer of H coincides, and so (A1) is equivalent to saying that H is a Cartan subalgebra (for a more general definition of Cartan subalgebras, see [NP]). In fact, a Lie algebra $L = (L, H)$ satisfying (A1) is called a *split Lie algebra* and H is called a *split Cartan subalgebra* of L in [NS, Def.II.1].

(2) If H is finite-dimensional, then (A3) is automatically satisfied, i.e., $H^*_{\mathcal{B}} = H^*$.

We call an element of the set

$$R^\times = R^\times(H,\mathcal{B}) = \{\xi \in R \mid (\xi,\xi) \neq 0\} \ \text{(resp. } R^0 = R^0(H,\mathcal{B}) = \{\xi \in R \mid (\xi,\xi) = 0\})$$

an *anisotropic root* (resp. an *isotropic root*). Note that R or $R^\times$ can be empty. For each nonzero $\xi \in R$, one chooses and fixes $x_\xi \in L_\xi$ and $x_{-\xi} \in L_{-\xi}$ such that $[x_\xi, x_{-\xi}] = t_\xi$ by (1.0). If $\xi \in R^\times$, the Lie algebra generated by x_ξ and $x_{-\xi}$ is isomorphic to $sl_2(F)$. Letting $y_\xi := \frac{2}{(\xi,\xi)}x_{-\xi}$ and $h_\xi := \frac{2}{(\xi,\xi)}t_\xi$, we call (x_ξ, h_ξ, y_ξ) an *sl_2-triplet for* $\xi \in R^\times$. Also, if $0 \neq \xi \in R^0$, the Lie algebra generated by x_ξ and $x_{-\xi}$ is a 3-dimensional Heisenberg Lie algebra. Letting $y_\xi := x_{-\xi}$, we call (x_ξ, t_ξ, y_ξ) a *Heisenberg-triplet for* $0 \neq \xi \in R^0$. Thus, an admissible triple is generated by a bunch of copies of $sl_2(F)$, 3-dimensional Heisenberg Lie algebras, and H.

Any symmetrizable Kac-Moody Lie algebra is an example of an admissible triple with finite-dimensional H.

Definition 1.2. An admissible triple $(L, H, \mathcal{B}) \in \mathcal{L}$ is called a *locally extended affine Lie algebra* or a *local EALA* for short, or simply a *LEALA* if it satisfies

(A4) $\operatorname{ad} x \in \operatorname{End}_F L$ is locally nilpotent for all $\xi \in R^\times$ and all $x \in L_\xi$,

(A5) $R^\times$ is irreducible, i.e., $R^\times = R_1 \cup R_2$ and $(R_1, R_2) = 0$ imply $R_1 = \emptyset$ or $R_2 = \emptyset$.

Also:

(i) If H is finite-dimensional, then L is called an *extended affine Lie algebra*, an *EALA* for short.

(ii) If $R^\times = \emptyset$, then $(L, H, \mathcal{B})$ is called a *null LEALA* (or a *null EALA*) or simply a *null system.*

(iii) If the triple $(L, H, \mathcal{B})$ is indecomposable, then it is called an *indecomposable EALA* (if H is finite-dimensional), an *indecomposable LEALA*, or an *indecomposable null system* (if $R^\times = \emptyset$).

Remark 1.3. Our EALAs generalize the original EALAs in [A-P]. For example, we only assume the base field F to be of characteristic 0, while the base field in [A-P] is $\mathbb{C}$ (the complex numbers). Also, even over $\mathbb{C}$, we will show in §5 that our class of EALAs is wider than the original one, but we would like to keep the same name EALA since the essential structure remains same. The EALAs defined by Neher in [N2] are exactly the tame EALAs of finite null rank in our sense (see Definition 3.11 and Definition 9.1).

§2 Basic Properties

Let $(L, H, \mathcal{B}) \in \mathcal{L}$ be a LEALA. Using sl_2-theory, one can prove the following lemma in the same way as in [A-P, Thm I.1.29]. (We do not need to assume (A5) in this section.)

Lemma 2.1. *Let $\alpha \in R^\times$. Then:*

(1) *For $\xi \in R$ we have $\xi(h_\alpha) = \frac{2(\xi,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.*
(2) *The reflection σ_α defined by $\sigma_\alpha(\mu) = \mu - \mu(h_\alpha)\alpha$ for all $\mu \in H^*$ preserves R, that is, $\sigma_\alpha(R) = R$. Also, σ_α is in the orthogonal group of the form on $H^*_{\mathcal{B}}$.*
(3) *If $k \in F$ and $k\alpha \in R$ then $k = 0, \pm 1$.*
(4) $\dim_F L_\alpha = 1$.
(5) *For any $\xi \in R$ there exist two non-negative integers d, u such that for any $n \in \mathbb{Z}$ we have $\xi + n\alpha \in R$ if and only if $-d \leq n \leq u$, where $\xi - d\alpha, \dots, \xi, \dots, \xi + u\alpha$ is called the α-string through ξ. Moreover, $d - u = \xi(h_\alpha) = \frac{2(\xi,\alpha)}{(\alpha,\alpha)}$.*

Also, we will use the following.

Lemma 2.2. *Let $\alpha, \beta \in R^\times$. If $\alpha + \beta$ and $\alpha - \beta \in R$, then $(\alpha, \alpha) = (\beta, \beta)$.*

Proof. Let $(x_{\alpha+\beta}, t_{\alpha+\beta}, x_{-\alpha-\beta})$ be a triplet satisfying $\mathcal{B}(x_{\alpha+\beta}, x_{-\alpha-\beta}) = 1$ (see (1.0)). Let $0 \neq x \in L_{\alpha-\beta}$. Since $2\alpha, -2\beta \notin R$, we have $[x_{\alpha+\beta}, x] = 0 = [x_{-\alpha-\beta}, x]$, and so $((\alpha, \alpha) - (\beta, \beta))x = (\alpha - \beta, \alpha + \beta)x = (\alpha - \beta)(t_{\alpha+\beta})x = [t_{\alpha+\beta}, x] = [[x_{\alpha+\beta}, x_{-\alpha-\beta}], x] = [x_{\alpha+\beta}, [x_{-\alpha-\beta}, x]] - [x_{-\alpha-\beta}, [x_{\alpha+\beta}, x]] = 0$. Hence, $(\alpha, \alpha) - (\beta, \beta) = 0$. □

§3 Kac conjecture

Let $(L, H, \mathcal{B}) \in \mathcal{L}$ be a LEALA. Here we scale the form $\mathcal{B}$ on L, which induces a scaling of the form $(\cdot, \cdot)$. Namely, we put $\mathcal{B}' = u\mathcal{B}$ for some nonzero element $u \in F$. This new form $\mathcal{B}'$ induces another form on $H^*_{\mathcal{B}} = H^*_{\mathcal{B}'}$, called $(\cdot, \cdot)'$. If $\alpha \in R$ with $\alpha \neq 0$, then $\mathcal{B}(t_\alpha, h) = \alpha(h) = \mathcal{B}'(t'_\alpha, h) = u\mathcal{B}(t'_\alpha, h) = \mathcal{B}(ut'_\alpha, h)$ and hence $t_\alpha = ut'_\alpha$, which implies $(\alpha, \alpha)' = \mathcal{B}'(t'_\alpha, t'_\alpha) = u\mathcal{B}(t'_\alpha, t'_\alpha) = \mathcal{B}(ut'_\alpha, t'_\alpha) = \mathcal{B}(t_\alpha, t_\alpha/u) = \mathcal{B}(t_\alpha, t_\alpha)/u = (\alpha, \alpha)/u$, where t'_α is the element of H corresponding to α defined by $\mathcal{B}'$. Modulo some scaling, we may assume that $(\alpha, \alpha) \in \mathbb{Q}$ for one $\alpha \in R^\times$. If $\beta \in R^\times$, we have $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ so that $(\beta, \alpha) \in \mathbb{Q}$, and hence, since $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, we get $(\beta, \beta) \in \mathbb{Q}$ if $(\alpha, \beta) \neq 0$. Thus, by (A5), our form is $\mathbb{Q}$-valued on the $\mathbb{Q}$-linear span of anisotropic roots. From now on *we assume that* *our form is scaled, if $R^\times \neq \emptyset$, so that there is at least one $\alpha \in R^\times$ with $(\alpha,\alpha) > 0$, and that $(\beta,\beta) \in \mathbb{Q}$ for all $\beta \in R^\times$.* But then we can immediately prove the following (shown in [A-P. Lem.I.2.3]).

Lemma 3.1. *Let $\gamma \in R^\times$. Then $(\gamma,\gamma) > 0$.*

We give an elementary proof which is different from the one in [A-P].

Proof. Suppose not, i.e., $(\gamma,\gamma) < 0$. Then, by (A5), there exists $\alpha, \beta \in R^\times$ such that $(\alpha,\alpha) > 0$, $(\beta,\beta) < 0$ and $(\alpha,\beta) \neq 0$. By Lemma 2.2, we have $\alpha+\beta \notin R$ or $\alpha-\beta \notin R$. If $\alpha+\beta \notin R$, then the root string $\beta - d\alpha, \dots, \beta-\alpha, \beta$ with $d = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} > 0$ gives $(\beta,\alpha) > 0$. However, the root string $\alpha - d'\beta, \dots, \alpha-\beta, \alpha$ with $d' = \frac{2(\alpha,\beta)}{(\beta,\beta)} > 0$ gives $(\alpha,\beta) < 0$, which is a contradiction. If $\alpha - \beta = \alpha + (-\beta) \notin R$, we also get a contradiction since $(-\beta,-\beta) < 0$ and $(\alpha,-\beta) \neq 0$. □

We later use the following. The proof is the same as in [A-P, Lem.I.2.6] using Lemma 2.1(5) and Lemma 3.1:

Lemma 3.2. *Let $\alpha \in R^\times$ and $\xi \in R$. Then $-4 \leq \xi(h_\alpha) \leq 4$.*

Now, we shall prove a crucial property that the isotropic roots are in the radical of the form. This was proved in [A-P, Prop.I.2.1] using two extra assumptions, namely, R is discrete in H^* (assuming the base field is $\mathbb{C}$) and R^0 is not isolated. We do not need such assumptions. The first part of our proof has already been established in in a few lines in the recent preprint [AKY] (in a different setup). It turns out that a small modification of the proof in [A-P] was enough to exclude the extra assumptions.

