Derivations and Invariant Forms of Jordan and Alternative Tori

Dedicated to Holger Petersson

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Abstract. Jordan and alternative tori are the coordinate algebras of extended affine Lie algebras of type A_1 and A_2. In this paper we show that the derivation algebra of a Jordan torus is a semidirect product of the ideal of inner derivations and the subalgebra of central derivations. In the course of proving this result, we investigate derivations of the more general class of division graded Jordan and alternative algebras. We also describe invariant forms of these algebras.

Introduction

This paper provides a detailed description of the derivation algebra $\operatorname{Der} J$ of a Jordan torus J. In particular, our main result 4.11 says that the derivation algebra $\operatorname{Der} J$ of a Jordan torus J is a semidirect product,

$$\operatorname{Der} J = \operatorname{IDer} J \rtimes \operatorname{CDer} J, \tag{1}$$

of the ideal $\operatorname{IDer} J$ of inner derivations and the subalgebra $\operatorname{CDer} J$ of central derivations of J.

Let us explain the concepts involved in the statement above. We consider (linear) unital Jordan algebras J over a field F of characteristic $\neq 2$. All our algebras are G-graded, i.e., $J = \bigoplus_{g\in G} J_g$, $J_g J_{g'} \subset J_{g+g'}$, where G is an abelian group. We call J *division graded* if every $0 \neq x_g \in J_g$ is invertible. A division-graded J is a *Jordan G-torus* if $\dim J_g \leq 1$ for all $g \in G$, and is simply called a *Jordan torus* if $G = \mathbb{Z}^n$. Examples of Jordan tori are the plus algebras of quantum tori (see 4.2) which recently have gained a lot of interest. Jordan tori have recently been classified in [**31**].

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Perhaps even more important than the decomposition (1) itself is in our opinion the concept of central derivations. In general, for a nonassociative G-graded algebra $A = \bigoplus_{g\in G} A_g$ with centre Z a *central derivation* ∂_Θ is associated to any group homomorphism $\Theta\colon G \to (Z, +)$ by defining $\partial_\Theta(a_g) = \Theta(g)a_g$ for $a_g \in A_g$. We denote by $\operatorname{CDer} J$ the subalgebra of all central derivations.

We learned the concept of central derivations from the paper [**24**] by Osborn and Passman where they have been introduced for twisted group algebras and where (1) is proven for these algebras. That the derivation algebra is a semidirect product of the ideal of inner derivations and a subalgebra had been proven before in [**4**] for quantum tori and in [**5**] for Cayley tori, see 4.3. One of the novelties of this paper is that we provide a conceptual description of this subalgebra as the algebra of central derivations.

Although our main motivation are Jordan tori, in the body of the paper we are considering more general algebras, namely Jordan or alternative division graded algebras in §2 and the corresponding G-tori in §3. We do so since the methods and results needed for the Jordan torus case easily generalize to the more general settings, essentially without any extra cost, and since we believe that the corresponding Lie algebras, studied in [**32**] and [**29**], are an interesting class of algebras. Therefore, our paper also provides information on $\operatorname{Der} T$ for any G-torus T, see for example 3.2 where a version of (1) is proven for G-tori satisfying an additional condition. In §4 we then prove (1) for all Jordan tori, by making use of their classification ([**31**]). As already mentioned, (1) has been proven for associative tori in [**4**] and for nonassociative alternative tori in [**5**]. Our paper provides a slightly more conceptual proof in the latter case. It is included here since it can be done witout any extra cost. Besides the decomposition (1) we also determine the precise structure of $\operatorname{CDer} T$ and $\operatorname{IDer} T$. For example, we show that if G is finitely generated and A is division graded, then $\operatorname{CDer} A$ is a generalized Witt algebra 2.9, more precisely a generalization of the recent generalization of Witt algebras by Doković and Zhao [**7**].

Our interest in Jordan tori and their derivations comes from the theory of extended affine Lie algebras: it is shown in [**31**] that the centreless cores of extended affine Lie algebras of type A_1 are precisely the Tits-Kantor-Koecher algebras of Jordan tori. Moreover, it is proven in the recent preprint [**2**] by Allison and Gao that special classes of Jordan tori enter in the description of the centreless cores of extended affine Lie algebras of reduced non-simply-laced types. In the spirit of the paper [**4**] by Berman, Gao and Krylyuk on extended affine Lie algebras of type $\mathrm{A}_l, l \geq 3$ (or [**5**] for type A_2) the description (1) is an essential ingredient in the classification of all tame extended affine Lie algebras of type A_1 and other types. Another ingredient in the construction of extended affine Lie algebras of type A_1 are invariant forms. It is well-known that invariant forms are determined by invariant forms on the corresponding coordinate algebras, see for example Koecher's work [**17**] dealing with Tits-Kantor-Koecher algebras or Benkart's more recent paper [**3**] for root-graded Lie algebras. We prove in 4.9 that for Jordan tori there exists, up to scalar multiples, only one nondegenerate graded invariant form. The consequences of our paper for extended affine Lie algebars will be elaborated in a sequel to this paper.

This paper generalizes results contained in the second author's Ph.D. thesis written at the University of Ottawa under the supervision of the first author.

1. Basic definitions and notations

In this section we will review some basic concepts. Unless specified otherwise, we will consider nonassociative (= not necessarily associative) algebras over some field F of arbitrary characteristic, denoted ch. F. Our primary interest are unital alternative and Jordan algebras. Whenever we consider Jordan algebras we will assume that ch. $F \neq 2$, unless explicitly stated otherwise.

1.1. Preliminaries. Let A be a nonassociative algebra with product written as ab, $a, b \in A$. For $a, b, c \in A$ we define the *commutator* as $[a, b] = ab - ba = (\operatorname{ad} a)(b)$ and the *associator* as $(a, b, c) = (ab)c - a(bc)$. The span of all commutators and associators will be denoted $[A, A]$ respectively (A, A, A).

The *centre* $Z = Z(A)$ *of* A is defined as $Z(A) = \{z \in A \mid [z, A] = 0 = (z, A, A) = (A, z, A)\}$. If A is unital $Z(A)$ is a unital associative commutative subalgebra of A and A is canonically an algebra over Z. The interplay between the F-algebra and Z-algebra structure will be an important feature of this paper. We have $Z(A) = \{z \in A \mid [z, A] = 0 = (z, A, A)\}$ if A is alternative (but see 2.5) and $Z(A) = \{z \in A \mid (z, A, A) = 0\}$ if A is Jordan.

An F-linear map $d\colon A \to A$ is called a *derivation of* A if $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. With the usual commutator the set of all derivations of A is a Lie algebra denoted $\operatorname{Der}_F A$, or $\operatorname{Der} A$ if F is clear form the context. Any $d \in \operatorname{Der}_F A$ leaves the centre $Z = Z(A)$ invariant and hence $d\,|_Z \in \operatorname{Der}_F Z$ for $d \in \operatorname{Der}_F A$. Moreover, we note that zd is a derivation for any $z \in Z$ and $d \in \operatorname{Der}_F A$.

The definition of a derivation of course makes sense if A is an algebra over a unital commutative associative ring, for example a unital F-algebra considered as algebra over its centre Z. The Lie algebra of Z-linear derivations will be denoted $\operatorname{Der}_Z A$. Note that for $d \in \operatorname{Der}_F A$, we have $d \in \operatorname{Der}_Z A \iff d\,|_Z = 0$. It is then easily seen that $\operatorname{Der}_Z A$ is an ideal of $\operatorname{Der}_F A$.

1.2. Central closure. Let A be a unital algebra such that its centre $Z = Z(A)$ does not contain any zero divisors of A, i.e., the Z-module A is torsion-free. An example of such an algebra is a division G-graded algebra where G is an ordered abelian group (2.3.(d)). Let $\overline{Z}$ be the field of fractions of the integral domain Z. The *central closure of* A is defined as $\overline{A} = \overline{Z} \otimes_Z A$ which we consider as an algebra over $\overline{Z}$. We note the following facts [**33**]:

(i) $x \mapsto 1 \otimes x$ is an embedding of A into $\overline{A}$.
(ii) $\overline{A}$ is central over $\overline{Z}$, i.e., the centre of $\overline{A}$ is $\overline{Z}$.
(iii) $\overline{A}$ is prime (resp a domain) $\iff$ A is prime (resp a domain).

Let

$$\overline{\operatorname{Der}_Z A} := \overline{Z} \otimes_Z \operatorname{Der}_Z A,$$

which we consider as a Lie algebra over $\overline{Z}$. There is a canonical map

$$\overline{\operatorname{Der}_Z A} \to \operatorname{Der}_{\overline{Z}} \overline{A} : u \otimes d \mapsto l_u \otimes d \tag{1}$$

where $l_u \otimes d$ acts on $\overline{Z} \otimes A$ via $(l_u \otimes d)(v \otimes a) = uv \otimes d(a)$. Using the fact that any element in $\overline{\mathrm{Der}_Z A}$ has the form $\frac{1}{z} \otimes d$ for some $0 \neq z \in Z$ and $d \in \mathrm{Der}_Z A$, it is easily seen that this map is injective. Similarly, $\mathrm{Der}_Z A$ embeds into $\overline{\mathrm{Der}_Z A}$. After an identification we therefore have

$$\mathrm{Der}_Z A < \overline{\mathrm{Der}_Z A} < \mathrm{Der}_{\overline{Z}} \overline{A}, \tag{2}$$

where $<$ indicates a subalgebra. Also, we note:

$$\textit{If } A \textit{ is a finitely generated algebra over } Z \textit{ then } \overline{\mathrm{Der}_Z A} = \mathrm{Der}_{\overline{Z}} \overline{A}. \tag{3}$$

Indeed, let $\{a_i \mid 1 \leq i \leq n\}$ be a generating set of the Z-algebra A. For $d \in \mathrm{Der}_{\overline{Z}} \overline{A}$ we have $d(1 \otimes a_i) = (1/z_i) \otimes b_i$ for some $0 \neq z_i \in Z$ and $b_i \in A$. Thus, putting $z = \prod_i z_i$, we obtain $zd(1 \otimes a_i) \in 1 \otimes A$. Since zd is Z-linear it follows that $zd(1 \otimes A) \subset 1 \otimes A$, so $zd|_{1 \otimes A} =: d_0 \in \mathrm{Der}_Z A$ and $d = \frac{1}{z} d_0 \in \overline{\mathrm{Der}_Z A}$, proving (3).

For an associative algebra A we denote by A^+ the Jordan algebra defined on A with U-operator $U_x y = xyx$. The bilinear product of A^+ is given by $a.b = \frac{1}{2}(ab + ba)$ where on the right side we have the associative product of A. Any subalgebra of A^+ is then also a Jordan algebra, in particular this is so for $H(A, *) := \{a \in A \mid a^* = a\}$ where $*$ is an involution of A.

1.3. Lemma. *Let A be an associative algebra with involution $*$. We abbreviate $J = H(A, *)$ and consider J as a Jordan algebra. We assume that*

(i) *A is a torsion-free $Z(A)$-module, and*
(ii) *$Z(J) = Z(A) \cap J$.*

We let $\overline{A}$ respectively $\overline{J}$ be the central closures of A and J.

Then $*$ extends uniquely to an involution $*$ of $\overline{A}$ such that $H(\overline{A}, *) \cong \overline{J}$ over the field of fractions L of $Z(J)$ and $L = H(\overline{A}, *) \cap K$ for K the field of fractions of $Z(A)$.

Proof. The involution $*$ on $\overline{A}$ is given by $(\frac{1}{z} \otimes a)^* = \frac{1}{z^*} \otimes a^*$ for $0 \neq z \in Z(A)$ and $a \in A$. Since $Z(J) \subset Z(A)$ we have a well-defined L-linear map

$$\varphi\colon \overline{J} = L \otimes_{Z(J)} J \longrightarrow \overline{A} = K \otimes_{Z(A)} A \ : \ \tfrac{1}{z} \otimes_{Z(J)} x \longmapsto \tfrac{1}{z} \otimes_{Z(A)} x\,.$$

By construction, its image is contained in $H(\overline{A}, *)$. We claim that in fact $\varphi(\overline{J}) = H(\overline{A}, *)$. Indeed, let $\frac{1}{z} \otimes_{Z(A)} a \in H(\overline{A}, *)$, with $0 \neq z \in Z(A)$ and $a \in A$. Then $z^* a = z a^*$ holds in A, hence $z a^* \in H(A, *) = J$. Since also $z^* z \in Z(A) \cap J = Z(J)$ it now follows that $\frac{1}{z} \otimes_{Z(A)} a = \frac{z}{z^* z} \otimes_{Z(A)} a^* = \frac{1}{z^* z} \otimes_{Z(A)} z a^* \in \varphi(\overline{J})$. □

1.4. Inner derivations. For an associative algebra A and any $x \in A$, the map $\mathrm{ad}\, x : A \to A : y \mapsto [x, y]$ is a so-called inner derivation. More generally, for an alternative algebra A over a field F with $\mathrm{ch}.\, F \neq 2$, the inner derivations are sums of the derivations $\Delta_{a,b} = R_{[a,b]} - L_{[a,b]} - 3[L_a, R_b]$, where L and R denote the left respectively right multiplication in A. For a Jordan algebra J, the inner derivations are sums of the derivations $\Delta_{a,b} = [L_a, L_b]$, $a, b \in J$ (see e.g. [**14**, page 35 and page 300]).

