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AUTOMORPHISMS OF A LOCALLY AFFINE ROOT SYSTEM OF TYPE $\mathrm{A}^{(1)}_{\mathfrak{I}-1}$

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Abstract. We determine the automorphism group of a locally affine root system of type $\mathrm{A}^{(1)}_{\mathfrak{I}-1}$, which particularly gives the corresponding outer automorphism group. As a corollary, the automorphism group of an affine root system of type $\mathrm{A}^{(1)}_{n-1}$ is well understood in a certain general picture.

1. Introduction

Let us start with the definition of locally extended affine root systems introduced in [MY1] (see also [Y]).

Definition 1.1. Let V be a vector space over $\mathbb{Q}$ with a positive semidefinite bilinear form $(\cdot,\cdot)$. A subset R of V is called a *locally extended affine root system* or a *LEARS* for short if

(A1) $(\alpha,\alpha)\neq 0$ for all $\alpha\in R$, and R spans V;

(A2) $\langle\alpha,\beta\rangle\in\mathbb{Z}$ for all $\alpha,\beta\in R$, where $\langle\alpha,\beta\rangle=\dfrac{2(\alpha,\beta)}{(\beta,\beta)}$;

(A3) $\sigma_\alpha(\beta)\in R$ for all $\alpha,\beta\in R$, where $\sigma_\alpha(\beta)=\beta-\langle\beta,\alpha\rangle\alpha$;

(A4) $R=R_1\cup R_2$ and $(R_1,R_2)=0$ imply $R_1=\emptyset$ or $R_2=\emptyset$. (R is irreducible.)

A LEARS R is called *reduced* if $2\alpha\notin R$ for all $\alpha\in R$.

Definition 1.2. LEARS (V,R) and (V',R') are called **isomorphic**, denoted by $R\cong R'$, if there is a linear isomorphism $\varphi: V\longrightarrow V'$ such that $\varphi(R)=R'$ and $\langle\alpha,\beta\rangle=\langle\varphi(\alpha),\varphi(\beta)\rangle$ for all $\alpha,\beta\in R$. In particular, we define the automorphism group, called $\operatorname{Aut} R$, of a LEARS R by $\operatorname{Aut} R=\{\varphi\mid\varphi: R\xrightarrow{\sim} R \text{ an isomorphism}\}$. We note that $\sigma_\alpha\in\operatorname{Aut}R$ and $-\mathbf{1}_V\in\operatorname{Aut}R$. Then, we define $W(R)$ to be the subgroup, called the Weyl group, of $\operatorname{Aut}R$ generated by σ_α for all $\alpha\in R$. Since $\varphi\circ\sigma_\alpha\circ\varphi^{-1}=\sigma_{\varphi(\alpha)}$, the Weyl group $W(R)$ is a normal subgroup of $\operatorname{Aut}R$, which allows us to put $\operatorname{Out}R=\operatorname{Aut}R/W(R)$ as the outer automorphism group of R.

Let $V^0:=\{x\in V\mid (x,y)=0 \text{ for all } y\in V\}$ be the radical of the form. We call a LEARS (R,V) an *extended affine root system* or an *EARS* for short, if $\dim_{\mathbb{Q}} V/V^0<\infty$ and $\langle R\rangle$ is free. This coincides with the concept, which was firstly introduced by Saito in 1985 [S]. The notion of an EARS was also used in a different sense in [AABGP], but Azam showed that there is a natural correspondence between the two notions in [A]. EARS in Saito's sense naturally generalize the Macdonald's *affine root systems* in [M].

When the torsion-free abelian group $\langle R\rangle\cap V^0$ is free, we say that R has *nullity*. Our LEARS are a natural generalization of the Saito's EARS. In fact, Saito's EARS are the same as our EARS embedded into the real vector space $\mathbb{R}\otimes_{\mathbb{Q}}V$. Similarly, irreducible affine root systems in the sense of Macdonald [M] are our EARS of nullity 1. Note that the reduced irreducible affine root systems are the real roots of affine Kac-Moody Lie algebras. The elliptic root systems defined by Saito [S] are our EARS of nullity 2. Also, the sets of nonisotropic roots of EARS in [AABGP] are our reduced EARS of finite rank (see [A]).

We call a LEARS of nullity 1 a *locally affine root system*, which generalizes irreducible affine root systems (see [N], [MY2]). Note that LEARS of nullity 0 are so-called locally finite irreducible root systems, which generalizes finite irreducible root systems (see [NS], [St]).

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For example,

$$R = \{\epsilon_i - \epsilon_j \mid i \neq j \in \mathfrak{I}\}$$

in V is a locally finite root system of $A_{\mathfrak{I}-1}$, where $\mathfrak{I}$ is any index set, $\{\epsilon_i \mid i \in \mathfrak{I}\}$ is an orthonormal basis of $V^\sharp = \oplus_{i\in\mathfrak{I}}\mathbb{Q}\epsilon_i$ and $V = \oplus_{j\in\mathfrak{I}\setminus\{1\}}\mathbb{Q}(\epsilon_1 - \epsilon_j) \subset V^\sharp$, and where $1 \in \mathfrak{I}$ is a fixed element. Also,

$$R = \{\epsilon_i - \epsilon_j + k\delta \mid i \neq j \in \mathfrak{I},\ k \in \mathbb{Z}\}$$

in $\tilde{V} = V \oplus \mathbb{Q}\delta$ with $\tilde{V}^0 = \mathbb{Q}\delta$ is a locally affine root system of type $A^{(1)}_{\mathfrak{I}-1}$. In this paper, we concentrate to determine the automorphism group of a locally affine root system of type $A^{(1)}_{\mathfrak{I}-1}$. As a corollary, the automorphism group of an affine root system of type $A^{(1)}_{n-1}$ is well understood in a certain general picture. The result, that is, the structure of the automorphism group of type $A^{(1)}_{n-1}$, was shown as a corollary of the conjugacy theorem of root bases (see e.g. [K, Prop.5.5, Cor.5.10]), but we show this directly as a special case of type $A^{(1)}_{\mathfrak{I}-1}$ in an elementary way.