We start with a lemma [A-P, Lem.I.1.30]. (This is well-known in terms of the representation theory for the 3-dimensional Heisenberg Lie algebra. Namely, there is no nontrivial finite-dimensional representation of the Lie algebra. [B])

Lemma 3.3. *Let $\delta \in R^0$ and $\xi \in R$. Suppose that $(\xi,\delta) \neq 0$. Then $\xi + n\delta \in R$ for infinitely many integers n.*

For the convenience of the reader, and also since [AKY] starts with different axioms, we repeat their argument hence showing that our axioms are enough. Also, note that it works for a null system.

Proposition 3.4. *Let $(L, H, \mathcal{B}) \in \mathcal{L}$ be a LEALA. Let $\xi \in R$ and $\delta \in R^0$. Then $(\xi,\delta) = 0$. That is, $(R, R^0) = 0$.*

Proof. First we show the statement for $\alpha := \xi \in R^\times$. Suppose that $(\alpha,\delta) \neq 0$. Then, by Lemma 3.3, we have $\alpha + n\delta \in R$ for infinitely many integers n. But $\alpha + n\delta$ can be isotropic for at most one n (since $(\alpha+n\delta, \alpha+n\delta) = 0 \Rightarrow (\alpha,\alpha) + 2n(\alpha,\delta) = 0 \Rightarrow n = -\frac{2(\alpha,\delta)}{(\alpha,\alpha)}$). Hence, $\alpha + n\delta \in R^\times$ for infinitely many integers n, and by Lemma 2.1(1),

$$\frac{2(\delta, \alpha+n\delta)}{(\alpha+n\delta, \alpha+n\delta)} = \frac{2(\delta,\alpha)}{(\alpha,\alpha)+2n(\alpha,\delta)} \in \mathbb{Z}$$

for infinitely many integers n, which is impossible. Therefore, $(\alpha,\delta) = 0$.

Suppose that $\delta' := \xi \in R^0$ and $(\delta', \delta) \neq 0$. Then, $(\delta \pm \delta', \delta \pm \delta') = \pm 2(\delta, \delta') \neq 0$. If $\delta + \delta' \in R$ (resp. $\delta - \delta' \in R$), then $\delta + \delta' \in R^\times$ (resp. $\delta - \delta' \in R^\times$). In this case, we have $0 = (\delta, \delta \pm \delta') = \pm(\delta, \delta') \neq 0$, which is a contradiction. If $\delta \pm \delta' \notin R$, then $[x_\delta, x_{\delta'}] = [y_\delta, x_{\delta'}] = 0$, and so $[t_\delta, x_{\delta'}] = 0$. (The notations are from Heisenberg triplets, defined in §1.) This implies $[t_\delta, x_{\delta'}] = (\delta, \delta')x_{\delta'}$, and hence $(\delta, \delta') = 0$, a contradiction again. $\square$

Let $(L, H, \mathcal{B}) \in \mathcal{L}$ be a LEALA or a null system. Let V be the $\mathbb{Q}$-subspace of H^* spanned by R. We will prove that the form on V is positive semidefinite (Kac conjecture). First as an immediate corollary of Proposition 3.4, we have:

Corollary 3.5. *Let $(L, H, \mathcal{B}) \in \mathcal{L}$ be a null system. Then the form $(\cdot, \cdot)$ on V is zero.*

Let $(L, H, \mathcal{B}) \in \mathcal{L}$ be a LEALA. Put $V^0 = \{v \in V \mid (v, w) = 0 \text{ for all } w \in V\}$, the radical of $(\cdot, \cdot)$, and $\bar{V} = V/V^0$. Let $^-\colon V \longrightarrow \bar{V}$ be the canonical map. Then the form $(\cdot, \cdot)$ on V induces a unique nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $\bar{V}$ so that $(\bar{v}, \bar{w}) = (v, w)$ for all $v, w \in V$. Define $\bar{R} = \{\bar{\xi} \mid \xi \in R\}$ and $\bar{R}^\times = \{\bar{\alpha} \mid \alpha \in R^\times\}$. Then $\bar{R} = \bar{R}^\times \cup \{\bar{0}\}$. Hence, $(\bar{\alpha}, \bar{\alpha}) > 0$ for all $\bar{\alpha} \in \bar{R}$ with $\bar{\alpha} \neq \bar{0}$, and $2(\bar{\beta}, \bar{\alpha})/(\bar{\alpha}, \bar{\alpha}) \in \mathbb{Z}$.

We note that the following result is true in general, which is trivial for the finite-dimensional case.

Lemma 3.6. *Let X be a vector space over any field with nondegenerate symmetric bilinear form $(\cdot, \cdot)$. Let Y be a finite-dimensional subspace of X. Then there exists a finite-dimensional subspace $\tilde{Y}$ of X containing Y such that $(\cdot, \cdot)\,|_{\tilde{Y} \times \tilde{Y}}$ is nondegenerate.*

Proof. Let $\{y_1, \ldots, y_r, u_1, \ldots, u_m\}$ be a basis of Y so that $\{u_1, \ldots, u_m\}$ is a basis of the radical of $(\cdot, \cdot)\,|_{Y \times Y}$. If $m = 0$ (i.e., the radical is zero), we can take $\tilde{Y} = Y$, and so we assume $m > 0$. Since $(\cdot, \cdot)$ is nondegenerate on X, there exists $x_1 \in X$ such that $(u_1, x_1) = 1$. Let $Y_1 := \langle y_1, \ldots, y_r, u_1, x_1 \rangle$. Let $y'_i := y_i - (y_i, x_1)u_1$ for $i = 1, \ldots, r$. Then $Y_1 = \langle y'_1, \ldots, y'_r \rangle \perp \langle u_1, x_1 \rangle$, and so $(\cdot, \cdot)\,|_{Y_1 \times Y_1}$ is nondegenerate. Let $\tilde{Y}_1 := Y_1 + \langle u_2, \ldots, u_m \rangle$. Let $u'_j := u_j - (x_1, u_j)u_1$ for $j = 2, \ldots, m$. Then $Y \subset \tilde{Y}_1 = Y_1 \perp \langle u'_2, \ldots, u'_m \rangle$, and the radical $\langle u'_2, \ldots, u'_m \rangle$ on $\tilde{Y}_1$ has dimension $m-1$. Hence, by induction, there exists a subspace $\tilde{Y}$ of X containing $\tilde{Y}_1$ such that $(\cdot, \cdot)\,|_{\tilde{Y} \times \tilde{Y}}$ is nondegenerate. $\square$

Using this lemma and a technique similar to [A-P, Lem.I.2.10], we can prove something new:

Proposition 3.7. *$(\bar{R}, \bar{V})$ is a locally finite irreducible root system (in the sense of [LN]). Also, $\bar{R}$ is finite if $\dim_F H < \infty$ (i.e., if L is an EALA), and hence $\bar{V}$ is finite-dimensional in this case, and so $(\bar{R}, \bar{V})$ is a finite irreducible root system (in the sense of [Bo, Ch.VI]).*

Proof. We choose a basis $\{\bar{\alpha}_i\}_{i \in I}$ of $\bar{V}$, where $\alpha_i \in R$. Let $\bar{W}$ be a finite-dimensional subspace of $\bar{V}$. Suppose that $\bar{R} \cap \bar{W} \neq 0$. Then, by Lemma 3.6, there exists a finite-dimensional subspace $\mathcal{U}$ of $\bar{V}$ containing $\bar{W}$ such that $(\cdot, \cdot)\,|_{\mathcal{U} \times \mathcal{U}}$ is nondegenerate. Let $\{\bar{w}_1, \ldots, \bar{w}_r\}$ be a basis of $\mathcal{U}$. Since $(\cdot, \cdot)\,|_{\mathcal{U} \times \mathcal{U}}$ is nondegenerate, the map $f : \bar{\beta} \mapsto ((\bar{w}_1, \bar{\beta}^\vee), \ldots, (\bar{w}_r, \bar{\beta}^\vee))$ of

$\bar{R}\cap\bar{W}$ into $\mathbb{Q}^r$ is injective, where $\bar{\beta}^\vee = 2\bar{\beta}/(\bar{\beta},\bar{\beta})$. Since $\bar{w}_i = \sum_{j\in I_i} a_{ij}\bar{\alpha}_j$ for all $1\le i\le r$, some finite subset I_i of I and $a_{ij}\in\mathbb{Q}$, we have, by Lemma 3.2,

$$(\bar{w}_i,\bar{\beta}^\vee) = \sum_{j\in I_i} a_{ij}(\bar{\alpha}_j,\bar{\beta}^\vee) = \sum_{j\in I_i} a_{ij}\alpha_j(h_\beta) \in \{\sum_{j\in I_i} m_j a_{ij} \mid -4\le m_j\le 4,\ m_j\in\mathbb{Z}\}.$$

Hence the image of f is finite and so is $\bar{R}\cap\bar{W}$, i.e., $\bar{R}$ is locally finite. By Lemma 2.1(2), we have $\sigma_{\bar{\alpha}}(\bar{R}) = \bar{R}$, and so $(\bar{R},\bar{V})$ is a locally finite root system relative to the map ${}^\vee : \bar{R}\longrightarrow \bar{V}^*$ defined by the pairing $\langle \bar{v},\bar{\alpha}\rangle := (\bar{v},\bar{\alpha}^\vee)$ for all $\bar{v}\in\bar{V}$ and $\bar{\alpha}\in\bar{R}$. The irreducibility follows from (A5).

For the second statement, let V_F be the subspace of H^* generated by R over F. Put $V_F^0 = \{v\in V_F \mid (v,w)=0 \text{ for all } w\in V_F\} = \{v\in V_F \mid (v,\beta)=0 \text{ for all } \beta\in R\}$, the radical of $(\cdot,\cdot)$ on V_F. Then we see that $V^0 = V\cap V_F^0$. Therefore, $\bar{V}_F := V_F/V_F^0 \supset V/V^0 = \bar{V}\supset\bar{R}$. Thus if H^* is finite-dimensional, then so is V_F or $\bar{V}_F$. We choose an F-basis $\{\bar{\alpha}_1,\dots,\bar{\alpha}_\ell\}$ of $\bar{V}_F$ from $\bar{R}$. Since $(\cdot,\cdot)$ is nondegenerate on $\bar{V}_F$, the map $\bar{\beta}\mapsto((\bar{\beta},\bar{\alpha}_1^\vee),\dots,(\bar{\beta},\bar{\alpha}_\ell^\vee))$ of $\bar{R}$ into $\{-4,\dots,0,\dots,4\}^\ell$ is injective (by Lemma 3.2). Hence $\bar{R}$ is finite. $\square$

One of the important notions for subsets of a locally finite root system is the fullness (see [LN] or [NS]).