For the classes of algebras considered above, the set of all inner derivations will be denoted by $\operatorname{IDer} A$. It is easily seen that $\operatorname{IDer} A$ is an ideal of the Lie algebra $\operatorname{Der}_F A$, and any inner derivation is Z-linear for $Z = Z(A)$:

$$\operatorname{IDer} A \lhd \operatorname{Der}_Z A. \tag{1}$$

Moreover, for any $z \in Z(A)$ we have $z\Delta_{a,b} = \Delta_{za,b} = \Delta_{a,zb}$ from which it easily follows that

$$\overline{Z} \otimes_Z \operatorname{IDer} A = \operatorname{IDer} \overline{A} \tag{2}$$

where we used the identification 1.2.2.

Example. When A is an associative algebra, any derivation of A is clearly a derivation of the Jordan algebra A^+. The inner derivations of A^+ are also inner derivations of A since

$$[L_u, L_v] = 4 \operatorname{ad}[u, v], \tag{3}$$

whence over a base field containing $\frac{1}{2}$,

$$\operatorname{Der} A \subset \operatorname{Der} A^+ \quad \text{and} \quad [A, A]/(Z(A) \cap [A, A]) \cong \operatorname{IDer} A^+ \subset \operatorname{IDer} A \tag{4}$$

(see 2.10.2 and 3.5 for equality).

1.5. Graded algebras. Let G be an abelian group, written additively and assume that A is *G-graded*, i.e., $A = \oplus_{g \in G} A_g$ and $A_g A_h \subset A_{g+h}$ for all $g, h \in G$. In this paper we will only consider algebras graded by an abelian group, although some results hold for non-abelian groups. For $a \in A$, written in the form $a = \sum_{g \in G} a_g$ with $a_g \in A_g$, the a_g will be referred to as the *homogeneous components* and any $a \in \bigcup_{g \in G} A_g$ will be called *homogenous*. The *support of* A is $\operatorname{supp} A := \{g \in G \mid A_g \neq (0)\}$. A subspace B of A is *graded* if $B = \oplus_{g \in G}(B \cap A_g)$ in which case we put $B_g = B \cap A_g$. Examples of graded subspaces are $[A, A]$, (A, A, A) and $Z(A)$. The subgroup of G generated by $\operatorname{supp} Z$, $Z = Z(A)$, is called the *central grading group*. If H is this group the Z-algebra A is G/H-graded: $A = \oplus_{\overline{g} \in G/H} A_{\overline{g}}$ for $A_{\overline{g}} = \oplus_{g \in \overline{g}} A_g$.

1.6. Derivations of graded algebras. Let $A = \oplus_{g \in G} A_g$ be a G-graded algebra. For $g \in G$ define

$$\begin{aligned} (\operatorname{Der}_F A)_g &= \{d \in \operatorname{Der}_F A \mid d(A_h) \subset A_{g+h} \text{ for all } h \in G\}, \text{ and} \\ \operatorname{grDer}_F A &= \bigoplus_{g \in G} (\operatorname{Der}_F A)_g. \end{aligned}$$

It is easily seen that $\operatorname{grDer}_F A$ is a subalgebra of $\operatorname{Der}_F A$,

$$\operatorname{grDer}_F A < \operatorname{Der}_F A \tag{1}$$

called the subalgebra of *graded derivations*. It is well-known that

$$\operatorname{grDer}_F A = \operatorname{Der}_F A \text{ if } A \text{ is a finitely generated } F\text{-algebra} \tag{2}$$

(see for example [**11**, Prop. 1] where this is proven for Lie algebras). For Z the centre of A we put

$$\mathrm{grDer}_Z A = \bigoplus_{g\in G} (\mathrm{Der}_Z A)_g \quad \text{where } (\mathrm{Der}_Z A)_g = (\mathrm{Der}_F A)_g \cap \mathrm{Der}_Z A.$$

It is easy to prove that

$$\mathrm{grDer}_F A \cap \mathrm{Der}_Z A = \mathrm{grDer}_Z A. \tag{3}$$

Let H be a subgroup of G (later this will be the grading group of Z) and put $\overline{G} = G/H$. We then have a $\overline{G}$-grading of A (see 1.5) and analogously a $\overline{G}$-grading of $\mathrm{grDer}_F A$:

$$\begin{aligned}\mathrm{grDer}_F A &= \bigoplus_{\overline{g}\in\overline{G}} (\mathrm{grDer}_F A)_{\overline{g}} \quad \text{where}\\ (\mathrm{grDer}_F A)_{\overline{g}} &= \bigoplus_{g'\in\overline{g}} (\mathrm{grDer}_F A)_{g'}\\ &= \{d \in \mathrm{grDer}_F A \mid d(A_{\overline{a}}) \subset A_{\overline{g}+\overline{a}} \text{ for all } a \in G\}.\end{aligned}$$

We will put

$$(\mathrm{grDer}_F A)_H := (\mathrm{grDer}_F A)_{\overline{0}} = \bigoplus_{h\in H} (\mathrm{Der}_F A)_h.$$

Concerning inner derivations of G-graded alternative or Jordan algebras A we have

$$\mathrm{IDer}\, A = \bigoplus_{g\in G} (\mathrm{IDer}\, A)_g \tag{4}$$

where $(\mathrm{IDer}\, A)_g = (\mathrm{Der}\, A)_g \cap \mathrm{IDer}\, A$. In particular, by 1.4.1,

$$\mathrm{IDer}\, A \subset \mathrm{grDer}_Z A\,. \tag{5}$$

1.7. Example of a leftsymmetric algebra. The algebra and its graded subalgebra, introduced in this subsection, will be used in 1.8 to define special types of derivations.

Let G be an abelian group and let $Z = \bigoplus_{g\in G} Z_g$ be a commutative associative G-graded algebra (later this will be the centre of a nonassociative algebra). We denote by $\mathrm{Hom}_{\mathbb{Z}}(G, Z)$ the F-vector space of abelian group homomorphisms from G into the additive group $(Z, +)$. Of course, $\mathrm{Hom}_{\mathbb{Z}}(G, Z)$ also carries a Z-module structure which however will not be important in the following. We define an F-algebra on $\mathrm{Hom}_{\mathbb{Z}}(G, Z)$ by assigning to $\Theta, \Phi \in \mathrm{Hom}_{\mathbb{Z}}(G, Z)$ the product $\Theta * \Phi$, given by

$$(\Theta * \Phi)(g) = \textstyle\sum_{h\in G} \Theta(h)\Phi(g)_h \tag{1}$$

Note that this is well-defined since $\Phi(g)_h \neq 0$ for finitely many $h \in G$ only. A straightforward calculation shows that the associator with respect to $*$ satisfies

$$(\Theta, \Phi, \Psi)(g) = -\textstyle\sum_{h\in G} \Theta(h)\Phi(h)\Psi(g)_h\,. \tag{2}$$

Therefore $(\mathrm{Hom}_{\mathbb{Z}}(G, Z), *)$ is a *leftsymmetric algebra*, i.e., $(\Theta, \Phi, \Psi) = (\Phi, \Theta, \Psi)$ holds for all $\Theta, \Phi, \Psi \in \mathrm{Hom}_{\mathbb{Z}}(G, Z)$. Leftsymmetric algebras, rather their opposite algebras have recently been studied in [**10**] and [**9**] where the reader can also find references to previous investigations. In particular, these two papers consider examples of rightsymmetric algebras closely related to the example above.

It is well-known and easily checked that for any leftsymmetric algebra the commutator defines a Lie algebra. It will follow from the results in 1.8, 1.9 and 2.9 that the Lie algebra $(\mathrm{Hom}_{\mathbb{Z}}(G,Z),[.,.])$ can be considered a generalization of the Witt algebra.

We denote by $\mathrm{Hom}_{\mathbb{Z}}(G,Z)_{\mathrm{fin}}$ the Z-submodule of all Θ for which $\Theta(G)$ is contained in the sum of finitely many Z_g's and by $\mathrm{Hom}_{\mathbb{Z}}(G,Z_g)$ the F-subspace of those $\Theta \in \mathrm{Hom}_{\mathbb{Z}}(G,Z)$ with $\Theta(G) \subset Z_g$, $g \in G$. It is easily seen that

$$\mathrm{Hom}_{\mathbb{Z}}(G,Z)_{\mathrm{fin}} = \textstyle\bigoplus_{g\in G} \mathrm{Hom}_{\mathbb{Z}}(G,Z_g) \quad (\text{for all } G) \tag{3}$$

$$= \mathrm{Hom}_{\mathbb{Z}}(G,Z) \quad \text{if } G \text{ is finitely generated.} \tag{4}$$

We have $\mathrm{Hom}_{\mathbb{Z}}(G,Z_{g_1}) * \mathrm{Hom}_{\mathbb{Z}}(G,Z_{g_1}) \subset \mathrm{Hom}_{\mathbb{Z}}(G,Z_{g_1+g_2})$, since

$$\Theta * \Phi = \Theta(g_2)\Phi \quad \text{for } \Theta \in \mathrm{Hom}_{\mathbb{Z}}(G,Z), \Phi \in \mathrm{Hom}_{\mathbb{Z}}(G,Z_{g_2}). \tag{5}$$

Hence $\mathrm{Hom}_{\mathbb{Z}}(G,Z)_{\mathrm{fin}}$ is a subalgebra of $(\mathrm{Hom}_{\mathbb{Z}}(G,Z),*)$ which is G-graded by (3). We note

$$\mathrm{Hom}(G,Z) * \mathrm{Hom}(G,Z_0) = 0. \tag{6}$$

1.8. Central derivations. Let A be a G-graded unital algebra over F with centre $Z = \bigoplus_{h\in H} Z_h$ and central grading group H. For the sake of uniqueness

we assume in this subsection that $\operatorname{supp} A$ *spans* G *as abelian group.* (1)

In order to apply the results of 1.7 we will also consider Z as a G-graded algebra. Any $\Theta \in \mathrm{Hom}_{\mathbb{Z}}(G,Z)$ gives rise to an F-linear derivation ∂_Θ of A, defined on A_g by

$$\partial_\Theta(a_g) = \Theta(g)a_g \quad (a_g \in A_g) \tag{2}$$

and called a *central derivation* (see [**24**, 2] where this concept has been introduced for the case of twisted group algebras). We denote by $\mathrm{CDer}_F A$ the space of central derivations. The formula

$$[\partial_\Theta, \partial_\Phi] = \partial_{\Theta*\Phi-\Phi*\Theta} \tag{3}$$

shows that $\mathrm{CDer}_F A$ is a subalgebra of the Lie algebra $\mathrm{Der}_F A$ and that

$$\partial\colon \mathrm{Hom}_{\mathbb{Z}}(G,Z) \to \mathrm{CDer}_F A \tag{4}$$

is an epimorphism of Lie algebras. It maps the subalgebra $\mathrm{Hom}_{\mathbb{Z}}(G,Z)_{\mathrm{fin}}$ of $(\mathrm{Hom}_{\mathbb{Z}}(G,Z),[.,.])$ onto the graded subalgebra

$$(\mathrm{CDer}_F A)_{\mathrm{fin}} := \partial(\mathrm{Hom}_{\mathbb{Z}}(G,Z)_{\mathrm{fin}}) = \textstyle\bigoplus_{h\in H}(\mathrm{CDer}_F A)_h$$

of $\mathrm{grDer}_F A$, where

$$\partial(\mathrm{Hom}_{\mathbb{Z}}(G,Z_h)) =: (\mathrm{CDer}_F A)_h \subset (\mathrm{Der}_F A)_h\,.$$

The derivations in

$$\mathcal{D} := \{\partial_\Theta \mid \Theta \in \mathrm{Hom}_{\mathbb{Z}}(G,Z_0)\} \tag{5}$$

will be called *degree derivations*. By 1.7.6, $\mathcal{D}$ is always abelian.

1.9. Generalized Witt algebras. Central derivations are related to a generalization of the Witt algebra. This Lie algebra can naturally be defined in the following setting:

(i) K is an associative commutative unital ring, H is an abelian group,
(ii) $Z = \bigoplus_{h\in H} Z_h$ is an associative commutative H-graded algebra over K (later this will be a twisted group ring $K^t[H]$ of H over K, or even the group algebra $K[H]$),
(iii) D is a K-module, and
(iv) $p\colon D \times H \to K : (d, h) \mapsto d(h)$ is a pairing which is K-linear in the first variable and additive in the second.

We define

$$W := W(Z, D, p) := Z \otimes_K D = \bigoplus_{h\in H} Z_h \otimes_K D$$

and use the abbreviation $z_h d = z_h \otimes_K d$ for $z_h \in Z_h, d \in D$. There exists a unique K-algebra product $*$ on W such that $(z_h d_1) * (z_i d_2) = z_h d_1(i) z_i d_2$ for homogenous $z_h, z_i \in Z$, $h, i \in H$ and $d_1, d_2 \in D$. It is easily seen that $(W, *)$ is a leftsymmetric algebra, hence W together with the commutator is a Lie algebra which we call the *generalized Witt algebra associated to* (Z, D, p). Thus, the Lie algebra product of W is given by

$$[z_h d_1, z_i d_2] = z_h z_i \left(d_1(i) d_2 - d_2(h) d_1\right). \tag{1}$$

For the special case of the group algebra $Z = K[H]$, this Lie algebra has been introduced in [**7**] where the reader can also find some background information and examples, justifying the terminology. Another special case has been considered in [**26**] and [**28**].