A locally extended affine root system is the set of anisotropic roots of a **locally affine Lie algebra** with a fixed maximal ad-diagonalizable subalgebra (see [MY2]). Therefore, it is very important to study those roots to establish the structure theorems as well as the classification theorems of the corresponding Lie algebras. Especially, to determine the automorphism of our locally affine root system of type $A^{(1)}_{\mathfrak{I}-1}$ here is very useful to obtain the classification theorems in [MY3]. In this paper, we will discuss this group as elementary as possible only using some set theoretical approaches. The most difficult point is that the cardinality of $\mathfrak{I}$ is infinite. We believe that the automorphism groups of locally affine root systems for other types will be also important in future research.

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2. Automorphisms of type $A^{(1)}_{\mathfrak{I}-1}$

Definition 2.1. Let $\mathfrak{I}$ be any index set. We denote $S_\mathfrak{I}$ the symmetric group of $\mathfrak{I}$, $S_{(\mathfrak{I})}$ the subgroup of $S_\mathfrak{I}$ generated by the transpositions, $\mathbb{Z}^\mathfrak{I}$ the ($|\mathfrak{I}|$ times) direct product of $\mathbb{Z}$, and $\mathbb{Z}^{(\mathfrak{I})}$ the direct sum of $\mathbb{Z}$. Also, for $x = (m_i)_{i\in\mathfrak{I}} \in \mathbb{Z}^{(\mathfrak{I})}$, we write $\mathrm{tr}(x) = \sum_{i\in\mathfrak{I}} m_i$, and it is called the **trace** of x.

Theorem 2.2. *Let $R = \{\epsilon_i - \epsilon_j + k\delta \mid i \neq j \in \mathfrak{I},\ k \in \mathbb{Z}\}$ be a locally affine root system of type $A^{(1)}_{\mathfrak{I}-1}$. Let φ be an automorphism of R. Then $\varphi(\delta) = \xi\delta$, where $\xi = \pm 1$, and there exists some permutation $\sigma \in S_\mathfrak{I}$ such that $\varphi(\epsilon_i - \epsilon_j) = \eta(\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + m_{ij}\delta)$ for some $m_{ij} \in \mathbb{Z}$, where $\eta = \pm 1$. Moreover, we have*

$$\begin{aligned} Aut\,R &\cong \{\pm 1_V\} \times \big((S_\mathfrak{I} \times \{\pm 1\}_\eta) \ltimes \mathbb{Z}^{\mathfrak{I}-1}\big) \cong \mathbb{Z}_2 \times \big((\mathbb{Z}_2 \times S_\mathfrak{I}) \ltimes \mathbb{Z}^{\mathfrak{I}-1}\big) \\ &\cong \{\pm 1_V\} \times \big((S_\mathfrak{I} \times \{\pm 1\}_\eta) \ltimes (\mathbb{Z}^\mathfrak{I}/\mathbb{Z}\iota)\big) \cong \mathbb{Z}_2 \times \big((\mathbb{Z}_2 \times S_\mathfrak{I}) \ltimes (\mathbb{Z}^\mathfrak{I}/\mathbb{Z}\iota)\big), \end{aligned}$$

where ι is the identity vector, i.e., all the coordinates of ι are 1, and the Weyl group of R is

$$W(R) \cong\ S_{(\mathfrak{I})} \ltimes \mathbb{Z}_0^{(\mathfrak{I})},$$

where $\mathbb{Z}_0^{(\mathfrak{I})} = \{x \in \mathbb{Z}^{(\mathfrak{I})} \mid tr(x) = 0\}$. In particular, if $|\mathfrak{I}| = n$, then

$$Aut\,R \cong \{\pm 1_V\} \times \big((S_n \ltimes \mathbb{Z}_0^n) \rtimes I_n\big) \cong \mathbb{Z}_2 \times \big((S_n \ltimes \mathbb{Z}_0^n) \rtimes I_n\big),$$

where $\mathbb{Z}_0^n = \{x \in \mathbb{Z}^n \mid tr(x) = 0\} \cong \mathbb{Z}^{n-1}$ and I_n is the dihedral group of order $2n$, and the Weyl group of R is $S_n \ltimes \mathbb{Z}_0^n$.

Proof. First, we show that $\varphi(\delta) = \pm\delta$. Since δ is in the radical of the positive semi-definite form on the vector space spanned by R over $\mathbb{Q}$, we know that $\varphi(\delta) \in \mathbb{Q}\delta$. Let $\varphi(\delta) = \ell\delta$ for $\ell \in \mathbb{Q}$. Since $\varphi(\epsilon_i - \epsilon_j + \delta) = (\epsilon_r - \epsilon_s + m\delta) + \ell\delta \in R$ for some $r \neq s \in \mathfrak{I}$ and $m \in \mathbb{Z}$, we have $\ell \in \mathbb{Z}$. Note that $\ell \neq 0$ since φ is one to one. So we have $\frac{1}{\ell}\varphi(\delta) = \delta$. If $|\ell| > 1$, then we have $\varphi(\epsilon_i - \epsilon_j + \frac{1}{\ell}\delta) = (\epsilon_r - \epsilon_s + m\delta) + \delta \in R$. This is a contradiction since $\epsilon_i - \epsilon_j + \frac{1}{\ell}\delta \notin R$. Therefore, we get $|\ell| = 1$, and so $\varphi(\delta) = \pm\delta$.

Let $1 \in \mathfrak{I}$ be fixed. Let $\varphi(\epsilon_1 - \epsilon_i) = \epsilon_x - \epsilon_y + m\delta$ and $\varphi(\epsilon_1 - \epsilon_j) = \epsilon_s - \epsilon_t + n\delta$ for some $x, y, s, t \in \mathfrak{I}$ and $m, n \in \mathbb{Z}$. Since $(\epsilon_1 - \epsilon_j) - (\epsilon_1 - \epsilon_i) = \epsilon_i - \epsilon_j \in R$, we have $(\epsilon_s - \epsilon_t) - (\epsilon_x - \epsilon_y) + (n - m)\delta \in R$.