Definition 3.8. Let Δ be a locally finite root system and M a subset of Δ. Then $\Delta_M := (\mathrm{span}_{\mathbb{Q}} M)\cap\Delta$ is called the *full subsystem generated by M*.

The following lemma [LN, Cor.3.16 or Stu, Prop.V4] is useful (see also [LN, Prop.3.13]).

Lemma 3.9. *Let M be a finite irreducible subset of a locally finite irreducible root system Δ. Then the full subsystem Δ_M is finite and irreducible.*

The Kac conjecture follows from [LN, Thm 4.2] or from an argument analogous to the one given in [A-P]: Let $0\ne\bar{v}\in\bar{V}$. Then $\bar{v} = \sum_{\bar{\alpha}_i\in M} a_i\bar{\alpha}_i$ for $a_i\in\mathbb{Q}$ and some finite irreducible subset M of $\bar{R}$. Then, by Lemma 3.9, the full subsystem $\bar{R}_M$ is finite and irreducible so that $\bar{W} := \mathrm{span}_{\mathbb{Q}} M$ contains $\bar{v}$. Then, $(\cdot,\cdot)\mid_{\bar{W}\times\bar{W}}$ is nondegenerate since the Cartan matrix of $\bar{R}_M$ is nonsingular. Then, apply for $(\bar{R}_M,\bar{W})$ instead of $(\bar{R},\bar{V})$ in [A-P, Thm I.2.14] to get $(\bar{v},\bar{v})>0$ (since $(\bar{\alpha},\bar{\alpha})>0$ for all $\bar{\alpha}\in\bar{R}_M$). Thus:

Theorem 3.10. *Let $(L,H,\mathcal{B})\in\mathcal{L}$ be a LEALA. Then the form $(\cdot,\cdot)$ on V is positive semidefinite.*

Definition 3.11. The dimension of the radical for V is called the *nullity* for a LEALA. If the additive subgroup of V generated by R^0 is free, we call the rank the *null rank* for a LEALA.

The null rank and the nullity coincide for the definition in [A-P] or [N2]. In our definition, null rank n implies nullity n, but nullity n does not imply null rank n. We will give an example of an EALA of nullity 1, which does not have any null rank in §5.

§4 Root systems

One can naturally consider a generalization of extended affine root systems defined by Saito [S], and the anisotropic root systems of our LEALAs (in $V \otimes_{\mathbb{Q}} \mathbb{R}$) are examples of the generalized root systems, defined in the following.

Definition 4.1. Let V be a vector space over $\mathbb{R}$ with (nonzero) positive semidefinite form $(\cdot,\cdot)$. A subset $\mathfrak{R}$ of V is called a *locally extended affine root system* or a *LEARS* for short if

(1) $(\alpha,\alpha) \neq 0$ for all $\alpha \in \mathfrak{R}$, and $\mathfrak{R}$ generates V;
(2) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \mathfrak{R}$;
(3) $\sigma_\alpha(\mathfrak{R}) \subset \mathfrak{R}$ for all $\alpha \in \mathfrak{R}$, where $\sigma_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$;
(4) $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2$ and $(\mathfrak{R}_1, \mathfrak{R}_2) = 0$ imply $\mathfrak{R}_1 = \emptyset$ or $\mathfrak{R}_2 = \emptyset$. (Or $\mathfrak{R}$ is irreducible.)

Let $(V, \mathfrak{R})$ be a LEARS. Let $(\bar{V}, \bar{\mathfrak{R}})$ be the canonical image onto V modulo the radical as in §3. Then $\bar{V}$ admits the positive definite form, and so as in [A-P, Lem.II.2.8], we get the following.

Proposition 4.2. *Let $(V, \mathfrak{R})$ be a LEARS. Then $(\bar{V}, \bar{\mathfrak{R}})$ is a locally finite irreducible root system.*

Proof. By Schwartz Inequality, we have

$$\left| \frac{2(\bar{\beta},\bar{\alpha})}{(\bar{\alpha},\bar{\alpha})} \frac{2(\bar{\alpha},\bar{\beta})}{(\bar{\beta},\bar{\beta})} \right| \leq 4$$

for all $\bar{\alpha}, \bar{\beta} \in \bar{\mathfrak{R}}$. Thus, for $\alpha, \beta \in \mathfrak{R}$, we get $-4 \leq \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \leq 4$, and repeat the argument in Proposition 3.7. □

Remark 4.3. A LEARS $\mathfrak{R}$ is called *reduced* if $2\alpha \notin \mathfrak{R}$ for all $\alpha \in \mathfrak{R}$. So, by Lemma 2.1(3), the set $R^\times$ of anisotropic roots of a LEALA is a reduced LEARS.

§5 Examples of new EALAs

First we construct an analogue of loop algebras. Let $\Lambda = (\Lambda, +, 0)$ be an abelian group. Let

$$F[\Lambda] = \bigoplus_{\lambda \in \Lambda} F\overline{\lambda}$$

be the group algebra of Λ over F. We define a bilinear form ε on $F[\Lambda]$ by

$$\varepsilon(\overline{\lambda}, \overline{\mu}) := \begin{cases} 1 & \text{if } \lambda + \mu = 0 \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda, \mu \in \Lambda$. Then ε is a nondegenerate symmetric invariant form. Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra of type Δ with a split Cartan subalgebra $\mathfrak{h}$. Let

$$M := \mathfrak{g} \otimes_F F[\Lambda]$$

be the Lie algebra with the bracket defined by

$$[x \otimes \overline{\lambda}, y \otimes \overline{\mu}] = [x, y] \otimes \overline{\lambda + \mu}$$

for $x, y \in \mathfrak{g}$ and $\lambda, \mu \in \Lambda$. Note that if $\Lambda = \mathbb{Z}$, then M is a loop algebra. Let

$$(\cdot, \cdot) := \kappa \otimes \varepsilon,$$

where κ is the Killing form of $\mathfrak{g}$. Then $(\cdot, \cdot)$ is a nondegenerate symmetric invariant bilinear form.

We assume that there exists a nonzero additive homomorphism φ of Λ into F. Let d_φ be a degree derivation of $F[\Lambda]$ determined by φ, i.e., $d_\varphi(\overline{\lambda}) = \varphi(\lambda)\overline{\lambda}$ for $\lambda \in \Lambda$. We define a Lie algebra L by

$$L = M \oplus Fc \oplus Fd_\varphi,$$

where c is a nonzero central element with multiplication as follows:

$$\begin{aligned} &[d_\varphi, x \otimes \overline{\lambda}] = x \otimes d_\varphi(\overline{\lambda}) = -[x \otimes \overline{\lambda}, d_\varphi] \quad \text{and} \\ &[x \otimes \overline{\lambda}, y \otimes \overline{\mu}] = [x, y] \otimes \overline{\lambda + \mu} + (x \otimes d_\varphi(\overline{\lambda}), y \otimes \overline{\mu})c \end{aligned} \tag{5.1}$$

for all $x \otimes \overline{\lambda}, y \otimes \overline{\mu} \in M$. (Indeed, this is a Lie algebra since d_φ is a skew derivation relative to ε.) Also, one can extend the form $(\cdot, \cdot)$ to a form $\mathcal{B}(\cdot, \cdot)$ on L by

$$\mathcal{B}(c, c) = \mathcal{B}(d_\varphi, d_\varphi) = \mathcal{B}(c, M) = \mathcal{B}(d_\varphi, M) = 0 \quad \text{and} \quad \mathcal{B}(c, d_\varphi) = 1.$$

Then $\mathcal{B}(\cdot, \cdot)$ is a nondegenerate symmetric invariant bilinear form. Let

$$H = \mathfrak{h} \oplus Fc \oplus Fd_\varphi.$$

Let δ_λ for $\lambda \in \Lambda$ be the element of H^* such that $\delta_\lambda(d_\varphi) = \varphi(\lambda)$ and $\delta_\lambda(\mathfrak{h}) = \delta_\lambda(c) = 0$. Then

$$[d_\varphi, h \otimes \overline{\lambda}] = d_\varphi(\lambda) h \otimes \overline{\lambda} = \varphi(\lambda) h \otimes \overline{\lambda} = \delta_\lambda(d_\varphi) h \otimes \overline{\lambda}$$

and $[\mathfrak{h} \oplus Fc, h \otimes \overline{\lambda}] = 0$ for all $h \in \mathfrak{h}$. So δ_λ is a root relative to H. Note that the centralizer of H is contained in $(\mathfrak{h} \otimes F[\Lambda]) \oplus Fc \oplus Fd_\varphi$. Thus H is self-centralizing if and only if φ is injective. We now assume that φ is injective so that $(L, H, \mathcal{B})$ is an admissible triple. Note that

$$\Lambda \text{ has to be torsion-free.}$$

Then one can check that if we denote the root system of $\mathfrak{g}$ by Δ, then

$$R^0 = \{\delta_\lambda \mid \lambda \in \Lambda\} \quad \text{and} \quad R^\times = \{\alpha + \delta_\lambda \mid \alpha \in \Delta, \ \lambda \in \Lambda\},$$

and L is an EALA. In particular, if we take Λ to be any additive subgroup of F and φ to be the inclusion, then $R^0 \cong \Lambda$ and the nullity of L is equal to the dimension of the $\mathbb{Q}$-span of Λ over $\mathbb{Q}$. For example, if $\Lambda = \mathbb{Q}$, then L is an EALA of nullity 1 and $R^0 = \langle R^0 \rangle$ is not a free abelian group (and so $\langle R \rangle$ is not a lattice). So L does not have null rank.

To give an example of null rank ∞ (so nullity ∞), let $\mathfrak{I}$ be an index set of an arbitrary cardinality. We assume that F contains a linearly independent subset S over $\mathbb{Q}$ with $|S| = |\mathfrak{I}|$. Let $\Lambda = \mathbb{Z}^{\oplus \mathfrak{I}}$. Then there is a group isomorphism from Λ into the additive subgroup of F generated by S, say φ. Thus our construction above gives an EALA with $R^0 = \mathbb{Z}^{\oplus \mathfrak{I}}$ and the null rank is $|\mathfrak{I}|$.

We note that this example is a natural generalization of untwisted affine Kac-Moody Lie algebras from the loop algebra $\mathfrak{g} \otimes F[t^{\pm 1}]$ to an infinite-loop algebra

$$\mathfrak{g} \otimes F[t_i^{\pm 1}]_{i \in \mathbb{N}}.$$

where $F[t_i^{\pm 1}]_{i \in \mathbb{N}}$ is the ring of Laurent polynomials in infinitely many variables. To see this more clearly, we give a slightly different description.