In the setting above, the left and right kernel of p are

$$D^H := \{d \in D : d(H) = 0\} \quad \textit{(left kernel)},$$
$$H^D := \{h \in H : p(D, h) = 0\} \quad \textit{(right kernel)}.$$

Example. In the setting of 1.8 we let $K = Z_0$, H the central grading group of $Z = Z(A)$, $D = \mathrm{Hom}_{\mathbb{Z}}(G, Z_0)$ and

$$p\colon \mathrm{Hom}_{\mathbb{Z}}(G, Z_0) \times H \to Z_0 : (\varphi, h) \mapsto \varphi(h) \tag{2}$$

the canonical pairing. In this case $D^H = \{\varphi \in \mathrm{Hom}_{\mathbb{Z}}(G, Z_0) \mid \varphi(H) = 0\}$ which is in general non-zero, e.g. for quantum tori. It follows from 1.7.5 that

$$\mu\colon W(Z, \mathrm{Hom}_{\mathbb{Z}}(G, Z_0), p) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(G, Z)_{\mathrm{fin}}\colon z_h \otimes \varphi \mapsto z_h \varphi \tag{3}$$

is a homomorphism of leftsymmetric algebras, hence also for the associated Lie algebras. Note that we now have constructed the following Lie algebra homomorphisms

$$W(Z, \mathrm{Hom}_{\mathbb{Z}}(G, Z_0), p) \xrightarrow{\mu} \mathrm{Hom}_{\mathbb{Z}}(G, Z)_{\mathrm{fin}} \xrightarrow{\partial} (\mathrm{CDer}_F A)_{\mathrm{fin}}. \tag{4}$$

We will show in 2.9 that for so-called division graded algebras both maps μ and ∂ are isomorphisms.

1.10. Lemma. *Let $W = W(Z, D, p)$ be a generalized Witt algebra as in* 1.9 *above.*

(a) *Any K-submodule D' of D gives rise to a subalgebra $Z \otimes_K D'$ of W. Moreover,*

$$Y := Z \otimes_K D^H \tag{1}$$

is an abelian ideal of W. If $D = D' \oplus D^H$ is a direct sum of K-modules, e.g. if K is a field, the generalized Witt algebra W is a semidirect product,

$$W = X \ltimes Y \quad \textit{for } X := Z \otimes_K D' \tag{2}$$

of an abelian ideal Y and a subalgebra X which is again a generalized Witt algebra, namely $X = W(Z, D', p')$, where $p' = p|D' \times H$ has zero left kernel.

(b) *If K is a field of characteristic $\neq 2$ and $p \neq 0$ then W is perfect.*

Proof. (a) is easy. For (b) we observe $[z_0 d, z_h d] = z_0 z_h d(h) d \in K z_h d$ for $h \in H$ and any $d \in D$. Hence $Zd \subset [W, W]$ for any $d \notin D^H$. But for $d \in D^H$ and any $d_1 \in D$ we have $[z_h d_1, z_i d] = z_h z_i d_1(i) d$. Since $p \neq 0$, there exist d_1 and $i \in H$ such that $d_1(i) \neq 0$ and so $z_k d \in [W, W]$ for any $k \in H$.

Remark. For the special case that K is a field of characteristic 0 and Z is the group algebra of H, the structure of W is determined in [**16**]. In particular, a more precise version of (2) is shown there: W is a semidirect product of the abelian ideal Y and a simple subalgebra X which is again a generalized Witt algebra. We will prove an analogue in our setting in 4.12.

1.11. Example: $H(A, *)$ for $*$ of second kind. Let $A = \oplus_{g \in G} A_g$ be a unital associative G-graded algebra over a field E of ch. $E \neq 2$ and suppose $*$ is a graded involution of A, i.e., $A_g^* = A_g$ for all $g \in G$, which is of second kind in the sense that $E.1 = (E.1)^* \not\subset H(A, *)$. For simpler notation let us put

$$H := H(A, *) \quad \text{and} \quad S := S(A, *) = \{a \in A \mid a^* = -a\}.$$

We identify $E = E.1$. With $F = E \cap H$ we have that E/F is a Galois extension of degree 2, so $E = F[s]$ for some $s = -s^* \in E$. Then

$$A = H \oplus sH \quad \text{where} \quad sH = S \tag{1}$$

as G-graded spaces. It follows that the supports of the associative algebra A and the Jordan algebras A^+ and H all coincide:

$$\operatorname{supp} A = \operatorname{supp} A^+ = \operatorname{supp} H\,. \tag{2}$$

Since $[a, b]^* = -[a^*, b^*]$ and the Lie product $[\,\cdot, \cdot\,]$ is E-linear, (1) implies

$$[A, A] = [H, H] \oplus [H, S] \quad \text{with} \quad [A, A] \cap S = [H, H] = [S, S] \tag{3}$$

and $[A, A] \cap H = [H, S] = s[H, H]$. We have the general formula

$$A \ni [x, [y, z]] = 4(y, x, z) \in A^+ \tag{4}$$

where the left and right side are calculated in A and A^+ respectively. Hence the centres of the Jordan algebras A^+ and H are given by

$$\begin{aligned} Z(A^+) &= \{z \in A \mid [A,[A,z]] = 0\} \quad \text{and} \\ Z(H) &= \{z \in H \mid [H,[H,z]] = 0\} = Z(A^+) \cap H \end{aligned} \tag{5}$$

where the second equality in (5) follows from E-linearity of $[\,\cdot,\cdot\,]$.

Let us now consider derivations. The involution $*$ induces a Lie algebra automorphism $\mathrm{Der}_E A^+ \to \mathrm{Der}_E A^+ : d \mapsto d^*$ of order 2, where d^* is defined by $d^*(a) = d(a^*)^*$. Hence, with obvious notation,

$$\mathrm{Der}_E A^+ = H(\mathrm{Der}_E A^+, *) \oplus S(\mathrm{Der}_E A^+, *)$$

Any $d \in H(\mathrm{Der}_E A^+, *)$ leaves H invariant, hence we have a well-defined map

$$\Psi\colon H(\mathrm{Der}_E A^+, *) \longrightarrow \mathrm{Der}_F H : d \mapsto d|_H\,.$$

In fact, Ψ is an isomorphism. Injectivity of Ψ follows from (1), and for surjectivity one notes that any $d \in \mathrm{Der}_F H$ extends to a $\tilde{d} \in \mathrm{Der}_E A^+$ with $\tilde{d}^* = \tilde{d}$ by defining $\tilde{d}(h + sh') = d(h) + sd(h')$ for $h, h' \in H$. In the following we will identify $\mathrm{Der}_F H = H(\mathrm{Der}_E A^+, *)$ via Ψ. We then have

$$\mathrm{Der}_E A^+ = \mathrm{Der}_F H \,\oplus\, s\,\mathrm{Der}_F H \tag{6}$$

where $s\,\mathrm{Der}_F H = S(\mathrm{Der}_E A^+, *)$. By construction, Ψ preserves the homogeneous spaces, hence $(\mathrm{Der}_F H)_g$ is the fixed point space of the automorphism $*$, and because of (6) we then have

$$\dim_E(\mathrm{Der}_E A^+)_g = \dim_F(\mathrm{Der}_F H)_g \tag{7}$$

for all $g \in G$. For inner derivations 1.4.3 implies

$$\mathrm{IDer}\, A^+ = \mathrm{ad}[A,A] = \mathrm{ad}[H,H] \oplus \mathrm{ad}[H,S] = \mathrm{ad}[H,H] \oplus s\,\mathrm{ad}[H,H]$$

where the second equality follows from (3) and $(\mathrm{ad}\, a)^* = \mathrm{ad}(-a^*)$. Hence again by 1.4.3

$$\mathrm{IDer}\, H = \mathrm{ad}[H,H] = \mathrm{IDer}\, A^+ \cap \mathrm{Der}_F H \tag{8}$$

so that (7) also holds for inner derivations. For a central derivation ∂_Θ of A^+ we have $(\partial_\Theta)^* = \partial_{\Theta^*}$ for $\Theta^*(g) = (\Theta(g))^*$. The map $\Theta \mapsto \Theta^*$ leaves $\mathrm{Hom}_{\mathbb{Z}}(G, Z(A^+))$ invariant. By (5), any Θ with $\Theta^* = \Theta$ maps G into $Z(H)$ and hence $\mathrm{Hom}_{\mathbb{Z}}(G, Z(A^+)) = \mathrm{Hom}_{\mathbb{Z}}(G, Z(H)) \oplus s\,\mathrm{Hom}_{\mathbb{Z}}(G, Z(H))$ which implies

$$\mathrm{CDer}_E A^+ = \mathrm{CDer}_F H \oplus s\,\mathrm{CDer}_F H\,. \tag{9}$$

Therefore $\mathrm{CDer}_F H = \mathrm{CDer}_E A^+ \cap \mathrm{Der}_F H$ and (7) also holds for central derivations.

1.12. Graded invariant forms. A *graded invariant form* on G-graded F-algebra $A = \oplus_{g\in G} A_g$ is a symmetric bilinear form $(.|.)$: $A \times A \to F$ satisfying

(i) $(A_g|A_h) = 0$ for $g + h \neq 0$, and
(ii) $(ab|c) = (a|bc)$ for all $a, b, c \in A$.

The F-vector space of all graded invariant forms on A will be denoted $\mathcal{GIF}(A)$. It is described in the following Lemma 1.13 whose proof will be left to the reader.

We will say that a graded invariant form $(.|.)$ is *nondegenerate* if its *radical* $\mathrm{rad}(.|.) = \{a \in A \mid (a|A) = 0\}$ vanishes. Clearly $\mathrm{rad}(.|.)$ is a graded ideal. Hence, if A is graded-simple, i.e., A does not have non-zero graded ideals, then any non-zero graded invariant form is nondegenerate.

1.13. Lemma (graded invariant forms). *Let A be a G-graded algebra over a field F, put $\mathrm{GIF}(A) = A_0/([A, A]_0 + (A, A, A)_0)$ and denote by π the map $A \to \mathrm{GIF}(A) : a = \sum_{g\in G} a_g \mapsto \overline{a_0}$ where $\overline{a_0}$ is the image of $a_0 \in A_0$ under the canonical projection of A_0 onto $\mathrm{GIF}(A)$. Then every linear form $f \in \mathrm{GIF}(A)^* = \mathrm{Hom}_F(\mathrm{GIF}(A), F)$ gives rise to a graded invariant form $(.|.)_f$ defined by $(a|b)_f = f(\pi(ab))$. The map*

$$\mathrm{GIF}(A)^* \to \mathcal{GIF}(A) : f \mapsto (.|.)_f \tag{1}$$

is F-linear, and is an isomorphism if A is unital. In particular,

$$[A, A]_0 = 0 = (A, A, A)_0 \text{ and } A \text{ unital } \implies \mathcal{GIF}(A) \cong A_0^* . \tag{2}$$

1.14. Skew derivations. Let again A be a G-graded F-algebra and let $(.|.)$ be a graded invariant form. A derivation D is called a *skew derivation* if $(d(a)|b) + (a|d(b)) = 0$ for all $a, b \in A$. The set of all skew derivations is a subalgebra of $\mathrm{Der}_F A$, denoted $\mathrm{SDer}_F A$. It is straigthforward to check that

$$\mathrm{IDer}_F A \subset \mathrm{SDer}_F A \quad \text{for } A \text{ alternative or Jordan, ch. } F \neq 2. \tag{1}$$

It is also easily seen that a graded derivation is skew if and only if every homogenous component is skew. Thus we have

$$\mathrm{grSDer}_F A := \mathrm{grDer}_F A \cap \mathrm{SDer}_F A = \bigoplus_{g\in G} (\mathrm{SDer}_F A)_g \tag{2}$$

where $(\mathrm{SDer}_F A)_g := \mathrm{SDer}_F A \cap (\mathrm{Der}_F A)_g$. Assume A is unital and let $Z = \bigoplus_{h\in H} Z_h$ be the centre of A. Then for any $h \in H$

$$\mathrm{CDer}_F A \cap (\mathrm{SDer}_F A)_h = \{\partial_\Theta \mid \Theta \in \mathrm{Hom}_{\mathbb{Z}}(G, Z_h), \Theta(h) = 0\}$$
$$\text{if } (.|.) \text{ is nondegenerate and } A \text{ is unital.} \tag{3}$$

Indeed, if $\Theta \in \mathrm{Hom}_{\mathbb{Z}}(G, Z_h)$ then $\partial_\Theta \in \mathrm{SDer}_F A$ if and only if for all $g \in G$ and $a_g \in A_g, b_{-g-h} \in A_{-g-h}$ we have $0 = (\Theta(g)a_g|b_{-g-h}) + (a_g|\Theta(-g-h)b_{-g-h}) = -(a_g b_{-g-h}|\Theta(h))$. Since A is unital and $(.|.)$ is nondegenerate this is equivalent to $\Theta(h) = 0$.