Hence $s = x$ or $t = y$. Suppose that $s = x$, and so we have $\varphi(\epsilon_1 - \epsilon_i) = \epsilon_x - \epsilon_y + m\delta$ and $\varphi(\epsilon_1 - \epsilon_j) = \epsilon_x - \epsilon_t + n\delta$. Note that if $y = t$, then $\varphi(\epsilon_i - \epsilon_j) = (n - m)\delta$, which is a cotradiction. Hence, $y \neq t$. If $|\mathfrak{I}| \geq 4$, then for any $k \in \mathfrak{I} \setminus \{1, i, j\}$, by the same argument above, we have $\varphi(\epsilon_1 - \epsilon_k) = \epsilon_x - \epsilon_z + \ell\delta$ or $\epsilon_u - \epsilon_y + \ell\delta$ for some $z, u \in \mathfrak{I}$ and $\ell \in \mathbb{Z}$, comparing with $\varphi(\epsilon_1 - \epsilon_i) = \epsilon_x - \epsilon_y + m\delta$, and $\varphi(\epsilon_1 - \epsilon_k) = \epsilon_x - \epsilon_z + \ell\delta$ or $\epsilon_v - \epsilon_t + \ell\delta$ for some $v \in \mathfrak{I}$, comparing with $\varphi(\epsilon_1 - \epsilon_j) = \epsilon_x - \epsilon_t + n\delta$. Thus, if $\varphi(\epsilon_1 - \epsilon_k) \neq \epsilon_x - \epsilon_z + \ell\delta$, then $\epsilon_u - \epsilon_y + \ell\delta = \epsilon_v - \epsilon_t + r\delta$. This forces $y = t$, which is a contradiction. Hence, we have $\varphi(\epsilon_1 - \epsilon_k) = \epsilon_x - \epsilon_z + \ell\delta$. Thus we can write $\varphi(\epsilon_1 - \epsilon_i) = \epsilon_x - \epsilon_{\sigma(i)} + m_i\delta$ for some $m_i \in \mathbb{Z}$, where σ is a map from $\mathfrak{I} \setminus \{1\}$ to $\mathfrak{I} \setminus \{x\}$. If $\sigma(i) = \sigma(j)$, then $\varphi(\epsilon_i - \epsilon_j) = (m_j - m_i)\delta$, which is a contradiction. Hence σ is injective. For any $s \in \mathfrak{I} \setminus \{x\}$, there exist some $p, q \in \mathfrak{I}$ and $m \in \mathbb{Z}$ such that $\varphi(\epsilon_p - \epsilon_q + m\delta) = \epsilon_x - \epsilon_s$ since φ is surjective. If $q = 1$, then $\varphi(\epsilon_p - \epsilon_1 + m\delta) = -\epsilon_x + \epsilon_{\sigma(p)}$, which forces $x = s$. But then $\varphi(\epsilon_p - \epsilon_q + m\delta) = \epsilon_s - \epsilon_s = 0$, which is a contradiction. Thus we assume that $q \neq 1$. Note that $\epsilon_p - \epsilon_q = (\epsilon_p - \epsilon_1) + (\epsilon_1 - \epsilon_q)$, and so if $p \neq 1$, then $\epsilon_x - \epsilon_\ell = \varphi(\epsilon_p - \epsilon_q + m\delta) = (\epsilon_{\sigma(p)} - \epsilon_x) + (\epsilon_x - \epsilon_{\sigma(q)}) = \epsilon_{\sigma(p)} - \epsilon_{\sigma(q)}$. Hence, we have $x = \sigma(p)$, which contradicts the definition of σ. Therefore, p has to be 1, and then $\ell = \sigma(q)$. Thus we have shown the surjectivity of σ, and so σ is a bijection from $\mathfrak{I} \setminus \{1\}$ onto $\mathfrak{I} \setminus \{x\}$. So, if we define $\sigma(1) = x$, then σ is a permutation on $\mathfrak{I}$. Then, for any $i \neq j \in \mathfrak{I}$, we have $\varphi(\epsilon_i - \epsilon_j) = \varphi(\epsilon_i - \epsilon_1) + \varphi(\epsilon_1 - \epsilon_j) = \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + (m_j - m_i)\delta$. Let us define

$$m_1 = 0 \quad \text{and} \quad m_{ij} := m_j - m_i.$$

Then we can write

$$\varphi(\epsilon_i - \epsilon_j) = \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + m_{ij}\delta,$$

even though $i = 1$ or $j = 1$. So $m_{1i} = m_i$ for all $i \in \mathfrak{I} \setminus \{1\}$.

If $t = y$, then $-\varphi(\epsilon_1 - \epsilon_i) = \epsilon_y - \epsilon_x - m\delta$ and $-\varphi(\epsilon_1 - \epsilon_j) = \epsilon_y - \epsilon_s - n\delta$ for $x \neq s$. Thus, by the same argument above, we get $-\varphi(\epsilon_1 - \epsilon_i) = \epsilon_y - \epsilon_{\sigma(i)} + m_i\delta$ for some $m_i \in \mathbb{Z}$, where σ is a permutation on $\mathfrak{I}$ with $\sigma(1) = y$, and get $-\varphi(\epsilon_i - \epsilon_j) = \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + (m_j - m_i)\delta$, i.e.,

$$\varphi(\epsilon_i - \epsilon_j) = -(\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} + m_{ij}\delta)$$

for some $\sigma \in S_{\mathfrak{I}}$ and all $i \neq j \in \mathfrak{I}$. Thus the first assertion is shown.