Let $F = \mathbb{C}$ for convenience. (One can take F to be $\mathbb{R}$ or any field with $[F : \mathbb{Q}] = \infty$.) In fact,

$$\mathbb{C}[t_i^{\pm 1}]_{i \in \mathbb{N}} = \bigoplus_{\boldsymbol{\alpha} \in \mathbb{Z}^{\oplus \mathbb{N}}} \mathbb{C} t_{\boldsymbol{\alpha}},$$

where $t_{\boldsymbol{\alpha}} = \prod_{i \in \mathbb{N}} t_i^{\alpha_i}$ for $\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{Z}^{\oplus \mathbb{N}}$ (only a finite number of α_i is nonzero), is the group algebra $\mathbb{C}[\mathbb{Z}^{\oplus \mathbb{N}}]$ by $\overline{\boldsymbol{\alpha}} = t_{\boldsymbol{\alpha}}$. Let $\{s_i\}_{i \in \mathbb{N}}$ be a linearly independent subset of $\mathbb{C}$ over $\mathbb{Q}$, and let

$$d := \sum_{i \in \mathbb{N}} s_i t_i \frac{\partial}{\partial t_i} \quad \text{(formal infinite sum)}$$

be the derivation of $\mathbb{C}[t_i^{\pm 1}]_{i \in \mathbb{N}}$, which can be interpreted as our d_φ above. Thus the second part of (5.1) can be rephrased as follows:

$$[x \otimes t_{\boldsymbol{\alpha}}, y \otimes t_{\boldsymbol{\beta}}] = [x, y] \otimes t_{\boldsymbol{\alpha}+\boldsymbol{\beta}} + \kappa(x, y) \sum_{i \in \mathbb{N}} s_i \mathrm{Res}_i\left(\frac{\partial t_{\boldsymbol{\alpha}}}{\partial t_i} t_{\boldsymbol{\beta}}\right) c$$

for all $x \otimes t_{\boldsymbol{\alpha}}, y \otimes t_{\boldsymbol{\beta}} \in M$, where Res_i gives the coefficient of t_i^{-1}.

Remark 5.2. (1) In the setting above, let $|I| = 2$, $s_1 = 1$ and $s_2 = \sqrt{2}$ (or any irrational number). Then $R^0 = \mathbb{Z}^2$ and so the null rank is 2, but R^0 is not discrete in the $\mathbb{R}$-span of R^0. So this is not an EALA in the sense of [A-P] (see also [G]). But in our sense this is just an EALA of null rank 2 over $\mathbb{C}$.

(2) The algebra $\mathbb{C}[t_i^{\pm 1}]_{i \in \mathbb{N}}$ can be generalized to a quantum torus of infinitely many variables $\mathbb{C}_{\boldsymbol{q}}[t_i^{\pm 1}]_{i \in \mathbb{N}}$, and from $sl_n(\mathbb{C}_{\boldsymbol{q}}[t_i^{\pm 1}]_{i \in \mathbb{N}})$, we get an EALA of null rank ∞ by the same construction. More generally, from Jordan or structurable Λ-tori ([Y1], [AY] or [Y3]), where Λ is a torsion-free abelian group, one can construct various new EALAs.

§6 Examples of LEALAs

Let $\mathfrak{g}_\infty$ be one of the so-called infinite rank affine algebras of type $\mathrm{A}_{+\infty}$, A_∞, B_∞, C_∞ and D_∞, and identify it with a subalgebra of $gl_\infty(F)$, the Lie algebra of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ $(a_{ij}\in F)$ such that the number of nonzero a_{ij} is finite, with the usual bracket, as in [K, §7.11]. Then it is easy to show that $\mathfrak{g}_\infty$ with the trace form and taking our Cartan to be the subalgebra consisting of diagonal matrices is a LEALA of nullity 0. Also, as in §5, one can construct a LEALA

$$(\mathfrak{g}_\infty \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d_\varphi \quad \text{(null rank } n\text{) or}$$

$$(\mathfrak{g}_\infty \otimes_{\mathbb{C}} \mathbb{C}[t_i^{\pm 1}]_{i\in\mathbb{N}}) \oplus \mathbb{C}c \oplus \mathbb{C}d_\varphi \quad \text{(null rank } \infty\text{)},$$

taking the trace form instead of the Killing form for the multiplication. Moreover, as in Remark 5.2(2),

$$sl_\infty(\mathbb{C}_{\boldsymbol{q}}[t_i^{\pm 1}]_{i\in\mathbb{N}}) \oplus \mathbb{C}c \oplus \mathbb{C}d_\varphi$$

is a LEALA of null rank ∞. ($sl_\infty(A)$ for any associative algebra A is defined in [N1].)

Locally finite split simple Lie algebras were classified by Neeb and Stumme in [Stu] and [NS]. They showed that locally finite split simple Lie algebras over a field of characteristic 0 are isomorphic to exactly one of infinite rank affine Lie algebras of type A_∞, B_∞ and C_∞. (They showed that $\mathrm{A}_{+\infty}\cong\mathrm{A}_\infty$ and $\mathrm{B}_\infty\cong\mathrm{D}_\infty$.) To classify LEALAs of nullity 0 in §8, we need more information about these algebras. Thus we precisely define them here. (The size of matrices is not only $\aleph_0$ but any cardinality.)

Definition 6.1. Let $\mathfrak{I}$ be any index set. Then the Lie algebra of type $\mathrm{X}_\mathfrak{I}$ is defined as a subalgebra of the matrix algebra $gl_\mathfrak{I}(F)$, $gl_{2\mathfrak{I}+1}(F)$ or $gl_{2\mathfrak{I}}(F)$ consisting of matrices having only a finite number of nonzero entries:

Type $\mathrm{A}_\mathfrak{I}$; $sl_\mathfrak{I}(F)=\{x\in gl_\mathfrak{I}(F)\mid \mathrm{tr}(x)=0\}$;
Type $\mathrm{B}_\mathfrak{I}$; $o_{2\mathfrak{I}+1}(F)=\{x\in gl_{2\mathfrak{I}+1}(F)\mid sx=-x^ts\}$;
Type $\mathrm{C}_\mathfrak{I}$; $sp_{2\mathfrak{I}}(F)=\{x\in gl_{2\mathfrak{I}}(F)\mid sx=-x^ts\}$;
Type $\mathrm{D}_\mathfrak{I}$; $o_{2\mathfrak{I}}(F)=\{x\in gl_{2\mathfrak{I}}(F)\mid sx=-x^ts\}$;

where

$$s=\begin{pmatrix}0 & I_\mathfrak{I} & 0\\ I_\mathfrak{I} & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\ \begin{pmatrix}0 & I_\mathfrak{I}\\ -I_\mathfrak{I} & 0\end{pmatrix} \quad\text{or}\quad \begin{pmatrix}0 & I_\mathfrak{I}\\ I_\mathfrak{I} & 0\end{pmatrix},$$

respectively for type $\mathrm{B}_\mathfrak{I}$, $\mathrm{C}_\mathfrak{I}$ or $\mathrm{D}_\mathfrak{I}$, and $I_\mathfrak{I}$ is the identity matrix of size $\mathfrak{I}$ (x^t = transpose of x).

We give interesting examples of LEALAs of nullity 0.

Example 6.2. (1) Let $d=\sum_{i\in\mathbb{N}} ie_{ii}$, where the e_{ij} are matrix units in $M_\mathbb{N}(F)$. Define a bilinear form $\mathcal{B}$ on the matrix Lie algebra $L=sl_\mathbb{N}(F)\oplus Fd$ by $\mathcal{B}(x,y):=\mathrm{tr}(xy)$ for $x\in sl_\mathbb{N}(F)$ and $y\in L$, and $\mathcal{B}(d,d):=0$. Let $\mathfrak{h}$ be the subalgebra of $sl_\mathbb{N}(F)$ consisting of diagonal matrices, and let $H:=\mathfrak{h}\oplus Fd$. Then $(L,H,\mathcal{B})$ is a centreless LEALA of nullity 0.

(2) Let $d=\sum_{i\in\mathbb{N}}(e_{ii}-e_{\mathbb{N}+i,\mathbb{N}+i})$. Define a bilinear form $\mathcal{B}$ on the natural Lie algebra $L=o_{2\mathbb{N}+1}(F)\oplus Fd$ by $\mathcal{B}(x,y):=\mathrm{tr}(xy)$ for $x\in o_{2\mathbb{N}+1}(F)$ and $y\in L$, and $\mathcal{B}(d,d):=0$. Let

$\mathfrak{h}$ be the subalgebra of $o_{2\mathbb{N}+1}(F)$ consisting of diagonal matrices, and let $H := \mathfrak{h} \oplus Fd$. Then $(L, H, \mathcal{B})$ is a centreless LEALA of nullity 0.

Note that one can construct similar kinds of Lie algebras of type $\mathrm{C}_{\mathfrak{J}}$ or $\mathrm{D}_{\mathfrak{J}}$. Also, for any case, you can choose any scalar for $\mathcal{B}(d, d)$ (not necessarily 0).

We will show in Proposition 8.3 that the *core* of a LEALA (which will be defined in 8.1) of nullity 0 is a locally finite split simple Lie algebra. To classify LEALAs in general, one may need some ideas from the theory of locally finite Lie algebras (e.g. [BB], [GN] or [St]).

§7 Examples of null systems

A typical example of null systems is a Heisenberg Lie algebra with derivations added. More precisely:

Example 7.1. Let $\Lambda = (\Lambda, +, 0)$ be an abelian group. Let S be a subset of Λ satisfying

$$0 \in S, \quad \text{and} \quad \delta \in S \Rightarrow -\delta \in S.$$

Let $N = \oplus_{\delta \in S} N_\delta$ be a graded vector space over F with a symmetric bilinear form $\mathcal{B}$ satisfying

(N1) $\mathcal{B}\,|_{N_\delta \times N_{-\delta}}$ is nondegenerate for each $\delta \in S$;
(N2) $N_0 = Fc + Fd$ for some $c, d \in N_0$ with $\mathcal{B}(c, c) = \mathcal{B}(d, d) = 0$ and $\mathcal{B}(c, d) = 1$ (or N_0 is a hyperbolic plane);
(N3) $\mathcal{B}\,|_{N_\delta \times N_{\delta'}} = 0$ unless $\delta' = -\delta$.