2. Division graded algebras

We specialize the set-up of the previous section 1 and consider a class of unital G-graded algebras which we assume to be alternative or Jordan. In the latter case we will suppose that ch. $F \neq 2$, unless explicitly stated otherwise. We begin with a result on reflection spaces which naturally arise as the support of division graded algebras.

2.1. Reflection spaces. A set M together with an operation $M \times M \to M : (m, n) \mapsto m \cdot n$ is a *reflection space* in the sense of Loos ([**19**]) if it satisfies the three axioms $m \cdot m = m$, $m \cdot (m \cdot n) = n$ and $m \cdot (n \cdot p) = (m \cdot n) \cdot (m \cdot p)$ for all $m, n, p \in M$. A *homomorphism* between reflection spaces is a map preserving the products.

Now let G be an abelian group. With respect to the operation $g \cdot h = 2g - h$, the set G becomes a reflection space. It has a distinguished base point, the zero element 0 of G. The following formulas hold in G:

$$0 \cdot g = -g \quad \text{and} \tag{1}$$

$$g_1 \cdot (g_2 \cdot (\cdots (g_{m-1} \cdot g_m) \cdots) = 2(g_1 - g_2 + g_3 - \cdots + (-1)^m g_{m-1}) + (-1)^{m+1} g_m \tag{2}$$

For a subset S of G we denote by $\langle S \rangle$ the subgroup spanned by S. The following conditions are equivalent for $S \subset G$:

(i) $0 \in S$ and $S \cdot S \subset S$;
(ii) $2\langle S \rangle \subset S$ and $2\langle S \rangle + S \subset S$;
(iii) S is a union of cosets modulo $2\langle S \rangle$ including the trivial coset $2\langle S \rangle$.

Indeed, if (i) holds we have $-S = S$ by (1) and hence every $g \in \langle S \rangle$ can be written in the form $g = s_1 - s_2 + s_3 - \cdots + (-1)^{n+1} s_n$ for suitable $s_i \in S$ and $n \geq 1$. For arbitrary $s \in S$ we then obtain $2g + s \in S$ from (2), thus $2\langle S \rangle + S \subset S$ and in particular $2\langle S \rangle \subset S$ since $0 \in S$. The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are immediate.

A subset S satisfying the equivalent conditions (i) – (iii) above will be called a *reflection subspace of* G. The special case of $S \subset G = (\mathbb{Z}^n, +)$ and $\langle S \rangle = \mathbb{Z}^n$ has been treated in [**1**, Ch.II §1] where S was called a *semilattice*. The description of reflection subspaces above is in fact a straightforward generalization of [**1**, Ch.II Prop. 1.4 and Remark 1.6].

2.2. Proposition. *Let $S \subset G$ be a reflection subspace.*

(a) *If G is finitely generated as abelian group, S is finitely generated as reflection space.*

(b) *Let H be an abelian group without 2-torsion. Then any homomorphism $\varphi\colon S \to H$ of reflection spaces preserving the base points uniquely extends to a group homomorphism $\Phi\colon \langle S \rangle \to H$.*

Proof. (a) Since a subgroup of a finitely generated group is again finitely generated we may assume $\langle S \rangle = G$. The quotient $G/2G$ is a finitely spanned $\mathbb{Z}/2\mathbb{Z}$-vector space, hence finite. It therefore follows from 2.1 that S is the union of a finite number of cosets modulo $2G$, say $\tau_i + 2G$, $0 \leq i \leq n$, where we can assume $\tau_0 = 0$.

We next note that S contains a finite generating set of the abelian group G. Indeed, by assumption, there exist finitely many elements $x_1, x_2, \dots, x_m \in G$ spanning G. But since $G = \langle S \rangle$ and $S = -S$, any x_i is a finite sum of elements of S, say $x_i = \sum_j \sigma_{ij}$ with $\sigma_{ij} \in S$. Hence the collection of all $\{\sigma_{ij}\}$ is a finite generating set of G contained in S.

Let $X = \{\sigma_{ij}\} \cup \{\tau_0, \dots \tau_n\}$ and denote by S' the reflection space generated by X. By 2.1(ii) applied to S', we have $\mathbb{Z}y \subset S'$ for any $y \in S'$. Hence, by choosing appropriate g_i in $\bigcup_i \mathbb{Z}\sigma_{ij} \cup \{\pm\tau_i\} \subset S'$, 2.1.2 shows $2g + \tau_i \in S'$ for every $g \in G$ and τ_i, $0 \le i \le n$, proving $S' = S$.

(b) We may again assume that $\langle S \rangle = G$. Since $\varphi(0) = 0$ by assumption we have $\varphi(-s) = \varphi(0 \cdot s) = 0 \cdot \varphi(s) = -\varphi(s)$ for every $s \in S$ and also $\varphi(2s) = \varphi(s \cdot 0) = 2\varphi(s)$. Now for any $g \in G$, $s \in S$ we obtain $\varphi(2s - 2g) = \varphi(s \cdot 2g) = \varphi(s) \cdot \varphi(2g) = 2\varphi(s) - \varphi(2g)$. Hence, if $\varphi(2g) \in 2H$, so is $\varphi(2(s - g))$. Since any $g \in G$ can be written in the form $g = s_1 - s_2 + s_3 - \dots + (-1)^{n+1} s_n$ it follows by induction on n that $\varphi(2g) \in 2H$. We define $\Phi\colon G \to H$ by $\Phi(g) = \frac{1}{2}\varphi(2g)$. Then Φ extends φ, and for $s \in S$ and $g \in G$ we have $2\Phi(s+g) = \varphi(2s + 2g) = \varphi(s \cdot (-2g)) = \varphi(s) \cdot \varphi(-2g) = 2\varphi(s) + \varphi(2g) = 2(\Phi(s) + \Phi(g))$, whence $\Phi(s + g) = \Phi(s) + \Phi(g)$. Since any $g \in G$ is a finite sum of elements in S, this implies, by induction, that Φ is a group homomorphism. □

2.3. Strongly graded and division graded algebras. *From now on we will assume that A is a unital alternative or a unital Jordan algebra over F. Unless specified otherwise we will assume that* ch. $F \neq 2$ *if A is Jordan.*

We will say that a G-graded algebra A is *strongly graded* or that A has a G-grading *of strong type* if $A_g A_h = A_{gh}$ for all $g, h \in G$. The algebra A is called a *division G-graded algebra* if all nonzero homogeneous elements are invertible. We list some known properties of division graded algebras.

(a) In any alternative algebra an element a is invertible if and only if the left multiplication L_a is invertible, and in this case $(L_a)^{-1} = L_{a^{-1}}$. Moreover, if a, b are invertible then so is their product ab, and then $(ab)^{-1} = b^{-1}a^{-1}$. This easily implies that a division G-graded alternative algebra is strongly graded and that its support is a subgroup of G ([**29**] or [**23**, I.4.5] for the case of associative algebras).

A division G-graded associative algebra, sometimes also called a graded division ring [**23**], is a crossed product algebra $D * G$ of G over an associative division algebra D, and conversely [**25**, 2]. The classification of division $\mathbb{Z}^n$-graded alternative algebras is given in [**29**].

(b) For a Jordan algebra A we denote by U_a the U-operator of a Jordan algebra, i.e., $U_a b = 2a(ab) - a^2 b$. Since A is supposed to have a unit element 1, one can recover the bilinear product from the U-operator in view of the formula $2ab = (U_{a+b} - U_a - U_b)1$. An element $a \in A$ is invertible if and only if U_a is invertible.

Let A be a division graded Jordan algebra and denote by $S = \operatorname{supp} A$ its support. Then $U_{A_g} A_h = A_{2g+h}$ for all $g, h \in S$. Hence *S is a reflection subspace of G*. We note that in general S is not a subgroup of G (see for example 4.4).

(c) ([**29**] or [**23**, I.4.2] for associative algebras) For a totally ordered abelian group G, e.g. $G = \mathbb{Z}^n$, any division G-graded algebra A is a domain in the sense that $xy = 0 \Rightarrow x = 0$ or $y = 0$ if A is alternative, and $U_x y = 0 \Rightarrow x = 0$

or $y = 0$ if A is Jordan. We recall that any domain is strongly prime, meaning nondegenerate and prime.

(d) The centre Z of a division G-graded algebra A is again division graded. In particular, Z_0 is a field and $Z = Z_0^t[H]$ is a twisted group algebra where H is the central grading group. Moreover, A is a division G-graded algebra over Z_0. If G is totally ordered, Z is an integral domain.

(e) Since A is graded-simple, any non-zero graded invariant form on A is nondegenerate. For the existence of such forms see 1.13, 3.3 and 4.9.

We next present some preliminary results on division graded algebras.

2.4. Lemma. *Let A be a division G-graded algebra such that*

(i) *G is finitely generated and*

(ii) *$\dim_F A_g < \infty$ for all $g \in G$.*

Then A is finitely generated as an algebra.

Proof. Let $X \subset S = \operatorname{supp} A$ be a finite generating set of the reflection space S and let Y be the union of vector space bases of all A_g, $g \in X$. The subalgebra A' generated by Y is graded and has support S, in view of 2.1 and (a), (b) of 2.3. By induction on the number of generators needed to express $s \in S$ as a product of factors from X one shows $A'_s = A_s$ for all $s \in S$, hence $A' = A$. □

2.5. Lemma. *If A is an alternative division graded algebra, then*

$$Z(A) = \{a \in A \mid [a, A] = 0\}. \tag{1}$$

Proof. It suffices to show that any homogeneous $a \in A$ satisfying $[a, A] = 0$ lies in the centre of A. It is proven in McCrimmon's unpublished book [**20**, III Lemma 4.1] that $(a, A, A)^2 = 0$ for all $a \in A$ with $[a, A] = 0$. In particular, for homogeneous $b, c \in A$ the equation $(a, b, c)^2 = 0$ forces $(a, b, c) = 0$. Since [**20**] is not published, we mention that for the special case of a prime alternative algebra over a field of characteristic $\neq 3$, the lemma follows from results in [**33**]. Indeed, for the proof of (1) we may assume that A is not associative. Then A is nondegenerate ([**33**, 9 Thm. 5']), hence its central closure $\overline{A}$ is a simple alternative algebra (proof of [**33**, 9 Thm.9]) and (1) holds for $\overline{A}$ by [**33**, 7 Cor. of Lemma 7 and 7, Cor. 1 of Lemma 1]. In particular, any $a \in A$ which commutes with A and hence with $\overline{A}$ associates with everything of $A \subset \overline{A}$ proving (1). □

2.6. Lemma. *Let A be a division G-graded algebra whose central closure $\overline{A}$ exists (for example this holds by 2.3(c) if G is ordered) and has only inner derivations. Then* $\operatorname{IDer} A = \operatorname{grDer}_Z A$.

Proof. By 1.6.5 we only need to prove $\operatorname{grDer}_Z A \subset \operatorname{IDer} A$. So let $0 \neq d \in \operatorname{grDer}_Z A$. We may assume that d is homogeneous. By 1.2.2, we have the derivation $\tilde{d} = \mathrm{Id} \otimes d$ of $\overline{A}$ which, by assumption and 1.4.2, has the form $\tilde{d} = \frac{1}{z}\Delta$ for some $0 \neq z \in Z$ and some $\Delta \in \operatorname{IDer} A$. Hence $z\tilde{d} = \Delta$ on $\overline{A}$ and so, by restriction, $zd = \Delta$ on A. Considering the homogeneous components of this equation we obtain a non-zero homogeneous $z' \in Z$ and a homogeneous $\Delta' \in \operatorname{IDer} A$ such that $z'd = \Delta'$. Since z' is invertible it follows that $d = \frac{1}{z'}\Delta' \in \operatorname{IDer} A$. □

2.7. Graded modules. Let $A = \oplus_{g\in G} A$ be an associative division G-graded algebra over F. Suppose that the group G acts on a set S. A left A-module M is called a *graded A-module of type S* if $M = \oplus_{s\in S} M_s$, and $A_g M_s \subset M_{g.s}$ for each $g \in G$ and $s \in S$. In this case, we let $\bar{s} := G.s \in S/G$ be the G-orbit of s and put $M_{\bar{s}} := \oplus_{g\in G} M_{g.s}$. Obviously, $M = \bigoplus_{\bar{s}\in S/G} M_{\bar{s}}$.

The following lemma is easy to prove. For related results see for example [**6**, Thm. 3], [**23**, I.3.4] and [**31**, Lemma 3.6].