We have shown that, for $\varphi \in \operatorname{Aut} R$, there exists

$$(\eta, \sigma, \xi, (m_i)_{i \in \mathfrak{I}\setminus\{1\}}) = (\eta, \sigma, \xi, (m_{1i})_{i \in \mathfrak{I}\setminus\{1\}}) \in \big(\{\pm 1\}, S_{\mathfrak{I}}, \{\pm 1\}, \mathbb{Z}^{\mathfrak{I}-1}\big)$$

such that

$$\varphi : \begin{cases} \epsilon_1 - \epsilon_k \mapsto \eta(\epsilon_{\sigma(1)} - \epsilon_{\sigma(k)} + m_{1k}\delta) \\ \delta \mapsto \xi\delta \ . \end{cases}$$

On the other hand, we notice that a quadruple of these η, σ, ξ and (m_i) determines an automorphism of R conversely. Let $\varphi = (\eta, \sigma, \xi, (m_{1i})_{i \in \mathfrak{I}\setminus\{1\}})$ and $\phi' = (\eta', \sigma', \xi', (m'_{1i})_{i \in \mathfrak{I}\setminus\{1\}}) \in \big(\{\pm 1\}, S_{\mathfrak{I}}, \{\pm 1\}, \mathbb{Z}^{\mathfrak{I}-1}\big)$. Then we have

$$\varphi \circ \psi : \begin{cases} \epsilon_1 - \epsilon_k \mapsto \eta'(\epsilon_{\sigma'(1)} - \epsilon_{\sigma'(k)} + m'_{1k}\delta) \mapsto \eta\eta'(\epsilon_{\sigma\sigma'(1)} - \epsilon_{\sigma\sigma'(k)} + m_{\sigma'(1)\sigma'(k)}\delta) + \xi\eta' m'_{1k}\delta \\ = \eta\eta'\big(\epsilon_{\sigma\sigma'(1)} - \epsilon_{\sigma\sigma'(k)} + (m_{\sigma'(1)\sigma'(k)} + \xi\eta m'_{1k})\delta\big) \\ \delta \mapsto \xi'\delta \mapsto \xi\xi'\delta, \end{cases}$$

and so

$$\varphi \circ \psi = \big(\eta\eta', \sigma\sigma', \xi\xi', (m_{\sigma'(1)\sigma'(i)} + \xi\eta m'_{1i})_{i \in \mathfrak{I}\setminus\{1\}}\big)\big). \tag{1}$$

Thus through this composite, we have the group epimorphism Ψ from the group

$$(\{\pm 1\}_\eta \times S_{\mathfrak{I}} \times \{\pm 1\}_\xi) \ltimes \mathbb{Z}^{\mathfrak{I}-1}$$

onto $\operatorname{Aut} R$. If $\varphi \in \ker \Psi$, then $\varphi(\epsilon_1 - \epsilon_k) = \eta(\epsilon_{\sigma(1)} - \epsilon_{\sigma(k)} + m_{1k}\delta) = \epsilon_1 - \epsilon_k$ for all k and $\varphi(\delta) = \xi\delta = \delta$, which implies that $\eta = 1$, $\sigma = 1$, $m_k = 0$ and $\xi = 1$. Therefore, we obtain that $\ker \Psi$ is trivial, which implies

$$\operatorname{Aut} R \cong (\{\pm 1\}_\eta \times S_{\mathfrak{I}} \times \{\pm 1\}_\xi) \ltimes \mathbb{Z}^{\mathfrak{I}-1}.$$

In particular, we obtain the following four subgroups of $\operatorname{Aut} R$:

$$\begin{array}{rcl} \{\pm 1\}_\eta & = & \{(\pm 1, \mathrm{id}, 1, \mathbf{0})\}, \\ S_{\mathfrak{I}} & = & \{(1, \sigma, 1, \mathbf{0}) \mid \sigma \in S_{\mathfrak{I}}\}, \\ \{\pm 1\}_\xi & = & \{(1, \mathrm{id}, \pm 1, \mathbf{0})\}, \\ \mathbb{Z}^{\mathfrak{I}-1} & = & \{(1, \mathrm{id}, 1, (m_i)_{i \in \mathfrak{I}-1}) \mid m_i \in \mathbb{Z}\}. \end{array}$$

Here, we determine the center Z of $\operatorname{Aut} R$. Let $\varphi = (\eta, \sigma, \xi, (m_i)_{i\in\mathfrak{I}-1})$ be a central element of $\operatorname{Aut} R$. Then, we obtain:

$$\begin{aligned}
&\varphi \text{ is central}\\
\Leftrightarrow\ & \varphi\circ\varphi' = \varphi'\circ\varphi,\ \forall\varphi' = (\eta',\sigma',\xi',(m'_i)_{i\in\mathfrak{I}-1})\\
\Leftrightarrow\ & \eta\eta' = \eta'\eta,\ \sigma\sigma' = \sigma'\sigma,\ \xi\xi' = \xi'\xi,\ m_{\sigma'(1),\sigma'(i)} + \xi\eta m'_{1i} = m'_{\sigma(1),\sigma(i)} + \xi'\eta' m_{1i},\ \forall\varphi'\\
\Leftrightarrow\ & \sigma\sigma' = \sigma'\sigma,\ m_{\sigma'(1),\sigma'(i)} + \xi\eta m'_{1i} = m'_{\sigma(1),\sigma(i)} + \xi'\eta' m_{1i},\ \forall\varphi'\\
\Leftrightarrow\ & \sigma = \mathrm{id},\ m_{\sigma'(1),\sigma'(i)} + \xi\eta m'_{1i} = m'_{1i} + \xi'\eta' m_{1i},\ \forall\varphi'\\
\Leftrightarrow\ & \sigma = \mathrm{id},\ \xi\eta = 1,\ m_{\sigma'(1),\sigma'(i)} = \xi'\eta' m_{1i},\ \forall\varphi'\\
\Leftrightarrow\ & \sigma = \mathrm{id},\ \xi\eta = 1,\ m_{1i} = 0\\
\Leftrightarrow\ & \varphi = (1,\mathrm{id},1,\mathbf{0}) \text{ or } (-1,\mathrm{id},-1,\mathbf{0}).
\end{aligned}$$