We assume that there exists an injective additive homomorphism φ from Λ into F. So Λ has to be torsion-free as in §7. Now we define the Lie bracket on $N = (N, S, \mathfrak{b}, \varphi)$ as follows. For any $0 \neq \delta, \delta' \in S$, $x \in N_\delta$ and $y \in N$, we define

$$[x, y] = \mathcal{B}(\varphi(\delta)x, y)c, \quad [c, N] = 0 = [N, c] \quad \text{and} \quad [d, x] = \varphi(\delta)x = -[x, d].$$

Then one can check that N is in fact a Lie algebra, and $(N, N_0, \mathcal{B})$ is a null system. Note that Fc is the centre of N, and $N = [N, N] \oplus Fd$. As in §5, the nullity of N is the $\mathbb{Q}$-dimension of the $\mathbb{Q}$-span of $\varphi(\Lambda)$ in F, and the null rank of N is the rank of Λ if Λ is free.

Note that this N is indecomposable unless $S = 0$. In fact, suppose that N is decomposable. Then $N = (L_1, H_1, \mathcal{B}_1) \oplus (L_2, H_2, \mathcal{B}_2)$, and $H_1 = F(c + ud)$ and $H_2 = F(c - ud)$ for some nonzero $u \in F$ since $H_1 \perp H_2$. Let $0 \neq x \in N_\delta$ for any $0 \neq \delta \in S$. Then $[c + ud, x] = u\varphi(\delta)x \neq 0$, and so $x \notin L_2$. Similarly, $x \notin L_1$. But we have $N_\delta = (N_\delta \cap L_1) \oplus (N_\delta \cap L_2)$, which implies $N_\delta = 0$, a contradiction.

Remark 7.2. (1) If $\Lambda = \mathbb{Z}$, $S = \{0, \pm 1\}$, and $\varphi = \mathrm{id}$, then $[N, N]$ is a Heisenberg Lie algebra (see [MP, §1.5]). If we also assume that $\dim_F N_1 = \dim_F N_{-1} = 1$, then N is 4-dimensional and $[N, N]$ is the 3-dimensional Heisenberg Lie algebra. Moreover, if one takes an n copies of this 4-dimensional N, i.e., $N^{(n)} := N \oplus \cdots \oplus N$, then $[N^{(n)}, N^{(n)}]$ is usually called a Heisenberg Lie algebra of order n (see [K, §2.9]).

(2) Let $L = (\mathfrak{g} \otimes F[t^{\pm 1}]) \oplus Fc \oplus Fd$ be an (untwisted) affine Kac-Moody Lie algebra, which is a special case in the previous section, and let $N_m := \mathfrak{h} \otimes t^m$ for $m \neq 0$ and $N_0 := Fc \oplus Fd$.

Then the $\mathbb{Z}$-graded subalgebra N of L is a null system ($S = \mathbb{Z}$) and $[N, N]$ is a $\mathbb{Z}$-graded Heisenberg Lie algebra.

(3) If $S = 0$, then $N = N_0$ is an abelian Lie algebra, which is a decomposable null system of nullity 0. Conversely, any abelian Lie algebra can be considered as a null system of nullity 0.

The next example is a generalization of the standard 'null part' of an (original) EALA of maximal type for the case of commutative associative coordinates.

Example 7.3. Consider $F[t_i^{\pm 1}]_{i\in\mathbb{N}} = \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^{\oplus\mathbb{N}}} Ft_{\boldsymbol{\alpha}}$ as in §5. Let d_i be the natural projection of $\mathbb{Z}^{\oplus\mathbb{N}}$ onto $\mathbb{Z}$ for $i \in \mathbb{N}$, i.e., $\boldsymbol{\alpha} = (\alpha_i) \mapsto \alpha_i$, and let

$$\mathcal{D} := \bigoplus_{i\in\mathbb{N}} Fd_i.$$

Let

$$W := \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^{\oplus\mathbb{N}}} t_{\boldsymbol{\alpha}}\mathcal{D} \subset \mathrm{Der}(F[t_i^{\pm 1}]_{i\in\mathbb{N}})$$

be a generalized Witt algebra over F so that

$$(t_{\boldsymbol{\alpha}}d)(t_{\boldsymbol{\beta}}) = d(\boldsymbol{\beta})t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$$

for $d \in \mathcal{D}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}^{\oplus\mathbb{N}}$. The Lie bracket satisfies

$$[t_{\boldsymbol{\alpha}}d, t_{\boldsymbol{\beta}}d'] = t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(d(\boldsymbol{\beta})d' - d'(\boldsymbol{\alpha})d), \tag{1}$$

and so W is a $\mathbb{Z}^{\oplus\mathbb{N}}$-graded Lie algebra. Let $W_{\boldsymbol{\alpha}} := t_{\boldsymbol{\alpha}}\mathcal{D}$ for $\boldsymbol{\alpha} \in \mathbb{Z}^{\oplus\mathbb{N}}$. Then $W_{\mathbf{0}}$ is a self-centralizing abelian ad-diagonalizable subalgebra. Let $\{\delta_i\}_{i\in\mathbb{N}}$ be the dual set of $\{d_i\}_{i\in\mathbb{N}}$. Identifying $\mathbb{Z}^{\oplus\mathbb{N}} \subset \mathcal{D}^*$ by $\boldsymbol{\alpha} = (\alpha_i) \mapsto \sum\ \alpha_i\delta_i$ (finite sum), the set of roots for $(W, W_{\mathbf{0}})$ is $\mathbb{Z}^{\oplus\mathbb{N}}$, i.e.,

$$[d, t_{\boldsymbol{\beta}}d'] = t_{\boldsymbol{\beta}}d(\boldsymbol{\beta})d' = \boldsymbol{\beta}(d)t_{\boldsymbol{\beta}}d'. \tag{2}$$

Let

$$\mathcal{C} := \bigoplus_{i\in\mathbb{N}} F\delta_i \subset \mathcal{D}^*,$$

and let

$$Y = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^{\oplus\mathbb{N}}} t_{\boldsymbol{\alpha}}\mathcal{C}$$

be a $\mathbb{Z}^{\oplus\mathbb{N}}$-graded vector space. For $c \in \mathcal{C}$, we define

$$(t_{\boldsymbol{\alpha}}c)(t_{\boldsymbol{\beta}}d) = \delta_{\boldsymbol{\alpha}+\boldsymbol{\beta},\mathbf{0}}c(d). \tag{3}$$

Then, $Y \subset W^*$. Consider W^* as a natural W-module. So we have

$$\begin{aligned}((t_{\boldsymbol{\alpha}} d).(t_{\boldsymbol{\beta}} c))(t_{\boldsymbol{\gamma}} d') &= (t_{\boldsymbol{\beta}} c)(-[t_{\boldsymbol{\alpha}} d, t_{\boldsymbol{\gamma}} d']) \\ &= (t_{\boldsymbol{\beta}} c)(t_{\boldsymbol{\alpha}+\boldsymbol{\gamma}}(d'(\boldsymbol{\alpha})d - d(\boldsymbol{\gamma})d')) \\ &= \delta_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma},\mathbf{0}}(d'(\boldsymbol{\alpha})c(d) - d(\boldsymbol{\gamma})c(d')) \\ &= (t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(c(d)c_{\boldsymbol{\alpha}} + d(\boldsymbol{\alpha}+\boldsymbol{\beta})c))(t_{\boldsymbol{\gamma}} d'),\end{aligned}$$

where $c_{\boldsymbol{\alpha}} \in \mathcal{C}$ is defined as $c_{\boldsymbol{\alpha}}(d') = d'(\boldsymbol{\alpha})$. (We could simply write $\boldsymbol{\alpha}$ instead of $c_{\boldsymbol{\alpha}}$ as in (2).) Hence,

$$[t_{\boldsymbol{\alpha}} d, t_{\boldsymbol{\beta}} c] = t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(c(d)c_{\boldsymbol{\alpha}} + d(\boldsymbol{\alpha}+\boldsymbol{\beta})c). \tag{4}$$

Letting $Y_{\boldsymbol{\alpha}} := t_{\boldsymbol{\alpha}}\mathcal{C}$, we get a graded W-module $Y = \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^{\oplus\mathbb{N}}} Y_{\boldsymbol{\alpha}}$.

Now, for $\boldsymbol{\alpha} \in \mathbb{Z}^{\oplus\mathbb{N}}$, let

$$W'_{\boldsymbol{\alpha}} := \{t_{\boldsymbol{\alpha}} d \in W_{\boldsymbol{\alpha}} \mid d(\boldsymbol{\alpha}) = 0\} \quad \text{and} \quad W' = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^{\oplus\mathbb{N}}} W'_{\boldsymbol{\alpha}}.$$

Then, for $t_{\boldsymbol{\alpha}} d, t_{\boldsymbol{\beta}} d' \in W'$, since $d(\boldsymbol{\alpha}) = d'(\boldsymbol{\beta}) = 0$, we have (see (1))

$$d(\boldsymbol{\beta})d'(\boldsymbol{\alpha}+\boldsymbol{\beta}) - d'(\boldsymbol{\alpha})d(\boldsymbol{\alpha}+\boldsymbol{\beta}) = d(\boldsymbol{\beta})d'(\boldsymbol{\alpha}) - d'(\boldsymbol{\alpha})d(\boldsymbol{\beta}) = 0,$$

and hence $[t_{\boldsymbol{\alpha}} d, t_{\boldsymbol{\beta}} d'] \subset W'_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$. So W' is a graded subalgebra of W. Let

$$Y'_{\boldsymbol{\alpha}} := \{(t_{\boldsymbol{\alpha}} c)|_{W'} \mid c \in \mathcal{C}\}.$$

Then $Y' = \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^{\oplus\mathbb{N}}} Y'_{\boldsymbol{\alpha}}$ is also a W'-graded module with

$$[t_{\boldsymbol{\alpha}} d, t_{\boldsymbol{\beta}} c] = t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(c(d)c_{\boldsymbol{\alpha}} + d(\boldsymbol{\beta})c), \tag{4'}$$

instead of (4). Note that $W'_{\mathbf{0}} = W'_{\mathbf{0}}$ and $Y'_{\mathbf{0}} = Y_{\mathbf{0}}$. Let

$$N := W' \oplus Y' = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^{\oplus\mathbb{N}}} (W'_{\boldsymbol{\alpha}} \oplus Y'_{\boldsymbol{\alpha}}).$$