2.8. Lemma. *Let $A = \oplus_{g\in G} A_g$ be an associative division G-graded algebra and $M = \oplus_{s\in S} M_s$ a graded A-module of type S. Then:*

(a) *$M_{g.s} = a_g M_s = A_g M_s$ for all $g \in G$, $s \in S$ and any $0 \neq a_g \in A_g$. Hence $AM_s = M_{\bar{s}}$.*

(b) *Suppose that G acts freely on S, i.e., $g.s = s$ for some $g \in G$ and $s \in S$ implies $g = 0$. Then $M_{\bar{s}}$ is a free A-module, namely any A_0-basis of the A_0-vector space M_s is also an A-basis of $M_{\bar{s}}$. Hence $\operatorname{rank}_A M_{\bar{s}} = \dim_{A_0} M_{s'}$ for all $s' \in \bar{s}$. Moreover, M is a free A-module, there exists an A-basis of M consisting of homogeneous elements and $M \cong A \otimes_{A_0} N$ for a suitable A_0-module N.*

(c) *Suppose that G and S are totally ordered and that the action preserves the order, i.e., $g < g'$ and $s < s'$ implies $g.s < g'.s'$. Then any graded submodule N of M is saturated in the sense that $N = \{m \in M \mid am \in N \text{ for some } 0 \neq a \in A\}$. Hence, for any multiplicatively closed subset B of A we have $B^{-1}M = B^{-1}N \iff M = N$.*

2.9. Proposition. *Let A be a division graded algebra with $G = \langle \operatorname{supp} A\rangle$, and $Z = \bigoplus_{h\in H} Z_h$ the centre of A with central grading group H. We choose $0 \neq z_h \in Z_h$ for all $h \in H$. Recall that $\mathcal{D}$ denotes the degree derivations as defined in* 1.8.5. *Then the maps μ and ∂ of* 1.9.4

$$W(Z, \operatorname{Hom}_{\mathbb{Z}}(G, Z_0), p) \xrightarrow{\mu} \operatorname{Hom}_{\mathbb{Z}}(G, Z)_{\mathrm{fin}} \xrightarrow{\partial} (\operatorname{CDer}_F A)_{\mathrm{fin}}$$

are isomorphisms of Lie algebras. Moreover,

$$\operatorname{Hom}_{\mathbb{Z}}(G, Z)_{\mathrm{fin}} = \bigoplus_{h\in H} z_h \operatorname{Hom}_{\mathbb{Z}}(G, Z_0) \cong Z \otimes_{Z_0} \operatorname{Hom}_{\mathbb{Z}}(G, Z_0), \quad (1)$$

$$(\operatorname{CDer}_F A)_{\mathrm{fin}} = \bigoplus_{h\in H} z_h \mathcal{D} = Z \otimes_{Z_0} \mathcal{D} \quad \text{and} \quad (2)$$

$$\operatorname{SDer}_F A \cap (\operatorname{CDer}_F A)_{\mathrm{fin}} = \bigoplus_{h\in H} z_h \{\partial_\Theta \in \mathcal{D} : \Theta(h) = 0\} \quad (3)$$

with respect to any nondegenerate graded invariant form on A.

Proof. (1) and (2) are immediate from 2.8, while (3) follows from 1.14.3. In particular, (1) implies that μ is an isomorphism. To show that ∂ is an isomorphism it suffices in view of 1.8.3 to prove injectivity. Assume therefore that $z_h \partial_\varphi = 0$ for $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(G, Z_0)$. Then $\varphi|_{\operatorname{supp} A} = 0$ follows and hence $\varphi = 0$ because of our assumption $G = \langle \operatorname{supp} A\rangle$. □

In the remainder of this section we will introduce division graded versions of the standard examples of Jordan algebras.

2.10. Proposition. *Let A be a division graded associative algebra over a field F of* ch. $F \neq 2$.

(a) *Any graded subalgebra J of A^+ is a division G-graded Jordan algebra as soon as $x^{-1} \in J$ for any non-zero homogeneous $x \in J$. In particular, A^+ itself is division G-graded.*

(b) *If A is associatively generated by a division graded subalgebra J of A^+ then*

$$Z(J) = Z(A) \cap J\,. \tag{1}$$

(c) *For $J = A^+$ we have*

$$\operatorname{Der}_F A = \operatorname{Der}_F A^+\,, \quad Z(A) = Z(A^+) \quad \textit{and} \tag{2}$$

$$\operatorname{CDer}_F A = \operatorname{CDer}_F A^+. \tag{3}$$

If A is semiprime (b) follows from [**22**, Thm. 3.3]. However, A is in general not semiprime. For example, even a group algebra over a field need not be semiprime (see [**18**, (10.17)]). Also, for an arbitrary prime algebra A the equality $\operatorname{Der}_F A = \operatorname{Der}_F A^+$ follows from [**13**, Thm. 3.3]. Although A is in general not prime, we can nevertheless use some of the methods of Herstein's proof. In particular, the crucial formula (4) below is taken from [**13**].

Proof. (a) follows from the fact that invertibility in A and A^+ are the same. For (b), the inclusion $Z(A) \cap J \subset Z(J)$ is obvious from 1.11.4. To prove the other inclusion we may assume that $0 \neq z \in Z(J)$ is a homogenous element. For any $x \in J$ we then have $[x,z]^2 = 2x.U_z x - U_x z^2 - U_z x^2 = 0$ since $z \in Z(J)$. If $z \notin Z(A)$ then $[x,z] \neq 0$ for some homogenous $x \in J$ and hence $[x,z]$ is invertible, contradicting $[x,z]^2 = 0$.

(c) It is clear that $\operatorname{Der} A \subset \operatorname{Der} A^+$. So, let $d \in \operatorname{Der} A^+$ and put $a^b = d(ab) - d(a)b - ad(b)$ for $a, b \in A$. By calculating $d((ab)ab + ba(ab))$ in two different way one obtains [**13**, Lemma 3.6]

$$a^b[a,b] = 0. \tag{4}$$

Linearizing in b gives

$$a^b[a,c] + a^c[a,b] = 0. \tag{5}$$

It suffices to prove $a^b = 0$ for all homogeneous $a, b \in A$. We will do so by distinguishing four cases.

Case 1) $a, b \in Z(A)$: Then $d(ab) = d(a.b) = d(a).b + a.d(b) = d(a)b + ad(b)$, i.e., $a^b = 0$. Case 2) $a \notin Z(A), b \in Z(A)$: By (5) we then have $a^b[a,c] = 0$ for all $c \in A$. Since a is homogenous there exists a homogeneous c such that $[a,c] \neq 0$ and hence $[a,c]$ invertible, forcing $a^b = 0$. Case 3) $a \in Z(A), b \notin Z(A)$: Since $a^b = -b^a$ this case follows from the previous one. Case 4) $a, b \notin Z(A)$: If $[a,b] \neq 0$ we obtain $a^b = 0$ from (4). Otherwise, we get $a^b[a,c] = 0$ from (5) for all $c \in A$, and again $a^b = 0$ follows.

Thus $\operatorname{Der}_F A = \operatorname{Der}_F A^+$. That $Z(A) = Z(A^+)$ follows from (1), and then $\operatorname{CDer}_F A = \operatorname{CDer}_F A^+$ by definition. □

2.11. Quadratic form Jordan algebras. Let Φ be an associative commutative ring, M a Φ-module and q: $M \to \Phi$ a quadratic form with base point, i.e., $q(1) = 1$ for some $1 \in M$. We let $q(x, y) = q(x + y) - q(x) - q(y)$ be the polar of q, and define $\overline{x} = q(x, 1)1 - x$ and $U_x y = q(x, \overline{y})x - q(x)\overline{y}$. Then M together with the quadratic operator U is a (quadratic) Jordan algebra denoted by J over Φ. We will often confuse J and M. It is well-known that any $x \in M$ with invertible $q(x)$ is invertible in J. Indeed, its inverse is $x^{-1} = \frac{1}{q(x)}\overline{x}$.

Suppose Φ is G-graded, M is a graded module of type G and q: $M \to \Phi$ is *graded* in the sense that $q(M_g) \subset \Phi_{2g}$, $q(M_g, M_h) \subset \Phi_{g+h}$ for $g, h \in G$ and $1 \in M_0$. The corresponding Jordan algebra J is then G-graded, i.e., $U_{J_g} J_h \subset J_{2g+h}$ and $\{J_g\, J_h\, J_i\} \subset J_{g+h+i}$ where $\{...\}$ is the Jordan triple product of J. Moreover, J is division G-graded if q is *anisotropic graded* in the sense that $q(m_g)$ is invertible for any $0 \neq m_g \in M_g = J_g$.

If $\frac{1}{2} \in \Phi$ we have $J = \Phi.1 \oplus V$ for $V = \{x \in M \mid q(x, 1) = 0\}$ and with respect to this decomposition the bilinear product of J satisfies

$$(s_1 1 \oplus v_1)(s_2 1 \oplus v_2) = (s_1 s_2 + f(v_1, v_2))1 \oplus (s_1 v_2 + s_2 v_1) \tag{1}$$

for $f = -\frac{1}{2} q(.,.)|_{V \times V}$. In this case, the associator of $x_i = s_i.1 \oplus v_i$ is, with obvious notation,

$$(s_1 \oplus v_1, s_2 \oplus v_2, s_3 \oplus v_3) = f(v_1, v_2)v_3 - f(v_3, v_2)v_1 \in V. \tag{2}$$

Hence $\Phi.1 \subset Z(J)$ and $(J, J, J) \subset V$. We note that J always has a non-zero invariant form, namely $(x|y) = q(xy, 1)$ is such a form. But in general this form is not graded. We now describe special classes of quadratic form algebras.

2.12. Lemma. *Let J be a quadratic form algebra over a G-graded Φ containing $\frac{1}{2}$, and decompose the underlying module $M = \Phi.1 \oplus V$ as in* 2.11 *above. Suppose that V is an orthogonal sum of non-zero G-graded Φ-modules $V^{(i)}$, $i \in I$ with $|I| \geq 2$ such that each $q|V^{(i)}$ is anisotropic graded (hence the type S_i of $V^{(i)}$ is contained in G). We extend the grading of V to M in the canonical way, so that J is a G-graded algebra.*

(a) *Then $Z(J) = \Phi.1$ and $(J, J, J) = V$, hence $J = Z(J) \oplus (J, J, J)$.*

(b) *Let $\mathfrak{o}(f) = \{d \in \mathrm{End}_\Phi V \mid f(du, v) + f(u, dv) = 0$ for all $u, v \in V\}$ be the orthogonal Lie algebra associated to f and let $\mathfrak{eo}(f)$ be the ideal spanned by all elementary orthogonal transformations $\mathbb{E}_{u,v}$, $u, v \in V$, given by $\mathbb{E}_{u,v}(w) = f(u, w)v - f(v, w)u = (u, w, v) = \Delta_{v,u}(w)$. Then*

$$\mathrm{Der}_\Phi J \to \mathfrak{o}(f) : d \mapsto d|_V \tag{1}$$

is an isomorphism of Lie algebras mapping $\mathrm{IDer}\, J$ *onto $\mathfrak{eo}(f)$.*

(c) *Suppose Φ is division graded, so that by* 2.8 *every $V^{(i)}$ is free and has a homogenous Φ-basis. Assume that V has finite rank with a homogeneous Φ-basis $\{v_1, \ldots, v_n\}$ satisfying $f(v_i, v_j) = 0$ for $i \neq j$. Then $\mathfrak{o}(f) = \mathfrak{eo}(f)$, and hence all Φ-linear derivations are inner. Moreover, $\{\mathbb{E}_{v_i,v_j} \mid 1 \leq i < j \leq n\}$ is a Φ-basis of $\mathfrak{o}(f)$.*

Proof. At least in special cases, this result is known to the experts. For the convenience of the reader we include a short proof.

(a) Let $z \in Z(J) \cap V$. By 2.11.2 we then have $0 = (z,u,v) = f(z,u)v$ for any u, v in different submodules $V^{(i)}$. Considering homogenous components then forces $z = 0$. Hence $Z(J) = \Phi 1$. If u, v belong to the same $V^{(i)}$ but w to a different $V^{(j)}$ we get $(u,v,w) = f(u,v)w$. Since $f(u,v)$ for suitable choices of u, v is invertible, this shows $w \in (V,V,V)$, and then $V = (J,J,J)$ follows.

(b) Any Φ-linear derivation d annihilates $1 \in J$ and leaves $V = (J,J,J)$ invariant. That $d|_V \in \mathfrak{o}(f)$ is immediate from the product formula 2.11.1. Thus the map (1) is well-defined and injective. That it is also surjective is an easy exercise. In case Φ is a field (c) is well-known. In the setting of this lemma it is proven in [**8**].

2.13. Construction of a division graded quadratic form Jordan algebra. For easier reference we describe a special case of the situation considered in 2.12. This setting will be specialized in 4.5 to define Clifford tori. Our construction uses the following data:

(i) Φ is a division graded commutative associative ring with grading group $H = \operatorname{supp}\Phi$ and $\frac{1}{2} \in \Phi_0$;
(ii) H is a subgroup of an abelian group G such that G/H is a 2-group, i.e., any element of G/H has order ≤ 2;
(iii) $I \subset G$ is a set of representatives of some cosets of G modulo H *different* from H, with at least two elements;
(iv) $(z_i)_{i\in I}$ is a family of non-zero (hence invertible) elements $z_i \in \Phi_{2i}$.