Let $\nu = (1,1,-1,\mathbf{0})$ for $\xi = -1$ and $\pi = (-1,1,1,\mathbf{0})$ for $\eta = -1$. Then

$$-\mathbf{1}_V = \nu\pi = (-1,\mathrm{id},-1,\mathbf{0})$$

is the automorphism of the multiplication by -1, which is in the center Z of $\operatorname{Aut} R$. Thus we get

$$\operatorname{Aut} R \cong \{\pm\mathbf{1}_V\} \times \big((S_{\mathfrak{I}} \times \{\pm 1\}_\eta) \ltimes \mathbb{Z}^{\mathfrak{I}-1}\big) \cong \mathbb{Z}_2 \times \big((S_{\mathfrak{I}} \times \mathbb{Z}_2) \ltimes \mathbb{Z}^{\mathfrak{I}-1}\big).$$

From now on, we want to determine $W(R)$ and $\operatorname{Out} R$. To do so, we need to have some identification. To show another isomorphism, we identify $\mathbb{Z}^{\mathfrak{I}-1}$ with $\mathbb{Z}^{\mathfrak{I}}/\mathbb{Z}\iota$ by the group isomorphisim

$$f : (x_2, x_3, \ldots) \mapsto (0, x_2, x_3 \ldots) + \mathbb{Z}\iota = \overline{(0, x_2, x_3 \ldots)}.$$

Thus, for $\varphi = (\eta, \sigma, \xi, (m_i)_{i\in\mathfrak{I}\setminus\{1\}})$, we can write

$$\varphi = (\eta, \sigma, \xi, \overline{(m_i)}_{i\in\mathfrak{I}}) \in (\{\pm1\}_\eta \times S_{\mathfrak{I}} \times \{\pm1\}_\xi) \ltimes (\mathbb{Z}^{\mathfrak{I}}/\mathbb{Z}\iota),$$

and in (1), we have $(m_{\sigma'(1)\sigma'(i)} + \xi\eta m'_{1i})_{i\in\mathfrak{I}\setminus\{1\}} = (m_{\sigma'(i)} - m_{\sigma'(1)} + \xi\eta m'_i)_{i\in\mathfrak{I}\setminus\{1\}}$. Hence, we get

$$f\big((m_{\sigma'(1)\sigma'(i)} + \xi\eta m'_{1i})_{i\in\mathfrak{I}\setminus\{1\}}\big) = \overline{(m_{\sigma'(i)} - m_{\sigma'(1)} + \xi\eta m'_i)}_{i\in\mathfrak{I}} = \overline{(m_{\sigma'(i)} + \xi\eta m'_i)}_{i\in\mathfrak{I}}$$

since

$$\overline{(x_i)}_{i\in\mathfrak{I}} = \overline{(x_i + x)}_{i\in\mathfrak{I}} \quad \text{for any } x \in \mathbb{Z}$$

and the coordinate of $1 \in \mathfrak{I}$ in the last expression, i.e., $m_{\sigma'(1)} + \xi\eta m'_1$, is equal to $m_{\sigma'(1)}$ (since $m'_1 = 0$ by definition). Therefore, we obtain

$$\operatorname{Aut} R \cong (\{\pm1\}_\eta \times S_{\mathfrak{I}} \times \{\pm1\}_\xi) \ltimes (\mathbb{Z}^{\mathfrak{I}}/\mathbb{Z}\iota),$$

and for $\varphi = (\eta, \sigma, \xi, \overline{(m_i)}_{i\in\mathfrak{I}})$ and $\psi = (\eta', \sigma', \xi', \overline{(m'_i)}_{i\in\mathfrak{I}}), \in \operatorname{Aut} R$,

$$\varphi\circ\psi = \big(\eta\eta', \sigma\sigma', \xi\xi', \overline{(m_{\sigma'(i)} + \xi\eta m'_i)}_{i\in\mathfrak{I}}\big). \tag{2}$$

Also, by the same reason above (using $-\mathbf{1}_V = \nu\pi$), we get

$$\operatorname{Aut} R \cong \{\pm\mathbf{1}_{\mathbf{V}}\} \times \big((S_{\mathfrak{I}} \times \{\pm1\}_\eta) \ltimes (\mathbb{Z}^{\mathfrak{I}}/\mathbb{Z}\iota)\big) \cong \mathbb{Z}_2 \times \big((S_{\mathfrak{I}} \times \mathbb{Z}_2) \ltimes (\mathbb{Z}^{\mathfrak{I}}/\mathbb{Z}\iota)\big).$$

The Weyl group $W(R)$ is generated by $\sigma_{\epsilon_1-\epsilon_j+r\delta}$ (since $\{\epsilon_1 - \epsilon_j \mid j \in \mathfrak{I}\setminus\{1\}\}$ is a reflectable basis of the locally finite root system $A_{\mathfrak{I}-1}$), and we observe that $\sigma_{\epsilon_1-\epsilon_j+r\delta}(\delta) = \delta$ and

$$\sigma_{\epsilon_1-\epsilon_j+r\delta}(\epsilon_1 - \epsilon_i) = \sigma_{\epsilon_1-\epsilon_j}(\epsilon_1 - \epsilon_i) - r\langle\epsilon_1 - \epsilon_i, \epsilon_1 - \epsilon_j\rangle\delta.$$

So we have

$$\sigma_{\epsilon_1-\epsilon_j+r\delta} = (1, (1,j), 1, \overline{(m_i)}_{i\in\mathfrak{I}}) = (1, (1,j), 1, f((m_{1i})_{i\in\mathfrak{I}\setminus\{1\}})),$$

where $m_{1j} = -2r$ and $m_{1i} = -r$ for $i \neq j$, and f is defined above. Thus,

$$f((-r,\ldots,-r,-2r,-r,\ldots)) = \overline{(0,-r,\ldots,-r,-2r,-r,\ldots)} = \overline{(r,0,\ldots,0,-r,0,\ldots)},$$

where the last representative has the trivial trace. (We simply put the first coordinate to be the coordinate of $1 \in \mathfrak{I}$ for covenience.) Thus, the Weyl group $W(R)$ of R is isomorphic to $S_{(\mathfrak{I})} \ltimes \mathbb{Z}_0^{(\mathfrak{I})}$ since $(\mathbb{Z}_0^{(\mathfrak{I})} + \mathbb{Z}\iota)/\mathbb{Z}\iota \cong \mathbb{Z}_0^{(\mathfrak{I})}$. Hence, we obtain

$$\operatorname{Out} R \cong \{\pm\mathbf{1}_V\} \times \Big(\big(S_{\mathfrak{I}}/S_{(\mathfrak{I})} \times \mathbb{Z}_2\big) \ltimes \big(\mathbb{Z}^{\mathfrak{I}}/(\mathbb{Z}_0^{(\mathfrak{I})} + \mathbb{Z}\iota)\big)\Big).$$