Consider Y as an abelian Lie algebra. Then, by the action (4′), $N = \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^{\oplus\mathbb{N}}} N_{\boldsymbol{\alpha}}$, where $N_{\boldsymbol{\alpha}} = W'_{\boldsymbol{\alpha}} \oplus Y'_{\boldsymbol{\alpha}}$, becomes a $\mathbb{Z}^{\oplus\mathbb{N}}$-graded Lie algebra. (In other words, it is the trivial extension of W' by the abelian Lie algebra Y'.) Also, $N_{\mathbf{0}} = W_{\mathbf{0}} \oplus Y_{\mathbf{0}}$ is a self-centralizing abelian ad-diagonalizable subalgebra. For we have

$$[c, t_{\boldsymbol{\alpha}} d](t_{\boldsymbol{\beta}} d') = -t_{\boldsymbol{\alpha}}(c(d)c_{\boldsymbol{\alpha}})(t_{\boldsymbol{\beta}} d') = -c(d)d'(\boldsymbol{\alpha})\delta_{\boldsymbol{\alpha}+\boldsymbol{\beta},\mathbf{0}} = c(d)d'(\boldsymbol{\beta})\delta_{\boldsymbol{\alpha}+\boldsymbol{\beta},\mathbf{0}} = 0,$$

and hence $[W'_{\boldsymbol{\alpha}}, Y_{\mathbf{0}}] = 0$. So $Y_{\mathbf{0}}$ is the centre of N. Since $[d, t_{\boldsymbol{\beta}} c] = t_{\boldsymbol{\beta}} d(\boldsymbol{\beta})c$ (see (4′)), the set of roots is $\mathbb{Z}^{\oplus\mathbb{N}}$ as in the case of W. Note that

$$[N_{\boldsymbol{\alpha}}, N_{-\boldsymbol{\alpha}}] = Fc_{\boldsymbol{\alpha}} \tag{5}$$

for all $\boldsymbol{\alpha} \in \mathbb{Z}^{\oplus \mathbb{N}}$. In fact, for $t_{\boldsymbol{\alpha}}d, t_{-\boldsymbol{\alpha}}d' \in W'$, since $d(\boldsymbol{\alpha}) = d'(\boldsymbol{\alpha}) = 0$, we have $[t_{\boldsymbol{\alpha}}d, t_{-\boldsymbol{\alpha}}d'] = 0$ (since $d(-\boldsymbol{\alpha})d' - d'(\boldsymbol{\alpha})d = 0$). Hence, $[W'_{\boldsymbol{\alpha}}, W'_{-\boldsymbol{\alpha}}] = 0$ for all $\boldsymbol{\alpha} \neq \mathbf{0}$, and also, $[t_{\boldsymbol{\alpha}}d, t_{-\boldsymbol{\alpha}}c] = c(d)c_{\boldsymbol{\alpha}}$ (by (4′)). Thus (5) holds.

We define a symmetric bilinear form $\mathcal{B}$ as follows:

$$\mathcal{B}(W', W') = \mathcal{B}(Y', Y') = 0 \quad \text{and} \quad \mathcal{B}(t_{\boldsymbol{\alpha}}c, t_{\boldsymbol{\beta}}d) = (t_{\boldsymbol{\alpha}}c)(t_{\boldsymbol{\beta}}d) = \delta_{\boldsymbol{\alpha}+\boldsymbol{\beta},\mathbf{0}}c(d).$$

Then one can easily show that $\mathcal{B}$ is invariant. To show the nondegeneracy of $\mathcal{B}$, let $\boldsymbol{\alpha}^{\perp} := \{d \in \mathcal{D} \mid d(\boldsymbol{\alpha}) = 0\}$ and pick some $d_{\boldsymbol{\alpha}} \in \mathcal{D}$ so that $d_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) \neq 0$. Then $\mathcal{D} = Fd_{\boldsymbol{\alpha}} + \boldsymbol{\alpha}^{\perp}$. Now, $W'_{\boldsymbol{\alpha}} = \{t_{\boldsymbol{\alpha}}d \in W_{\boldsymbol{\alpha}} \mid d \in \boldsymbol{\alpha}^{\perp}\}$ and

$$Y'_{-\boldsymbol{\alpha}} = \{(t_{-\boldsymbol{\alpha}}c)|_{W'} \mid c \in \mathcal{C}\} = \{(t_{-\boldsymbol{\alpha}}c)|_{W'_{\boldsymbol{\alpha}}} \mid c \in \mathcal{C}\} = \{t_{-\boldsymbol{\alpha}}c|_{\boldsymbol{\alpha}^{\perp}} \mid c \in \mathcal{C}\}.$$

Identifying $c|_{\boldsymbol{\alpha}^{\perp}}$ with $\tilde{c} \in \mathcal{C}$, where $\tilde{c}(d_{\boldsymbol{\alpha}}) = 0$ and $\tilde{c}|_{\boldsymbol{\alpha}^{\perp}} = c|_{\boldsymbol{\alpha}^{\perp}}$,

$$\boldsymbol{\alpha}^{\perp} \times \{c|_{\boldsymbol{\alpha}^{\perp}} \mid c \in \mathcal{C}\} \longrightarrow F$$

is nondegenerate (since the pairing $\mathcal{D} \times \mathcal{C} \longrightarrow F$ is nondegenerate). So $W'_{\boldsymbol{\alpha}}$ and $Y'_{-\boldsymbol{\alpha}}$ give a nondegenerate pair, and $\mathcal{B}$ is nondegenerate. (We also see that $W'_{\boldsymbol{\alpha}} \cong Y'_{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \mathbb{Z}^{\oplus \mathbb{N}}$ and $\dim_F W_{\boldsymbol{\alpha}}/W'_{\boldsymbol{\alpha}} = \dim_F Y_{\boldsymbol{\alpha}}/Y'_{\boldsymbol{\alpha}} = 1$ for all $\boldsymbol{\alpha} \neq \mathbf{0}$.) Thus $(N, N_{\mathbf{0}}, \mathcal{B})$ is an admissible triple. Since $Y_{\mathbf{0}}$ is the centre, (5) implies that N is a null system of null rank ∞.

We show that this N is indecomposable. In fact, suppose that N is decomposable. Then $N = (L_1, H_1, \mathcal{B}_1) \oplus (L_2, H_2, \mathcal{B}_2)$. If $d_1 \in H_1$, then $d_1 \in H_2$ since $(d_1, d_1) = 0$, which is impossible. Hence, $d_1 \notin H_1$ and $d_1 \notin H_2$. So there exist $h_1 := \sum_i (u_i d_i + v_i \delta_i) \in H_1$ with $u_1 \neq 0$ and $h_2 := \sum_i (p_i d_i + q_i \delta_i) \in H_2$ with $p_1 \neq 0$ ($u_i, v_i, p_i, q_i \in F$). Note that $N_{\delta_1} = Ft_1 d + Ft_1 c$. For $0 \neq at_1 d + bt_1 c \in N_{\delta_1}$ ($a, b \in F$), we have $[h_1, at_1 d + bt_1 c] = u_1(at_1 d + bt_1 c) \neq 0$ since δ_i are central. Hence, $at_1 d + bt_1 c \notin L_2$. Similarly, $[h_2, at_1 d + bt_1 c] = p_1(at_1 d + bt_1 c) \neq 0$, and hence $at_1 d + bt_1 c \notin L_1$. But we have $N_{\delta_1} = (N_{\delta_1} \cap L_1) \oplus (N_{\delta_1} \cap L_1)$, which implies $N_{\delta_1} = 0$, a contradiction.

Remark 7.4. In the notation from §5, let $M = \mathfrak{g} \otimes_F F[t_i^{\pm 1}]_{i \in \mathbb{N}}$. Then $M \oplus N$ becomes a LEALA with suitable multiplication. ($H = \mathfrak{h} \oplus N_{\mathbf{0}}$ is infinite-dimensional.) Also, the subalgebra $M \oplus Y'$ is a central extension of M. More precisely, Y' is the centre of $M \oplus Y'$, and the multiplication is given by

$$[x \otimes t_{\boldsymbol{\alpha}}, y \otimes t_{\boldsymbol{\beta}}] = [x, y] \otimes t_{\boldsymbol{\alpha}+\boldsymbol{\beta}} + \kappa(x, y) t_{\boldsymbol{\alpha}+\boldsymbol{\beta}} c_{\boldsymbol{\beta}}$$

for $x \otimes t_{\boldsymbol{\alpha}}, y \otimes t_{\boldsymbol{\beta}} \in M$. The bracket of W' and Y' is the same as in N, and the bracket of W' and M is just the action of W' on $F[t_i^{\pm 1}]_{i \in \mathbb{N}}$ as derivations, i.e.,

$$[t_{\boldsymbol{\alpha}}d, x \otimes t_{\boldsymbol{\beta}}] = x \otimes d(\boldsymbol{\beta}) t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$$

for $t_{\boldsymbol{\alpha}}d \in W'$ and $x \otimes t_{\boldsymbol{\beta}} \in M$.

Note that this $M \oplus N$ corresponds to the trivial cocycle from $W' \times W' \longrightarrow Y'$ or in other words the trivial abelian extension of W' by Y'. There exists a nontrivial extension

$N_\tau = W' \oplus Y'$ determined by the Moody-Rao cocycle τ [EM] so that N_τ is a null system and $M \oplus N_\tau$ is a LEALA. More precisely, $\tau : W' \times W' \longrightarrow Y'$ is defined by

$$\tau(t_{\boldsymbol{\alpha}} d, t_{\boldsymbol{\beta}} d') = d'(\boldsymbol{\alpha}) d(\boldsymbol{\beta}) t_{\boldsymbol{\alpha}+\boldsymbol{\beta}} c_{\boldsymbol{\alpha}-\boldsymbol{\beta}},$$

and the new bracket $[\cdot,\cdot]_\tau$ of N_τ is defined as

$$(1_\tau) \qquad [t_{\boldsymbol{\alpha}} d, t_{\boldsymbol{\beta}} d']_\tau = [t_{\boldsymbol{\alpha}} d, t_{\boldsymbol{\beta}} d'] + \tau(t_{\boldsymbol{\alpha}} d, t_{\boldsymbol{\beta}} d'),$$

and the rest of brackets remain same as in N. As stated in [BGK, Rem.3.76], the classification of such cocycles seems to be interesting (see also [BeB]).

§8 Classification for LEALAs of nullity 0

Using the results in [NS] and [Stu], we can classify LEALAs of nullity 0. We define the core in general, and then classify the cores of nullity 0 as the first step.

Definition 8.1. Let L be a LEALA. The *core* of L, denoted by L_c, is the subalgebra of L generated by the root spaces L_α for all $\alpha \in R^\times$.