Assuming these data, we put $z_0 = 1 \in \Phi_0$ and let $V^{(i)}$, $i \in I \cup \{0\}$, be the graded Φ-module of type $i + H$ and rank 1 with basis $v_i \in V_i^{(i)} = \Phi_0 v_i$ where $v_0 = z_0 = 1 \in \Phi_0$. Thus

$$V^{(i)} = \textstyle\bigoplus_{h\in H} V_{i+h}^{(i)} \quad \text{with} \quad V_{i+h}^{(i)} = \Phi_h v_i.$$

For $i \in I\cup\{0\}$ we let $q^{(i)}\colon V^{(i)} \to \Phi$ be the Φ-quadratic form given by $q^{(i)}(v_i) = z_i \in \Phi_{2i}$. We then have a G-graded Φ-module $M = \bigoplus_{i\in I\cup\{0\}} V^{(i)} = \bigoplus_{g\in G} M_g$ where

$$M_g = \begin{cases} V_{i+h}^{(i)} & \text{if } g = i + h \text{ for } i \in I \cup \{0\}, h \in H \\ 0 & \text{otherwise} \end{cases}$$

with an anisotropic graded quadratic form $q = \bigoplus_{i\in I\cup\{0\}} q^{(i)}$, the orthogonal sum of the quadratic forms $q^{(i)}$. Hence the corresponding quadratic form Jordan algebra J, considered as algebra over $F = \Phi_0$, is division G-graded with $\operatorname{supp} J = I + H$. Moreover, Lemma 2.12 applies. In particular, it follows from 1.13.2 and 2.12(a) that

$$\dim_F \mathcal{GIF}(J) = 1. \tag{1}$$

Since by construction $\Delta_{v_i,v_j} \in (\operatorname{IDer} J)_{i+j}$ we obtain, using 2.8, the following more precise description of the inner derivation algebra.

2.14. Lemma. *In the setting of* 2.13 *suppose I is finite, and let $<$ be a well-ordering on I. Also, denote by $\overline{g}$ the canonical image of g in G/H.*

Then $\{\Delta_{v_i,v_j} \mid i,j \in I, i < j\}$ is a Φ-basis of $\operatorname{IDer} J = \operatorname{Der}_\Phi J$, and $\{\Delta_{v_i,v_j} : i,j \in I, i < j, \overline{i} + \overline{j} = \overline{g}\}$ is a Φ-basis of $\bigoplus_{h\in H} (\operatorname{IDer} J)_{g+h} = (\operatorname{IDer} J)_{\overline{g}}$. In particular,

$$\dim_F (\operatorname{IDer} J)_g = |\{\{i,j\} : i,j \in I, i \neq j, \overline{i} + \overline{j} = \overline{g}\}| \tag{1}$$

In a special case, these dimensions have been calculated in [**27**, Lemma 2.4].

2.15. Graded Albert algebras. Let A be a prime associative algebra over F whose central closure $\overline{A}$ has (generic) degree 3, and let $\mu \in Z = Z(A)$ be invertible. Assume that $\mathrm{tr}(A) \subset Z$ where tr is the generic trace of $\overline{A}$. Then, the subset $J = (A, \mu) := A \oplus A \oplus A$ of the first Tits construction $(\overline{A}, \mu) = \overline{A} \oplus \overline{A} \oplus \overline{A}$ is a Z-subalgebra such that $\overline{(A, \mu)} = (\overline{A}, \mu)$ ([**31**, Lemma 6.5]). Hence J and $\overline{J}$ are Albert algebras in the sense of [**21**]. Since $\overline{J} = (\overline{A}, \mu)$ has degree 3, Schafer's theorem [**14**, Exercise 1, page 319] implies $\overline{J} = \overline{Z} \oplus (\overline{J}, \overline{J}, \overline{J})$ if ch. $F \neq 3$.

To obtain a grading on J we assume the following data:

(i) $A = \oplus_{g \in G} A_g$ is a G-graded prime associative algebra with centre Z;
(ii) $|G/S| = 3$ where $S = \mathrm{supp}\, A$;
(iii) $\mu \in Z_{3g_0}$ for some $g_0 \in G \setminus S$;
(iv) tr is G-graded, i.e., $\mathrm{tr}(A_g) \subset Z_g$ for all $g \in G$.

Then $J = (A, \mu)$ is G-graded with homogeneous spaces defined as follows:

$$J_g := \begin{cases} A_g \oplus 0 \oplus 0 & \text{if } g \in S \\ 0 \oplus A_{g-g_0} \oplus 0 & \text{if } g - g_0 \in S \\ 0 \oplus 0 \oplus A_{g+g_0} & \text{if } g + g_0 \in S. \end{cases}$$

Moreover, J is division graded if A is so. If, in addition, G is totally ordered and ch. $F \neq 3$ then $J = Z \oplus (J, J, J)$. Indeed, since $(\overline{J}, \overline{J}, \overline{J}) = \overline{Z} \otimes_Z (J, J, J)$ this follows from Schafer's Theorem and 2.8(c) applied to the Z-modules $Z + (J, J, J)$ and $Z \cap (J, J, J)$.

3. G-tori

In this section we will introduce a special class of division graded algebras, so-called G-tori. Our main interest will be the study of their derivation algebras. The basic assumptions of the previous sections remain in place: we consider unital alternative and Jordan algebras over a field F, which in case of Jordan algebras will be assumed to have ch. $F \neq 2$.

3.1. G-tori. Let G be an abelian group. A division G-graded algebra $T = \bigoplus_{g \in G} T_g$ over F is called a *G-torus* if

(i) $\dim_F T_g \leq 1$ for all $g \in G$, and
(ii) $\mathrm{supp}\, T$ generates G as abelian group.

If the G-torus is associative, alternative or Jordan, it is called an *associative*, an *alternative* or a *Jordan* G-torus. We will frequently identify $T_0 = F$. We say that a G-torus is of *strong type* if T is strongly graded. In this case, the support $\mathrm{supp}\, T$ is a subgroup of G. By 2.3.(a), any alternative G-torus is of strong type. An associative G-torus is the same as a twisted group algebra $F^t[G]$. In particular, if T is a G-torus and H its central grading group, the centre $Z(T)$ is a twisted group algebra of H. An example of a Jordan G-torus is the quadratic form Jordan algebra of 2.13, viewed as algebra over $F = \Phi_0$.

3.2. Theorem. *Let T be a G-torus with central grading group H. Recall that $\mathcal{D}$ denotes the degree derivations of T, see* 1.8.5. *Then:*

$$(\mathrm{Der}_F T)_0 = \mathcal{D}\,, \tag{1}$$

$$(\mathrm{CDer}_F T)_{\mathrm{fin}} = \bigoplus_{h\in H} (\mathrm{Der}_F T)_h = (\mathrm{grDer}_F T)_H\,, \tag{2}$$

$$\mathrm{IDer}\, T \subset \bigoplus_{g\notin H} (\mathrm{Der}_F T)_g\,, \tag{3}$$

$$(\mathrm{CDer}_F T)_{\mathrm{fin}} \cap \mathrm{IDer}\, T = 0\,. \tag{4}$$

If G is finitely generated then so is T, and hence $\mathrm{Der}_F T = \mathrm{grDer}_F T$. *In this case we also have* $\mathrm{CDer}_F T = (\mathrm{CDer}_F T)_{\mathrm{fin}}$ *which is a generalized Witt algebra by* 2.9.

Proof. Let $d \in (\mathrm{Der}_F T)_0$, and define scalars φ_g for $g \in S = \mathrm{supp}\, T$ by $d(t_g) = \varphi_g t_g$, $t_g \in T_g$. The fact that d is a derivation means

$$\varphi_g + \varphi_h = \varphi_{g+h} \quad \text{for } g,h \in S \text{ with } T_g T_h \neq 0. \tag{5}$$

In particular, if T is alternative, it is a torus of strong type with $S = G$ and we have $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(G, Z_0)$ which means that d is a degree derivation. For a Jordan torus T we have $dU_x = U_{dx,x} + U_x d$ for $x \in T$ which implies

$$\varphi_{2g+h} = 2\varphi_g + \varphi_h \quad \text{for } g,h \in S. \tag{6}$$

Since $\varphi_0 = 0$ it follows from (5) that $\varphi_{-h} = -\varphi_h$ and then (6) shows that $\varphi : S \to F$ is a homomorphism of reflection spaces preserving the base points. By 2.2.b, φ extends uniquely to a group homomorphism Φ: $G \to F$ and so d is a degree derivation also in this case. This implies (1) and then (2) is immediate from 2.9.2 and the definition of $(\mathrm{grDer}_F T)_H$.

(3): Since $\mathrm{IDer}\, T$ is G-graded, it suffices to show that $\mathrm{IDer}\, T \cap (\mathrm{Der}_F T)_h = 0$ for all $h \in H$. Suppose there exists a non-zero $d \in (\mathrm{Der}_F T)_h$ for some $h \in H$. If T is Jordan we can assume that $d = [L_x, L_y]$ where $x \in T_g$, $y \in T_{g'}$ and $g + g' = h$. Hence $y = zx^{-1}$ for some non-zero $z \in Z_h$. But then $[L_x, L_y] = [L_x, L_{zx^{-1}}] = L_z[L_x, L_x^{-1}] = 0$, contradiction. The proof for alternative algebras is similar, using the the form of inner derivations (see 1.4). Thus (3) holds, and this implies (4). The last statement follows from 2.4, 1.6.2 and 1.7.4. □

Remark. We have

$$\mathrm{grDer}_F T = \mathrm{IDer}\, T \rtimes (\mathrm{CDer}_F T)_{\mathrm{fin}}$$
$$\textit{as soon as any } Z\textit{-linear derivation is inner.} \tag{7}$$

Indeed, by the result above we only have to show $(\mathrm{Der}_F T)_g \subset \mathrm{IDer}\, T$ for $g \notin H$. But any $d \in (\mathrm{Der}_F T)_g$ has $d(Z) \subset Z \cap \oplus_{g'\in g+H} (\mathrm{Der}_F T)_{g'} = 0$, so is Z-linear.

By 2.12 the criterion (7) is fulfilled for the G-torus of 2.13 with a finite I. This result is generalized in [**8**]. Other examples of G-tori satisfying (7) will be discussed in 3.5. In the next section, we will use this criterion to establish $\mathrm{Der}_F T = \mathrm{IDer}\, T \rtimes \mathrm{CDer}_F T$ for Jordan $\mathbb{Z}^n$-tori, even if $\mathrm{IDer}\, T = \mathrm{Der}_Z T$ is not always true there.

3.3. Proposition. *Let T be an alternative G-torus with central grading group H. Then*

$$(T,T,T) \subset [T,T] = \bigoplus_{g\notin H} T_g \tag{1}$$

hence

$$T = Z(T) \oplus [T,T] \quad \textit{and} \quad \mathcal{GIF}(T) \cong T_0^* \tag{2}$$

is 1*-dimensional. If T is associative, we have*

$$\operatorname{IDer}_F T = \operatorname{IDer}_F T^+ \cong \bigoplus_{g\notin H} T_g. \tag{3}$$

Proof. For easier notation we put $B = \bigoplus_{g\notin H} T_g$. We will first prove that $(T,T,T) \subset B$. Observe that (T,T,T) is G-graded and invariant under $Z = Z(T)$. Hence, either $(T,T,T) \subset B$ or there exist homogeneous $a,b,c \in T$ such that $1 = (a,b,c)$. Then $(ab)^{-1}$ and c have the same degree, and therefore $c = z(ab)^{-1} = zb^{-1}a^{-1}$ for some non-zero $z \in T_0 = Z_0$ by 2.8.a. But then we obtain the contradiction $1 = z(ab)(ab)^{-1} - za(b(b^{-1}a^{-1})) = z - z = 0$. Similarly, if $[T,T]$ is not contained in B we have $1 = [a,b]$ for suitable homogeneous $a,b \in T$ where $b = z'a^{-1}$ for some $z' \in Z_0$. Then $1 = [a,a^{-1}]z' = 0$ gives a contradiction.

For (1) it remains to prove $B \subset [T,T]$. By 2.5, any non-zero $a \in B$ satisfies $[a,T] \neq 0$. It is of course not harmful to assume that a is homogeneous. Then there exists a homogeneous $b \in T$ such that $0 \neq ab - ba = ab - (ba)(b^{-1}b) = ab - (b(ab^{-1}))b$ (by the Moufang identity) $= (a - b(ab^{-1}))b$. Since b is invertible, it follows that $0 \neq a - b(ab^{-1}) = za$ for some $0 \neq z \in Z_0$. But then $[ab^{-1}, z^{-1}b] = z^{-1}((ab^{-1})b - b(ab^{-1})) = z^{-1}(za) = a \in [T,T]$ proving our result.