Next we suppose $|\mathfrak{I}| = n$. Then we have $S_{\mathfrak{I}} = S_{(\mathfrak{I})} = S_n$ and $\operatorname{Out} R \cong \mathbb{Z}_2 \ltimes \big(\mathbb{Z}^n/(\mathbb{Z}_0^n + \mathbb{Z}\iota)\big)$. In fact, we obtain:

$$\mathbb{Z}^n/(\mathbb{Z}^n_0 + \mathbb{Z}\iota) \cong C_n = \langle s\rangle \text{ (a cyclic group)} \quad \text{via} \quad \mathbf{x} + \mathbb{Z}^n_0 + \mathbb{Z}\iota \mapsto s^{\mathrm{tr}(\mathbf{x})} \quad \text{and} \tag{3}$$

$$\mathbb{Z}_2 \ltimes C_n \cong I_n = \langle s, t\rangle \text{ (a dihedral group)} \quad \text{via} \quad \theta : (\eta, \mathbf{x} + \mathbb{Z}^n_0 + \mathbb{Z}\iota) \mapsto s^{\mathrm{tr}(\mathbf{x})}t^{\varepsilon(\eta,-1)}, \tag{4}$$

where $\varepsilon(i, j)$ means 1 (resp. 0) if $i = j$ (resp. $i \neq j$), and s and t are generators satisfying $s^n = t^2 = 1$ and $tst = s^{-1}$. In fact, the map defined in (3) is clearly a well-defined epimorphism. To show one to one, if $\mathrm{tr}(\mathbf{x}) = nk$ for some $k \in \mathbb{Z}$, then

$$\begin{aligned}\bar{\mathbf{x}} &= \overline{\mathbf{x} + (0, \dots, 0, -\mathrm{tr}(\mathbf{x})) + (0, \dots, 0, \mathrm{tr}(\mathbf{x}))} \\ &= \overline{(0, \dots, 0, \mathrm{tr}(\mathbf{x}))} \\ &= \overline{(k, \dots, k) - (k, \dots, k, -nk + k)} \\ &= \bar{\mathbf{0}} \mod \mathbb{Z}^n_0 + \mathbb{Z}\iota,\end{aligned}$$

and hence one to one. The map θ in (4) is clearly well-defined and surjective. That θ is a homomorphism follows from

$$\begin{aligned}\theta((\eta, \bar{\mathbf{x}})(\eta', \bar{\mathbf{x}}')) &= \theta((\eta\eta', \overline{\mathbf{x} + \eta\mathbf{x}'})) \\ &= s^{\mathrm{tr}(\mathbf{x}+\eta\mathbf{x}')}t^{\varepsilon(\eta\eta',-1)} \\ &= \begin{cases} s^{\mathrm{tr}(\mathbf{x})}s^{\mathrm{tr}(\mathbf{x}')}t^{\varepsilon(\eta',-1)} & \text{if } \eta = 1 \\ s^{\mathrm{tr}(\mathbf{x})}s^{-\mathrm{tr}(\mathbf{x}')}tt^{\varepsilon(-\eta',-1)} = s^{\mathrm{tr}(\mathbf{x})}ts^{\mathrm{tr}(\mathbf{x}')}t^{\varepsilon(\eta',-1)} & \text{if } \eta = -1 \end{cases} \\ &= \theta((\eta, \bar{\mathbf{x}}))\theta((\eta', \bar{\mathbf{x}}'))\end{aligned}$$

for $(\eta, \bar{\mathbf{x}}),\ (\eta', \bar{\mathbf{x}}') \in \mathbb{Z}_2 \ltimes \big(\mathbb{Z}^n/(\mathbb{Z}^n_0 + \mathbb{Z}\iota)\big)$. If $\theta((\eta, \bar{\mathbf{x}})) = s^{\mathrm{tr}(\mathbf{x})}t^{\varepsilon(\eta,-1)} = 1$, then $\mathrm{tr}(\mathbf{x}) = nk$ for some $k \in \mathbb{Z}$ and $\eta = 1$. Thus $\bar{\mathbf{x}} = \bar{\mathbf{0}}$ as above, and so θ is an isomorphism.

Here we notice that the following exact sequence splits.

$$0 \longrightarrow S_n \ltimes \mathbb{Z}^n_0 \hookrightarrow (S_n \times \{\pm 1\}_\eta) \ltimes (\mathbb{Z}^n/\mathbb{Z}\iota) \longrightarrow I_n \longrightarrow 0$$

In fact, let $\psi : I_n \longrightarrow (S_n \times \{\pm 1\}_\eta) \ltimes (\mathbb{Z}^n/\mathbb{Z}\iota)$ by

$$\begin{aligned} s &\mapsto S = \big(1, v, 1, \overline{(0, \dots, 0, 1)}\big) \\ t &\mapsto T = \big(-1, w, 1, \bar{\mathbf{0}}\big),\end{aligned}$$

where

$$\begin{aligned} v &= (1, 2, \dots, n) \quad \text{and} \\ w &= (1, n)(2, n-1)\cdots(p-1, p+2)(p, p+1) \quad \text{if } n = 2p \quad \text{or} \\ w &= (1, n)(2, n-1)\cdots(p-1, p+3)(p, p+2) \quad \text{if } n = 2p+1 \end{aligned}$$

Then, we can find

$$S^n = T^2 = \big(1, \mathrm{id}, 1, \bar{\mathbf{0}}\big) \quad \text{and} \quad TST = S^{-1},$$

which is confirmed by the following direct calculation.