Remark 8.2. One can show that the core L_c of a LEALA L is an ideal of L and the centralizer $C_L(L_c)$ of L_c in L is orthogonal to L_c relative to $\mathcal{B}$ (see [BGK, Lem.3.6] where they gave a proof for EALAs).

To classify the cores of LEALAs, we may need to consider a local version of a *Lie torus*, defined in [Y2] or [N3]. For the case of nullity 0, we have the following:

Proposition 8.3. *Let $(L, H, \mathcal{B})$ be a LEALA (resp. an EALA) of nullity 0. Then the core L_c is a locally finite split simple Lie algebra (resp. a finite-dimensional split simple Lie algebra), and $(L_c, H \cap L_c, \mathcal{B}|_{L_c})$ is also a LEALA (resp. an EALA) of nullity 0.*

Proof. Since L has nullity 0, $L_c = H' \oplus (\oplus_{\alpha \in R^\times} L_\alpha)$, where $H' := H \cap L_c$. Hence H' is self-centralizing, i.e., L_c is split. Let S be a finite subset of L_c. Then S is contained in a subalgebra generated by a finite number of homogenous elements, say $S \subset \langle x_{\alpha_1}, \dots, x_{\alpha_r} \rangle$, where $x_{\alpha_i} \in L_{\alpha_i}$. Since $\bar{R}^\times = R^\times$, $R^\times$ is a reduced locally finite irreducible root system by Proposition 3.7 (or Proposition 4.2) and Remark 4.3. Hence there exists a reduced finite irreducible subsystem Φ of $R^\times$ containing $\{\alpha_1, \dots, \alpha_r\}$ (see Lemma 3.9). Then the subalgebra M generated by L_α for all $\alpha \in \Phi$ contains S, and one can show that M is a finite-dimensional split simple Lie algebra of type Φ, using Serre's Theorem (cf. [AG, Prop.1.16]). Hence L_c is locally finite and simple. Note that $\mathcal{B}|_{L_c}$ is nondegenerate since L_c is simple. Thus the second statement is clear.

For the case where L is an EALA, $\bar{R}^\times = R^\times$ is a reduced finite irreducible root system by Proposition 3.7 and Remark 4.3. Hence again, using Serre's Theorem, one can show that L_c is a finite-dimensional split simple Lie algebra of type $R^\times$. □

To complete the classification of LEALAs of nullity 0, we need to determine the complement of the core. We first prove the following:

Lemma 8.4. *Let $(L, H, \mathcal{B})$ be a LEALA of nullity 0 and Z the centre of L. Then Z is split and $L = L_c \oplus D \oplus Z$ with $H = (H \cap L_c) \oplus D \oplus Z$, where D is an abelian subalgebra acting on L_c. In particular, $L_c \oplus D$ is a centreless Lie algebra.*

Proof. Note first that the centre Z is always contained in H for any LEALA. Also, we have $Z \cap L_c = 0$ since L_c is simple (by Lemma 8.3). We fix a complement D of $(H \cap L_c) \oplus Z$ in H as a vector space (which is automatically an abelian subalgebra). Then, $L = L_c \oplus D \oplus Z$ since L has nullity 0. Now the rest of statements are clear. □

Thus the problem now is the classification of centreless Lie algebras $L_c \oplus D$, where L_c is a locally finite split simple Lie algebra and D is an abelian Lie algebra consisting of derivations of L_c preserving the root spaces of L_c with $[H \cap L_c, D] = 0$. If L_c is finite-dimensional, then any derivation is inner. Hence, $L_c \oplus D$ cannot be centreless unless $D = 0$. Thus $D = 0$ in this case. For the infinite-dimensional case, we prove the following general lemma.

Lemma 8.5. *Let $L = \mathfrak{g} \oplus D$ be a Lie algebra, where $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta} \mathfrak{g}_\mu$ is a locally finite split simple Lie algebra with a split Cartan subalgebra $\mathfrak{h}$, and D is an abelian subalgebra acting on $\mathfrak{g}$ with $[\mathfrak{h}, D] = 0$ and $[d, \mathfrak{g}_\mu] \subset \mathfrak{g}_\mu$ for all $d \in D$ and $\mu \in \Delta$. Let σ be a symmetric invariant bilinear form on L so that $\sigma |_{\mathfrak{g}\times\mathfrak{g}} \neq 0$. Then $\operatorname{rad}\sigma \subset Z$, where $\operatorname{rad}\sigma$ is the radical of σ and Z is the centre of L. In particular, if $Z = 0$, then σ is nondegenerate.*

Also, we have $Z = 0 \iff D \cap Z = 0$ and $(\operatorname{ad}_{\mathfrak{g}} \mathfrak{g}) \cap (\operatorname{ad}_{\mathfrak{g}} D) = 0$, i.e., D consists of outer derivations of $\mathfrak{g}$.

Proof. Clearly, one can assume that $\operatorname{rad}\sigma \subset \mathfrak{h} \oplus D$. Let $h + d \in \operatorname{rad}\sigma$ for $h \in \mathfrak{h}$ and $d \in D$. Let ρ be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$. For each $\mu \in \Delta$, let $(x_\mu, t_\mu, x_{-\mu})$ be a triplet so that $\rho(x_\mu, x_{-\mu}) = 1$ (see (1.0)). Note that a symmetric invariant bilinear form on $\mathfrak{g}$ is unique up to scalars (see [NS, Lem.II.11]), and so $\sigma |_{\mathfrak{g}\times\mathfrak{g}} = u\rho$ for some $0 \neq u \in F$. Also, $\operatorname{ad} d(x_\mu) = v x_\mu$ for some $v \in F$. Now, we have $0 = \sigma(h + d, t_\mu) = u\rho(h, t_\mu) + \sigma(d, [x_\mu, x_{-\mu}]) = u\mu(h) + u\rho([d, x_\mu], x_{-\mu}) = u\mu(h) + uv\rho(x_\mu, x_{-\mu}) = u\mu(h) + uv$. Hence, $v = -\mu(h)$, and so $\operatorname{ad} d = -\operatorname{ad} h$, that is, $h + d \in Z$.

For the second statement, note that $Z \subset \mathfrak{h} \oplus D$. Thus $0 \neq h + d \in Z \Leftrightarrow h = 0$ and $0 \neq d \in Z$ or $h \neq 0$ and $d \neq 0$ with $\operatorname{ad}(h + d) = 0 \Leftrightarrow D \cap Z \neq 0$ or $(\operatorname{ad}_{\mathfrak{g}} \mathfrak{g}) \cap (\operatorname{ad}_{\mathfrak{g}} D) \neq 0$. □

Thus, in our situation, D has to be contained in the Lie algebra of outer derivations of L_c.

Conversely, for any abelian subalgebra D of outer derivations of a locally finite split simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta} \mathfrak{g}_\mu$ preserving the root spaces with $[\mathfrak{h}, D] = 0$, any nondegenerate symmetric invariant bilinear form ρ on $\mathfrak{g}$ and any symmetric bilinear form ψ on D, one can define a centreless LEALA $(\mathfrak{g} \oplus D, \mathfrak{h} \oplus D, \mathcal{B})$, where $\mathcal{B} = \rho + \psi$ with $\mathcal{B}(d, \mathfrak{g}_\mu) = \mathcal{B}(\mathfrak{g}_\mu, d) = 0$ and $\mathcal{B}(d, t_\mu) = \mathcal{B}(t_\mu, d) = v_\mu$ for all $d \in D$ and $\mu \in \Delta$, and v_μ is a unique element in F so that $[d, x_\mu] = v_\mu x_\mu$ for all $x_\mu \in \mathfrak{g}_\mu$. Indeed, all you need to check is the invariance of $\mathcal{B}$, which is easy, e.g., $\mathcal{B}([d, x_\mu], x_{-\mu}) = \mathcal{B}(v_\mu x_\mu, x_{-\mu}) = v_\mu \rho(x_\mu, x_{-\mu}) = v_\mu$, on the other hand, $\mathcal{B}(d, [x_\mu, x_{-\mu}]) = \mathcal{B}(d, t_\mu) = v_\mu$, choosing a triplet $(x_\mu, t_\mu, x_{-\mu})$ so that $\rho(x_\mu, x_{-\mu}) = 1$ (see (1.0)). Or $\mathcal{B}([x_\mu, d], x_{-\mu}) = \mathcal{B}(x_\mu, [d, x_{-\mu}])$ since $[d, x_{-\mu}] = -v_\mu x_\mu$.

Now, assuming that the core L_c is infinite-dimensional, by [NS, Thm IV.6 and Cor.IV.5], $(L_c, L_c \cap H)$ is isomorphic to $(\mathfrak{g}_{\mathfrak{I}}, \mathfrak{h})$, where $\mathfrak{g}_{\mathfrak{I}}$ is one of $sl_{\mathfrak{I}}(F)$, $o_{2\mathfrak{I}+1}(F)$, $sp_{2\mathfrak{I}}(F)$ or

$o_{2\mathfrak{I}}(F)$ for an infinite index set $\mathfrak{I}$ as defined in Definition 6.1, and $\mathfrak{h}$ is its standard Cartan subalgebra, that is, the subalgebra consisting of diagonal matrices. (They showed that $o_{2\mathfrak{I}+1}(F) \cong o_{2\mathfrak{I}}(F)$ but $(o_{2\mathfrak{I}+1}(F), \mathfrak{h}) \not\cong (o_{2\mathfrak{I}}(F), \mathfrak{h})$.) We identify them, and so D is an abelian subalgebra consisting of outer derivations of $\mathfrak{g}_{\mathfrak{I}}$ preserving the root spaces of $\mathfrak{g}_{\mathfrak{I}}$ with $[\mathfrak{h}, D] = 0$.