The first equation in (2) is obvious, and the second follows from 1.13.2. For an associative T we always have $\operatorname{IDer} T = \operatorname{ad} T \cong T/Z(T)$ and $\operatorname{IDer} T^+ = \operatorname{ad}[T,T] \cong T/(Z(T) \cap [T,T])$. □

3.4. Corollary. *Let A be an associative G-torus over a field F of* ch. $F \neq 2$ *and let $*$ be a graded F-linear involution of A such that $J := H(A,*)$ is a generating set of the associative algebra A. Denote by H the central grading group of A. Then there is a graded isomorphism of Lie algebras*

$$\operatorname{IDer} J \cong [J,J] \;=\; \bigoplus_{g\in G\setminus(H\cup \operatorname{supp} J)} A_g \quad \textit{(graded isomorphism)} \tag{1}$$

and hence for all $g \in G$

$$\dim_F (\operatorname{IDer} J)_g = \begin{cases} 0 & \textit{if } g \in H \cup \operatorname{supp} J \\ 1 & \textit{otherwise.} \end{cases} \tag{2}$$

Proof. Because of 1.4.3 we have an epimorphism

$$[J,J] \to \operatorname{IDer} J : x \mapsto \operatorname{ad} x|_J \tag{3}$$

of graded Lie algebras. Its kernel consists of all $x \in [J,J]$ which commute with J and hence also with A. But then $x \in Z(A)$ so that 3.3.2 shows $x = 0$. Hence (3) is an isomorphism. For the proof of the second part of (1) let $S(A,*) = \{a \in A \mid a^* = -a\}$. Then $A = J \oplus S(A,*)$ and $[J,J] \subset S(A,*) =$

$\bigoplus_{g\notin \operatorname{supp} J} A_g$ follows. By 3.3.2 we also have $[J,J]\subset \bigoplus_{g\notin H} A_g$. Conversely, for $g\notin (H\cup \operatorname{supp} J)$ and $0\neq y\in A_g$ we get $[y,x]\neq 0$ for some $x\in J_j=A_j$ because $[y,J]=0$ would imply $y\in Z(A)=\bigoplus_{h\in H} A_h$. Observe $[y,x]\in J$ and hence $g+j\in \operatorname{supp} J$. Because $\operatorname{supp} J$ is a reflection subspace of G we then obtain $g-j=0\cdot(j\cdot(g+j))\in \operatorname{supp} J$ (of course, this can also be seen directly). But $[y,x]\neq 0$ implies $[yx^{-1},x]\neq 0$, so $yx^{-1}\in J_{g-j}$ and $0\neq [yx^{-1},x]\in A_g\subset [J,J]$ follows. Finally, (2) follows immediately from (1). □

3.5. Proposition. *Let T be an associative G-torus where G is finitely generated and let H be the central grading group.*

(a) *(Osborn-Passman* [**24**, Cor. 2.3]*) The derivation algebra of T is a semidirect product*

$$\operatorname{Der}_F T=\operatorname{IDer} T\rtimes \operatorname{CDer} T \tag{1}$$

where

$$\operatorname{CDer}_F T=(\operatorname{CDer}_F T)_{\text{fin}}=(\operatorname{grDer}_F T)_H=\bigoplus\nolimits_{h\in H}(\operatorname{Der}_F T)_h \quad \textit{and}$$
$$\operatorname{IDer} T=\bigoplus\nolimits_{g\notin H}(\operatorname{Der}_F T)_g\cong \bigoplus\nolimits_{g\notin H} T_g\,.$$

(b) *For the associated Jordan algebra T^+ we have $Z(T)=Z(T^+)$,*

$$\operatorname{CDer}_F T^+=\operatorname{CDer}_F T \quad \textit{and} \quad \operatorname{IDer} T^+=\operatorname{IDer} T,$$

and hence also $\operatorname{Der}_F T^+=\operatorname{IDer} T^+\rtimes \operatorname{CDer} T^+$ is a semidirect product.

Proof. (a) (1) is proven in [**24**, loc. cit.]. The remaining equations follow from 1.7.4, 3.2.2 and 3.2.3.

(b) is a corollary to (a), 2.10.2 and 3.3.3.

3.6. Proposition. *Let T be an associative G-torus over a field E with* ch. $E\neq 2$ *and let H be the central grading group of T. Let $J=\bigoplus_{g\in G} J_g\subset T^+$ be a Jordan G-torus over a subfield F of E, and assume that J is a generating set of the associative algebra T. Then $Z(J)=Z(T)\cap J=\bigoplus_{h\in H} J_h$ and $(J,J,J)=\bigoplus_{h\notin H} J_h$. In particular*

$$J=Z(J)\oplus (J,J,J) \quad \textit{and} \quad \mathcal{GIF}(J)\cong J_0.$$

Proof. By 3.3 we have $\bigoplus_{g\notin H} T_g=[T,T]=[T,Z(T)+[T,T]]=[T,[T,T]]$. Because of 1.11.4 this implies $(J,J,J)\subset J\cap [T,T]=\bigoplus_{g\notin H} J_g$ by . For the other inclusion let $0\neq x\in J_g$, $g\notin H$. Then $x\notin Z(T)$ and there exists a homogeneous $y\in J$ such that $[x,y]\neq 0$. Hence, $yx=zxy$ for some $0\neq z\in E$. Then $(x,y,y^{-1})=\frac{1}{4}(2+z+z^{-1})x\in J_g$, i.e., $(x,y,y^{-1})=tx$ for some non-zero $t\in F$, whence $x\in (J,J,J)$. □

3.7. Corollary. *Let A be an associative G-torus over a field E of* ch. $E\neq 2$. *Suppose that G is finitely generated and that $*$ is a graded involution of second kind. With the notation of* 1.11 *we then have for the Jordan G-torus $H=H(A,*)$ over F:*

(a) $H=Z(H)\oplus (H,H,H)$,

(b) $\operatorname{Der}_F H=\operatorname{IDer} H\rtimes \operatorname{CDer} H$.

Proof. (a) is a special case of 3.6. For (b) we have $\operatorname{Der}_E A^+=\operatorname{IDer} A^+\rtimes \operatorname{CDer} A^+$ by 3.5, and both summands are left invariant under the automorphism $d\mapsto d^*$ of $\operatorname{Der}_E A^+$. By 1.11.9 and 1.11.7 the fixed point spaces under $*$ are the corresponding subalgebras for H, whence the result. □

4. $\mathbb{Z}^n$-tori

4.1. Tori. A $\mathbb{Z}^n$-torus will be called an *n-torus* or simply a *torus*. Hence an n-torus is a division $\mathbb{Z}^n$-graded algebra $T = \oplus_{\alpha\in\mathbb{Z}^n} T_\alpha$ such that $\dim_F T_\alpha \le 1$ for all $\alpha \in \mathbb{Z}^n$ and $\operatorname{supp} T$ generates $\mathbb{Z}^n$. The reader is reminded that all algebras considered here are alternative or Jordan over a field F of ch. $F \ne 2$ if T is Jordan. In the following the notion "torus" will always mean an alternative or a Jordan torus.

Let $T = \oplus_{\alpha\in\mathbb{Z}^n} T_\alpha$ and $T' = \oplus_{\alpha\in\mathbb{Z}^n} T'_\alpha$ be two n-tori. We will say that T and T' are *graded isomorphic*, abbreviated as $T \cong_{\mathbb{Z}^n} T'$, if there exists an isomorphism $\varphi : T \to T'$ such that $\varphi(T_\alpha) = T'_\alpha$ for all $\alpha \in \mathbb{Z}^n$.

We recall from 2.3 that $\operatorname{supp} T = \mathbb{Z}^n$ if T is alternative and that $\operatorname{supp} T$ is a semilattice in $\mathbb{Z}^n$ if T is Jordan. Also, by 2.4,

$$\textit{any torus is finitely generated,} \tag{1}$$

and any torus is a domain, and hence strongly prime (2.3.(c)).

The structure of alternative tori has been determined up to graded isomorphism in [**5**, Thm. 1.25] and in improved form in [**29**, 4.11]. Besides the associative tori, see 4.2 below, there is one more type, the so-called Cayley torus in ch. $F \ne 2$, see 4.3.1.

4.2. Associative tori. An $n \times n$ matrix $q = (q_{ij})$ over a field F satisfying $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ is called a *quantum matrix*. For a quantum matrix q the associated *quantum torus* $F_q = F_q[t_1^{\pm1}, \dots, t_n^{\pm1}]$ is the associative algebra over F with $2n$ generators $t_1^{\pm1}, \dots, t_n^{\pm1}$, and relations $t_i t_i^{-1} = t_i^{-1} t_i = 1$ and $t_j t_i = q_{ij} t_i t_j$ for all $1 \le i, j \le n$. Note that F_q is commutative if and only if $q = \mathbf{1}$ where $\mathbf{1}$ is the quantum matrix whose entries are all 1. In this case, the quantum torus $F_{\mathbf{1}}$ becomes the algebra of Laurent polynomials $F[t_1^{\pm1}, \dots, t_n^{\pm1}]$ in n variables.

Let $\langle \sigma_1, \dots, \sigma_n \rangle$ be a basis of $\mathbb{Z}^n$, and define the degree of $t_\alpha := t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, where $\alpha = \alpha_1\sigma_1 + \cdots + \alpha_n\sigma_n \in \mathbb{Z}^n$, to be α. Then $F_q = \oplus_{\alpha\in\mathbb{Z}^n} F t_\alpha$ is an n-torus. We call this grading a *toral $\mathbb{Z}^n$-grading of F_q*, or, if one needs to specify a basis of $\mathbb{Z}^n$, a *$\langle \sigma_1, \dots, \sigma_n \rangle$-grading of F_q*.

It is shown in [**5**, 1.8] that any associative torus is graded isomorphic to some F_q with a suitable toral grading. Any commutative associative torus is graded isomorphic to a Laurent polynomial ring $F_{\mathbf{1}} = F[t_1^{\pm1}, \dots, t_n^{\pm1}]$.

4.3. Alternative tori. Suppose ch. $F \ne 2$. For $n \ge 3$ the *Cayley n-torus*

$$\mathbb{O}_t = (F[t_1^{\pm1}, \dots, t_n^{\pm1}], t_1, t_2, t_3)$$

is the F-algebra obtained by the Cayley-Dickson process over the Laurent polynomial ring $F[t_1^{\pm1}, \dots, t_n^{\pm1}]$, using the structure constants t_1, t_2 and t_3. This becomes an alternative torus with $\operatorname{supp} \mathbb{O}_t = \mathbb{Z}^n$ and centre $F[t_1^{\pm1}, \dots, t_n^{\pm1}]$ graded by

$$2\mathbb{Z}\sigma_1 + 2\mathbb{Z}\sigma_2 + 2\mathbb{Z}\sigma_3 + \mathbb{Z}\sigma_4 + \cdots + \mathbb{Z}\sigma_n, \tag{1}$$

for a basis $\langle \sigma_1, \dots, \sigma_n \rangle$ of $\mathbb{Z}^n$ with $\deg t_i = 2\sigma_i$ for $i = 1, 2, 3$ and $\deg t_i = \sigma_i$ for $i = 4, \dots, n$.

We will next describe Jordan tori. For details, see [**31**].

4.4. Jordan tori of hermitian type. Let $\varepsilon = (\varepsilon_{ij})$ be an elementary quantum matrix, i.e., a quantum matrix with all $\varepsilon_{ij} = 1$ or -1. On the quantum torus $F_\varepsilon = F_\varepsilon[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, there exists a unique involution $*$ such that $t_i^* = t_i$ for all i. The symmetric elements $H(F_\varepsilon, *)$ form a Jordan torus with respect to the grading induced from a toral grading of F_ε. One can show that $\operatorname{supp} H(F_\varepsilon, *) = \mathbb{Z}^n$ if and only if $q = \mathbf{1}$ if and only if $H(F_\varepsilon, *)$ is of strong type. If $q \neq \mathbf{1}$, the central grading group of $H(F_\varepsilon, *)$ is

$$2\mathbb{Z}\sigma_1 + \cdots + 2\mathbb{Z}\sigma_m + \mathbb{Z}\sigma_{m+1} + \cdots + \mathbb{Z}\sigma_n, \tag{1}$$

for a suitable toral $\langle \sigma_1, \dots, \sigma_n \rangle$-grading of F_ε and $2 \leq m \leq n$ ([**30**, 2.5] and 2.10.1), while by [**30**, 1.8] the central grading group of F_ε is always of type

$$2\mathbb{Z}\sigma_1 + \cdots + 2\mathbb{Z}\sigma_{2l} + \mathbb{Z}\sigma_{2l+1} + \cdots + \mathbb{Z}\sigma_n. \tag{2}$$

Let E be a quadratic field extension of F. Let σ_E be the non-trivial Galois automorphism of E over F (recall that we assume ch. $F \neq 2$ for Jordan tori). Let $\xi = (\xi_{ij})$ be a quantum matrix over E such that $\sigma_E(\xi_{ij})\xi_{ij} = 1$, or, equivalently, $\sigma_E(\xi_{ij}) = \xi_{ji}$ for all i, j. On the quantum torus $E_\xi = E_\xi[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over E, there exists a unique σ_E-semilinear involution σ such that $\sigma(t_i) = t_i$ for all i. The symmetric elements $H(E_\xi, \sigma)$ form a Jordan torus over F with respect to the grading induced from a toral grading of E_ξ. We always have $\operatorname{supp} H(E_\xi, \sigma) = \mathbb{Z}^n$, and the central grading groups of $H(E_\xi, \sigma)$ and E_ξ coincide.