$$\begin{aligned} S^n &= S^{n-2} \circ \big(1, v, 1, \overline{(0, \dots, 0, 0, 1)}\big) \circ \big(1, v, 1, \overline{(0, \dots, 0, 1)}\big) \\ &= S^{n-2} \circ \big((1, v^2, 1, \overline{(0, 0, \dots, 0, 1, 1)}\big) \\ &= \cdots\cdots\cdots \\ &= S \circ \big((1, v^{n-1}, 1, \overline{(0, 1, \dots, 1, 1, 1)}\big) \\ &= \big((1, \mathrm{id}, 1, \bar{\mathbf{0}}\big), \\ T^2 &= \big((1, w^2, 1, \bar{\mathbf{0}}\big) \\ &= \big((1, \mathrm{id}, 1, \bar{\mathbf{0}}\big), \\ TST &= \big(-1, w, 1, \bar{\mathbf{0}}\big) \circ \Big(\big(1, v, 1, \overline{(0, \dots, 0, 1)}\big) \circ \big(-1, w, 1, \bar{\mathbf{0}}\big)\Big) \\ &= \big(-1, w, 1, \bar{\mathbf{0}}\big) \circ \big(-1, vw, 1, \overline{(0, \dots, 0, 1)}\big) \\ &= \big(1, wvw, 1, \overline{(0, 1, \dots, 1)}\big) \\ &= S^{-1} \end{aligned}$$

Therefore, we see that $\mathrm{Aut}\, R \cong \{\pm \mathbf{1}_V\} \ltimes \big((S_n \ltimes \mathbb{Z}^n_0) \rtimes \mathrm{I}_n\big) \cong \mathbb{Z}_2 \ltimes \big((S_n \ltimes \mathbb{Z}^n_0) \rtimes \mathrm{I}_n\big)$. Thus we have shown all the statements. □

Example 2.3. The following are concrete examples of so-called diagram automorphisms relative to the Dynkin diagram of type $\mathrm{A}^{(1)}_4$ or $\mathrm{A}^{(1)}_5$.

(1) Let φ be the automorphism of $\mathrm{A}^{(1)}_4$ defined by $\varphi(\delta)=\delta$ and

$$\epsilon_1-\epsilon_2\mapsto\epsilon_2-\epsilon_3,\quad \epsilon_2-\epsilon_3\mapsto\epsilon_3-\epsilon_4,\quad \epsilon_3-\epsilon_4\mapsto\epsilon_4-\epsilon_5\quad \textit{and}\quad \epsilon_4-\epsilon_5\mapsto\epsilon_5-\epsilon_1+\delta.$$

Then, we have $\epsilon_1-\epsilon_3\mapsto\epsilon_2-\epsilon_4$, $\epsilon_1-\epsilon_4\mapsto\epsilon_2-\epsilon_5$ and $\epsilon_1-\epsilon_5\mapsto\epsilon_2-\epsilon_1+\delta$, and hence,

$$\varphi=S=(1,v,1,\overline{(0,0,0,0,1)}),\ \text{where } v=(1,2,3,4,5).$$

Let φ' be the automorphism of $\mathrm{A}^{(1)}_4$ defined by $\varphi'(\delta)=\delta$ and

$$\epsilon_1-\epsilon_2\leftrightarrow\epsilon_4-\epsilon_5,\quad \epsilon_2-\epsilon_3\leftrightarrow\epsilon_3-\epsilon_4,\quad \textit{and}\quad \epsilon_5-\epsilon_1+\delta\mapsto\epsilon_5-\epsilon_1+\delta.$$

Then, we have $\epsilon_1-\epsilon_3\mapsto\epsilon_3-\epsilon_5$, $\epsilon_1-\epsilon_4\mapsto\epsilon_2-\epsilon_5$ and $\epsilon_1-\epsilon_5\mapsto\epsilon_1-\epsilon_5$, and hence,

$$\varphi'=T=(-1,w,1,\bar{\mathbf{0}}),\ \text{where } w=(1,5)(2,4).$$

(2) Similarly, let φ be the automorphism of $\mathrm{A}^{(1)}_5$ defined by $\varphi(\delta)=\delta$ and

$$\epsilon_i-\epsilon_{i+1}\mapsto\epsilon_{i+1}-\epsilon_{i+2}\quad (i=1,2,3,4)\quad \textit{and}\quad \epsilon_5-\epsilon_6\mapsto\epsilon_6-\epsilon_1+\delta.$$

Then, we get

$$\varphi=S=(1,w,1,\overline{(0,0,0,0,0,1)}),\ \text{where } v=(1,2,3,4,5,6).$$

Let φ' be the automorphism of $\mathrm{A}^{(1)}_5$ defined by $\varphi'(\delta)=\delta$ and

$$\epsilon_1-\epsilon_2\leftrightarrow\epsilon_5-\epsilon_6,\quad \epsilon_2-\epsilon_3\leftrightarrow\epsilon_4-\epsilon_5,\quad \epsilon_3-\epsilon_4\mapsto\epsilon_3-\epsilon_4,\quad \textit{and}\quad \epsilon_6-\epsilon_1+\delta\mapsto\epsilon_6-\epsilon_1+\delta.$$

Then, we have $\epsilon_1-\epsilon_3\mapsto\epsilon_4-\epsilon_6$, $\epsilon_1-\epsilon_4\mapsto\epsilon_3-\epsilon_6$, $\epsilon_1-\epsilon_5\mapsto\epsilon_2-\epsilon_6$ and $\epsilon_1-\epsilon_6\mapsto\epsilon_1-\epsilon_6$, and hence,

$$\varphi'=T=(-1,w,1,\bar{\mathbf{0}}),\ \text{where } w=(1,6)(2,5)(3,4).$$

(3) In case of type $\mathrm{A}^{(1)}_1$, we see $T=\mathbf{1}_V$ and, so we need to say $I_2=\langle S\rangle=\mathbb{Z}_2$.