By a communication with Karl-Hermann Neeb, we can describe our D more concretely. For each $d \in D$, consider $\tau(x, y) := \mathcal{B}([d, x], y) = \rho([d, x], y)$ for $x, y \in \mathfrak{g}_{\mathfrak{I}}$, which is a 2-cocycle of $\mathfrak{g}_{\mathfrak{I}}$ into the trivial module F. Note that the second cohomology $H^2(\mathfrak{g}_{\mathfrak{I}}, F)$ has to be 0 since $\mathfrak{g}_{\mathfrak{I}}$ is a direct limit of finite-dimensional split simple Lie algebras. Thus there exists a linear form f of $\mathfrak{g}_{\mathfrak{I}}$ such that $\tau(x, y) = f([x, y])$. Let $A = (a_{ij})$ be a unique matrix (of size $\mathfrak{I}$, $2\mathfrak{I} + 1$ or $2\mathfrak{I}$) so that $f(x) = \mathrm{tr}(Ax)$ for all $x \in \mathfrak{g}_{\mathfrak{I}}$, where tr is the trace of the matrix Ax. (Note that only finitely many diagonal entries of Ax are nonzero, and so the trace is defined.) As mentioned in §6, the trace form $\mathrm{tr}(x, y) := \mathrm{tr}(xy)$ is a symmetric nondegenerate invariant form on $\mathfrak{g}_{\mathfrak{I}}$. Hence, by [NS, Lem.II.11], $\rho = u\,\mathrm{tr}$ for some $0 \neq u \in F$. Then we have

$$u\,\mathrm{tr}([d, x], y) = \mathcal{B}([d, x], y) = \tau(x, y) = f([x, y]) = \mathrm{tr}(A[x, y]) = \mathrm{tr}([A, x]y)$$

for all $x, y \in \mathfrak{g}_{\mathfrak{I}}$. Hence $\mathrm{ad}\, d = u\, \mathrm{ad}\, A$ if $[A, x] \in \mathfrak{g}_{\mathfrak{I}}$. First, we note that

$$\text{(8.6)} \qquad \text{if } \mathfrak{g}_{\mathfrak{I}} \neq sl_{\mathfrak{I}}(F), \text{ then } a_{ii} = -a_{\mathfrak{I}+i,\mathfrak{I}+i}, \text{ and } a_{2\mathfrak{I}+1,2\mathfrak{I}+1} = 0 \text{ for } o_{2\mathfrak{I}+1}(F).$$

Thus, for any $\mathfrak{g}_{\mathfrak{I}}$ and distinct $i \neq j$, there exists k distinct from i, j such that $e_{jj} - e_{kk} \in \mathfrak{h}$. ($e_{ij}$'s are matrix units.) So we have

$$a_{ij} = \mathrm{tr}(Ae_{ji}) = \mathrm{tr}(A[e_{jj} - e_{kk}, e_{ji}]) = u\,\mathrm{tr}([d, e_{jj} - e_{kk}]e_{ji}) = 0$$

since $[D, \mathfrak{h}] = 0$. Therefore, A is diagonal, and in particular, $[A, x] \in \mathfrak{g}_{\mathfrak{I}}$.

Thus we have shown that D consists of diagonal matrices satisfying (8.6) since our Lie algebra is centreless. Also, infinitely many a_{ii} are nonzero since d is outer. For the case $\mathfrak{g}_{\mathfrak{I}} = sl_{\mathfrak{I}}(F)$, if the diagonal matrix A is almost scalar, i.e., $a_{ii} = a$ for some $a \in F$ except for finitely many i, then $\mathrm{ad}\, A = \mathrm{ad}\, B$ for $B := A - \sum_{i \in \mathfrak{I}} ae_{ii}$, and $\mathrm{ad}\, B = \mathrm{ad}\, B'$ for some matrix B' with $\mathrm{tr}(B') = 0$. So $\mathrm{ad}\, A$ is inner on $\mathfrak{g}_{\mathfrak{I}}$. Thus A cannot be an almost scalar matrix.

Conversely, if $\mathfrak{g}_{\mathfrak{I}} \neq sl_{\mathfrak{I}}(F)$, then the adjoint of any diagonal matrix satisfying (8.6) with infinitely many nonzero entries is outer, and if $\mathfrak{g}_{\mathfrak{I}} = sl_{\mathfrak{I}}(F)$, then the adjoint of any diagonal matrix which is not almost scalar is outer.

Thus we obtain a complete classification of LEALAs of nullity 0.

Theorem 8.7. *Let $(L, H, \mathcal{B})$ be a LEALA of nullity 0 and Z the centre of L.*

(1) If L_c is finite-dimensional, then $(L, H, \mathcal{B}) \cong (\mathfrak{g} \oplus Z, \mathfrak{h} \oplus Z, u\kappa \perp \psi)$ for some nonzero $u \in F$, where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra, $\mathfrak{h}$ is a split Cartan subalgebra, κ is the Killing form of $\mathfrak{g}$, and ψ is a nondegenerate symmetric bilinear form on Z. In particular, L is a split central extension of $\mathfrak{g}$.

Conversely, any such Z, u and ψ give a LEALA $(\mathfrak{g} \oplus Z, \mathfrak{h} \oplus Z, u\kappa \perp \psi)$ (an EALA if $\dim_F Z < \infty$).

(2) If L_c is infinite-dimensional, then $(L, H, \mathcal{B}) \cong (\mathfrak{g}_{\mathfrak{I}} \oplus D \oplus Z, \mathfrak{h} \oplus D \oplus Z, u\,\mathrm{tr} + \psi)$ for some nonzero $u \in F$, where $\mathfrak{g}_{\mathfrak{I}} = sl_{\mathfrak{I}}(F)$, $o_{2\mathfrak{I}+1}(F)$, $sp_{2\mathfrak{I}}(F)$ or $o_{2\mathfrak{I}}(F)$ with its standard Cartan subalgebra $\mathfrak{h}$ for an infinite index set $\mathfrak{I}$, and D is:

(i) *a subspace consisting of diagonal matrices satisfying (8.6) with infinitely many nonzero entries if $\mathfrak{g}_{\mathfrak{I}} \neq sl_{\mathfrak{I}}(F)$;*

(ii) *a subspace consisting of diagonal matrices which are not almost scalar if $\mathfrak{g}_{\mathfrak{I}} = sl_{\mathfrak{I}}(F)$,*

and ψ is a symmetric bilinear form on $D \oplus Z$ satisfying $Z \cap \mathrm{rad}\,\psi = 0$. In particular, L is a split central extension of the centreless LEALA $(\mathfrak{g}_{\mathfrak{I}} \oplus D, \mathfrak{h} \oplus D, u\,\mathrm{tr} + \psi\,|_{D \times D})$.

Conversely, any such Z, D, u and ψ give a LEALA $(\mathfrak{g}_{\mathfrak{I}} \oplus D \oplus Z, \mathfrak{h} \oplus D \oplus Z, u\,\mathrm{tr} + \psi)$.

Proof. We only need to prove that $u\,\mathrm{tr} + \psi$ is nondegenerate if and only if $Z \cap \mathrm{rad}\,\psi = 0$. Clearly, if $Z \cap \mathrm{rad}\,\psi \neq 0$, then $u\,\mathrm{tr} + \psi$ is degenerate. Thus we need to prove that $Z \cap \mathrm{rad}\,\psi = 0$ implies the nondegeneracy of $u\,\mathrm{tr} + \psi$. Let R be the radical of $u\,\mathrm{tr} + \psi$, which is clearly contained in $D \oplus Z$. Let $A + z \in R$ for $A \in D$ and $z \in Z$. If $A \neq 0$, then there exists $0 \neq x \in \mathfrak{g}_{\mathfrak{I}}$ such that $[x, A + z] = vx \in R$ for some $0 \neq v \in F$ since the radical is an ideal. This implies that $\mathfrak{g}_{\mathfrak{I}} \subset R$, which is a contradiction. Hence, $A = 0$, and so $R \subset Z$, which implies that $R \subset Z \cap \mathrm{rad}\,\psi$. Thus we get $R = 0$. □

Note that we gave some smallest nontrivial examples in Example 6.2.

§9 A note for tame LEALAs

We can define the tameness for a LEALA as in [A-P] (or [N2]).

Definition 9.1. Let L be a LEALA. Let $C = C_L(L_c)$ be the centralizer of the core L_c in L. Then L is called *tame* if $C \subset L_c$.

By Theorem 8.7, if the nullity is 0, then the centre has to be 0. So:

Corollary 9.2. *Let L be a tame LEALA of nullity 0. Then $L \cong \mathfrak{g}$ or $\mathfrak{g}_{\mathfrak{I}} \oplus D$ in the description of Theorem 8.7. In particular, tame $\iff$ centreless $\iff$ indecomposable, in the case of nullity 0.*

However, this is not the case if the nullity is bigger than 0. There are examples of LEALAs which are indecomposable but not tame in [BGK]. For the convenience of the reader, we give one such example.

Example 9.3. Let $M = \mathfrak{g} \otimes F[t^{\pm 1}]$ be a loop algebra with form $\kappa \otimes \varepsilon$ (see §5) and $A = M \oplus Fc \oplus Fd$ its (untwisted) affine Kac-Moody Lie algebra (cf. Remark 7.2). Let $(N, N_0, \mathcal{B}) = \oplus_{\delta \in S} N_\delta$ be a null system constructed from $\Lambda = \mathbb{Z}$, S and $\varphi = \mathrm{id}$ in Example 7.1. We identify N_0 with the subalgebra $Fc \oplus Fd$ of A. Let L be the Lie algebra containing A and N as subalgebras so that $L = A \oplus (\oplus_{\delta \neq 0} N_\delta)$ declaring $[M, \oplus_{\delta \neq 0} N_\delta] = 0$. Let $H := (\mathfrak{h} \otimes 1) \oplus N_0$. Then, $(L, H, \kappa \otimes \varepsilon + \mathcal{B})$ is an EALA of null rank 1. Since an affine Kac-Moody Lie algebra is indecomposable, H is never decomposed into a direct sum of orthogonal subspaces. Hence L is an indecomposable EALA. However, the subspace $\oplus_{\delta \neq 0} N_\delta$ centralizes the core of L, i.e., $L_c = M \oplus Fc$. So L is not tame unless $S = 0$.

Thus, to classify even EALAs of null rank 1, we need to classify null systems of null rank 1. So it may be natural to assume the tameness. Also, by the following lemma, the notion of indecomposability disappears if the tameness is assumed.

Lemma 9.4. *A tame LEALA is indecomposable. Also, an EALA is completely decomposable with the factors of one indecomposable EALA and some indecomposable null systems.*

Proof. Let $L = (L, H, \mathcal{B})$ be a LEALA. Suppose that $L = (L_1, H_1, \mathcal{B}_1) \oplus (L_2, H_2, \mathcal{B}_2)$. Then $R(H) = R(H_1) \cup R(H_2)$ and $(R(H_1), R(H_2)) = 0$. So by (A5), $R^\times \subset R(H_1)$ or $R(H_2)$. Thus the core L_c has to sit in one of the factors, say $L_c \subset L_1$. Then L_2 is a null system. So if L is tame, then $L_2 = 0$ since L_2 centralizes L_c.

For the second statement, if L is an EALA, then L is completely decomposable since $\dim H < \infty$. Thus, by the same reason as above, the core L_c has to be in one of the factors, and the rest are null systems. □

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