(4) In the case when $\mathfrak{I}=\mathbb{Z}$, which is a countable set, we let R be of type $\mathrm{A}_{\mathfrak{I}-1}$ or of type $\mathrm{A}^{(1)}_{\mathfrak{I}-1}$. Then, there are at least two typical outer automorphisms. That is, we have

$$\begin{aligned}&V=\oplus_{i\in\mathbb{Z}\setminus\{1\}}\mathbb{Q}(\epsilon_1-\epsilon_i)=\oplus_{i\in\mathbb{Z}}\mathbb{Q}(\epsilon_i-\epsilon_{i+1})\subset\hat{V}=V\oplus\mathbb{Q}\delta,\\ &\varphi:\epsilon_{i-1}-\epsilon_i\mapsto\epsilon_i-\epsilon_{i+1},\\ &\varphi':\epsilon_i-\epsilon_{i+1}\mapsto\epsilon_{-i}-\epsilon_{-i+1}\quad\text{or}\quad \epsilon_i-\epsilon_{i+1}\mapsto\epsilon_{-i-1}-\epsilon_{-i}.\end{aligned}$$

Then, the subgroup generated by φ and φ' is isomorphic to an infinite dihedral group I_∞.

(5) We suppose the same situation as in (4). Let us take and fix an integer $p\geq 2$. For $0\leq i\leq p-1$, we define

$$\begin{aligned}v_i&=(\ldots,i-3p,i-2p,i-p,i,i+p,i+2p,i+3p,\ldots)\\ w_i&=(i-p,i+p)(i-2p,i+2p)(i-3p,i+3p)\cdots.\end{aligned}$$

Then, the subgroup H of $\operatorname{Aut}R$ generated by v_i and w_i for all $i=0,1,\ldots,p-1$ is isomorphic to

$$I^p_\infty=\underbrace{I_\infty\times\cdots\times I_\infty}_{p}=\langle v_0,w_0\rangle\times\cdots\times\langle v_{p-1},w_{p-1}\rangle.$$

(6) Also in the same situation as in (4), it is true that every finite group G can be found in $\operatorname{Aut}R$ and in $\operatorname{Out}R$ at the same time as subgroups.

(7) Let R be an affine root system of type $\mathrm{A}^{(1)}_1$. We take $\varphi=(1,\mathrm{id},1,\overline{(1)})$, satisfying $\alpha_1\mapsto\alpha_1+\delta$ and $\delta\mapsto\delta$, and we take $\varphi'=(1,(1,2),1,\overline{(-2)})$, satisfying $\alpha_1\mapsto-\alpha_1-2\delta$ and $\delta\mapsto\delta$. Then, φ and φ' generate an infinite dihedral subgroup of $\operatorname{Aut}R$. Furthermore, we take $\varphi''=(1,\mathrm{id},-1,\bar{\mathbf{0}})$, satisfying $\alpha_1\mapsto\alpha_1$ and $\delta\mapsto-\delta$. Then, φ and φ'' also generate an infinite dihedral subgroup of $\operatorname{Aut}R$.

Remark 2.4. (1) One can determine the automorphisms of a locally finite root sysytem of type $\mathrm{A}_{\mathfrak{I}-1}$ by the same way as in the proof of Theorem 2.2. Namely, let φ be an automorphism of the root system $R=\{\epsilon_i-\epsilon_j\mid i\neq j\in\mathfrak{I}\}$. Then there exists some permutation $\sigma\in S_{\mathfrak{I}}$ such that $\varphi(\epsilon_i-\epsilon_j)=\epsilon_{\sigma(i)}-\epsilon_{\sigma(j)}$ for all $i\neq j\in\mathfrak{I}$ or such that $\epsilon_{\sigma(j)}-\epsilon_{\sigma(i)}$ for all $i\neq j\in\mathfrak{I}$. Hence, $\operatorname{Aut}R\cong S_{\mathfrak{I}}\times\{\pm1\}$, and the Weyl group of R is $S_{(\mathfrak{I})}$. These results are known in [LN].

(2) Let R be a locally affine root system in $\hat{V}$ as before. Put $\bar{V}=\hat{V}/\mathbb{Q}\delta$, and $\omega:\hat{V}\to\bar{V}$ be a canonical linear map. Then, $\bar{R}=\omega(R)$ is a locally finite root system in $\bar{V}$. The map ω induces

a group homomorphism, again called ω, of $\operatorname{Aut} R$ to $\operatorname{Aut} \bar{R}$. It is easily confirmed that this group homomorphism ω is surjective. Set $\operatorname{Aut}_0 R = \operatorname{Ker} \omega$. Therefore, we obtain the exact sequence

$$1 \longrightarrow \operatorname{Aut}_0 R \longrightarrow \operatorname{Aut} R \longrightarrow \operatorname{Aut} \bar{R} \longrightarrow 1.$$

We notice that this exact sequence is split.

(3) In the same situation as in Remark (1), we also obtain the following exact sequence:

$$1 \longrightarrow W_0(R) \longrightarrow W(R) \longrightarrow W(\bar{R}) \longrightarrow 1,$$

where $W_0(R) = W(R) \cap \operatorname{Aut}_0 R$. Therefore, we obtain the following diagram of exact sequences.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & W_0(R) & \rightarrow & W(R) & \rightarrow & W(\bar{R}) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \operatorname{Aut}_0 R & \rightarrow & \operatorname{Aut} R & \rightarrow & \operatorname{Aut} \bar{R} & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \operatorname{Out}_0 R & \rightarrow & \operatorname{Out} R & \rightarrow & \operatorname{Out} \bar{R} & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Here, $\operatorname{Out}_0 R = (\operatorname{Aut}_0 R)/W_0(R) \cong (\operatorname{Aut}_0 R)\, W(R)/W(R)$. Then, we have

$$W(R) = W(\bar{R}) \ltimes W_0(R), \quad \operatorname{Aut} R = (\operatorname{Aut} \bar{R}) \ltimes (\operatorname{Aut}_0 R) \quad \text{and} \quad \operatorname{Out} R = (\operatorname{Out} \bar{R}) \ltimes (\operatorname{Out}_0 R).$$

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