4.5. Jordan tori of Clifford type. Let $2 \leq m \leq n$ and let $S^{(m)}$ be a semilattice in $\mathbb{Z}^m$. We consider the following data:

(i) Φ is the Laurent polynomial ring $\Phi = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ which we view as a division $\mathbb{Z}^n$-graded algebra of $\operatorname{supp} \Phi = 2\mathbb{Z}^m \oplus \mathbb{Z}^{n-m}$ with respect to the canonical grading assigning t_i the degree $(0, \dots, 0, 2, 0, \dots, 0)$ with 2 in the ith component in case $1 \leq i \leq m$ and degree $(0, \dots, 0, 1, 0, \dots, 0)$ for $i > m$;

(ii) $I \subset \mathbb{Z}^m \subset \mathbb{Z}^m \oplus \mathbb{Z}^{n-m} = \mathbb{Z}^n$ is a set of representatives of $S^{(m)}/2\mathbb{Z}^m$, excluding the class $2\mathbb{Z}^m$;

(iii) $(z_i)_{i \in I}$ is a family of non-zero elements in Φ_{2i}.

The *Clifford torus* $J(S^{(m)}, (z_i)_{i \in I})$ is the quadratic form Jordan algebra constructed in 2.13 using the data above. In particular, the results mentioned in 2.12, 2.13 and 2.14 hold.

4.6. The Albert torus. Let $n \geq 3$. We assume that F contains a primitive 3rd root of unity ω, in particular ch. $F \neq 3$, and denote by w the quantum $n \times n$-matrix with $(1,2)$-entry equal to ω, $(2,1)$-entry equal to ω^{-1} and all other entries equal to 1.

Let $F_w = F_w[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ be the quantum torus determined by w, and let $Z = Z(F_w)$ be the centre of F_w. One finds $Z = F[u_1^{\pm 3}, u_2^{\pm 3}, u_3^{\pm 1}, \dots, u_n^{\pm 1}]$, the algebra of Laurent polynomials in the variables $u_1^3, u_2^3, u_3, \dots, u_n$. Let $\langle \sigma_1, \dots, \sigma_n \rangle$ be a basis of $\mathbb{Z}^n$ and put

$$S := \mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2 + 3\mathbb{Z}\sigma_3 + \mathbb{Z}\sigma_4 + \cdots + \mathbb{Z}\sigma_n.$$

We give F_w a toral S-grading, i.e., $F_w = \oplus_{\sigma \in S} Fu_\sigma$ where

$$u_\sigma = u_1^{m_1} \cdots u_n^{m_n} \text{ for } \sigma = m_1\sigma_1 + m_2\sigma_2 + 3m_3\sigma_3 + m_4\sigma_4 + \cdots + m_n\sigma_n.$$

One knows ([**31**]) that $A = F_w$ fulfills the assumptions of 2.15 for $G = \mathbb{Z}^n$ and S as above. The exceptional Jordan algebra $\mathbb{A}_t = (F_w, u_3)$ of 2.15 is called the *Albert torus*. The grading of $\mathbb{A}_t$ defined in 2.15 is called the toral grading. We note that

$$\mathbb{A}_t \textit{ is of strong type.} \tag{1}$$

4.7. Classification of Jordan tori [31]. *Let* ch. $F \neq 2$. *Then any Jordan torus is graded isomorphic to one of the five tori*

$$F_q^+,\ H(F_\varepsilon, *),\ H(E_\xi, \sigma),\ J(S^{(m)}, (z_i)_{i\in I}) \textit{ or } \mathbb{A}_t$$

endowed with suitable toral gradings.

4.8. Central closures of tori. By 2.3.(d), the centre Z of a torus T is an integral domain, so that we can form the central closure $\overline{T}$ of T (see 1.2). It follows that $\overline{T}$ is always a central domain. By [**31**, 3.9], $\overline{T}$ is a $\mathbb{Z}^n/H$-torus over $\overline{Z}$, where H denotes the central grading group. Moreover, we have

$$\overline{T} \textit{ is a division algebra } \iff \overline{T} \textit{ is finite dimensional.} \tag{1}$$

Since both Jordan and alternative algebras are power associative, the implication $\Leftarrow$ is a special case of the general fact that a power associative finite dimensional domain is a division algebra. To prove this general result, we consider the subalgebra generated by a single element and are then reduced to showing that a finite dimensional associative domain is a division algebra. This is of course well-known, it is for example an immediate consequence of Wedderburn's Structure Theorem. For the other direction, suppose that $\overline{T}$ is infinite dimensional. Then the rank of the central grading group H of T is less than n. Hence the $\mathbb{Z}^n/H$-torus $\overline{T}$ contains a subalgebra which is a 1-torus. But a 1-torus is not a division algebra by [**31**, 3.6] for Jordan tori and [**29**, §2] in the alternative case, and then $\overline{T}$ is not a division algebra either.

We list here the central closures of Jordan tori and of the Cayley torus:

(a) The central closure of F_q^+ is the +-algebra of $\overline{F_q}$, i.e., $\overline{F_q^+} = \overline{F_q}^+$. (This holds for any prime associative A by [**22**, Cor. 3.4]).

(b) By 1.3 the central closures of the hermitian tori $H(F_\varepsilon, *)$ and $H(E_\xi, \sigma)$ are $H(\overline{F_\varepsilon}, *)$ and $H(\overline{E_\xi}, \sigma)$ respectively. Here $H(\overline{F_\varepsilon}, *)$ is a finite-dimensional central Jordan division algebra of degree 2^m for a suitable m. Indeed, by 4.4.2 and (1), $\overline{F_\varepsilon}$ is a central-simple division algebra of dimension 2^{2l}, hence of degree 2^l. By the discussion on [**14**, p. 209] the degree of $H(\overline{F_\varepsilon}, *)$ is therefore either 2^l of 2^{l-1}.

(c) The central closure of a Clifford torus $J(S^{(m)}, (z_i)_{i\in I})$ is a $(|I| + 1)$-dimensional Jordan division algebra of a symmetric bilinear form.

(d) The central closure of an Albert torus is a 27-dimensional Albert division algebra over a field of characteristic $\neq 2, 3$. Similarly, the central closure of the Cayley torus is an 8-dimensional octonion algebra over a field of characeristic $\neq 2$, hence a division algebra by (1).

As a consequence of the above, we have

$$\dim_{\overline{Z}} \overline{T} < \infty \text{ for } T = H(F_\varepsilon, *),\ J(S^{(m)}, (z_i)_{i\in I}), \mathbb{O}_t \text{ and } \mathbb{A}_t. \tag{2}$$

We note that the central closures of F_q, F_q^+ and $H(E_\xi, \sigma)$ are in general not finite-dimensional over $\overline{Z}$.

4.9. Proposition. *For a Jordan torus J over F with* ch. $F \neq 3$ *in case J is an Albert torus we have $J = Z(J) \oplus (J, J, J)$. Hence $\mathcal{GIF}(J) \cong F1$ is* 1*-dimensional.*

Proof. This follows from and 1.13.2 and the classification of Jordan tori 4.7, using 3.6 for $J = F_q^+$, $H(F_\varepsilon, *)$, 3.7 for $J = H(E_\xi, \sigma)$, 2.13.1 for $J = J(S^{(m)}, (z_i)_{i\in I})$ and 2.15 for $J = \mathbb{A}_t$. □

The analogous result for alternative G-tori is proven without classification in 3.3.

4.10. Lemma. *Let T be a Jordan torus but not of type F_q^+ or $H(E_\xi, \sigma)$ or let T be a Cayley torus over a field F of* ch. $F \neq 2, 3$. *Then* $\operatorname{IDer} T = \operatorname{Der}_Z T$.

Proof. For a Clifford torus this is a special case of 2.12(c). For the remaining cases we use 1.6.1 and 2.6. So it suffices to show that $\overline{T}$ has only inner derivations. Given the structure of $\overline{T}$ described above this is known. Indeed, for $T = H(F_\varepsilon, *)$ we can use [**14**, Exercise 1, page 258] for algebras of degree 2 and [**14**, Theorem 9, page 254] for algebras of degree ≥ 3, for the Albert torus this follows from [**14**, Theorem 17, page 408] and for the Cayley torus this is a consequence of [**14**, page 301, Lemma 3] (see also [**5**, Lemma 1.39(b)]). □

If F_q, $T = F_q^+$ or $H(E_\xi, \sigma)$, then $\operatorname{IDer} T = \operatorname{Der}_Z T$ is not true in general. For example, if T is simple, then $\operatorname{Der}_Z T = \operatorname{Der}_F T$ because $Z = F$ in this case.

We are now ready to prove our main result.

4.11. Theorem. *Let T be an n-torus over a field F and assume that* ch. $F \neq 2$ *if T is a Jordan torus and that* ch. $F \neq 2, 3$ *if T is a Cayley torus or an Albert torus. Then*

$$\operatorname{Der} T = \operatorname{IDer} T \rtimes \operatorname{CDer} T. \tag{1}$$

With respect to any nondegenerate graded invariant form we have

$$\operatorname{SDer} T = \operatorname{IDer} T \rtimes (\operatorname{SDer} T \cap \operatorname{CDer} T). \tag{2}$$

If $H \subset \mathbb{Z}^n$ is the central grading group of T then

$$\operatorname{IDer} T = \bigoplus_{\alpha\in\mathbb{Z}^n\setminus H} (\operatorname{Der}_F T)_\alpha \quad \text{and} \quad \operatorname{CDer} T = \bigoplus_{\alpha\in H} (\operatorname{Der}_F T)_\alpha. \tag{3}$$

Proof. For $T \cong_{\mathbb{Z}^n} F_q$ and $T \cong_{\mathbb{Z}^n} F_q^+$ (1) has been proven in 3.5 and for $T \cong_{\mathbb{Z}^n} H(E_\xi, \sigma)$ in 3.7. The remaining cases follow from 3.2.7 and 4.10 keeping in mind that $\operatorname{grDer}_F T = \operatorname{Der}_F T$ since T is finitely generated. The structure of $\operatorname{SDer} T$ is immediate from 1.14, and (3) follows from 3.2. □

Remarks. For an alternative torus (1) was known before. Indeed, an alternative torus is either a quantum torus, in which case this was proven in [**4**, Lemma 2.48, page 364] for the field $F = \mathbb{C}$ of complex numbers, but the proof works over any field or a Cayley torus, in which case (1) was proven for ch. $F \neq 2, 3$ in [**5**, Theorem 1.40]. We have reproven the result here to show how it fits in the more general framework of tori considered in this paper, and also because it can be done without extra effort. Since (1) holds for Jordan and alternative tori, it is natural to conjecture that it also holds for structurable tori.

The ideal $\operatorname{IDer} T$ will be described in more detail in 4.13 – 4.15 below. The structure of the central derivation algebra $\operatorname{CDer} T$ is determined in 4.12. Concerning $\operatorname{SDer} T \cap \operatorname{CDer} T$, see 1.14.

4.12. Theorem. *Let T be an n-torus with centre Z and central grading group H of rank r. Then the central derivations $\operatorname{CDer} T$ are isomorphic to the generalized Witt algebra associated to $Z = F[H]$, $\mathcal{D} \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, F)$ and the canonical pairing p: $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, F) \times H \to F : (\Phi, h) \mapsto \Phi(h)$. Moreover,*

$$\operatorname{CDer} T \cong \operatorname{Der}_F Z \ltimes Y \tag{1}$$

where

(i) $Z \otimes_F \operatorname{Hom}_{\mathbb{Z}}(H, F) \cong \operatorname{Der}_F Z \cong \operatorname{Der} F[z_1^{\pm 1}, \dots, z_r^{\pm 1}]$ *is a generalized Witt algebra and*

(ii) $Y \cong Z \otimes_F \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n/H, F)$ *is an abelian ideal.*

If $r \geq 1$ and ch. $F \neq 2$ *then $\operatorname{CDer} T$ is a perfect Lie algebra.*

Proof. Since $\operatorname{CDer} T_{\text{fin}} = \operatorname{CDer} T$ by 3.2, it follows from 2.9 that $\operatorname{CDer} T$ is a generalized Witt algebra. Note that in our case the centre Z is actually isomorphic to the group algebra of H with a suitable grading. The splitting (1) will be a consequence of

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, F) = \operatorname{Hom}_{\mathbb{Z}}(H, F) \oplus \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n/H, F). \tag{2}$$

To establish (2), we consider the canonical sequence of F-linear maps

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n/H, F) \xrightarrow{\iota} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, F) \xrightarrow{\varrho} \operatorname{Hom}_{\mathbb{Z}}(H, F) \to 0 \tag{3}$$

The imbedding ι is given by $\varphi \mapsto \varphi \circ \pi$ where π: $\mathbb{Z}^n \to \mathbb{Z}^n/H$ is the canonical epimorphism. The map ϱ is the restriction map. Obviously, the image of ι equals the kernel of ϱ. We write $H = m_1\mathbb{Z}\varepsilon_1 + \dots + m_n\mathbb{Z}\varepsilon_n$ where $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ and where $m_1, \dots, m_n \geq 0$. Then surjectivity of ϱ will follow from

$$\text{ch. } F \text{ does not divide } m_i \text{ for any nonzero } m_i. \tag{4}$$

Indeed, for a Cayley torus or $H(F_\varepsilon, *)$ or a Clifford torus, all $m_i = 1$ or 2, and ch. $F \neq 2$ in these cases (see 4.3.1, 4.4.1 and 2.12 and 4.5). For an Albert torus, all $m_i = 1$ or 3, but also ch. $F \neq 3$ in this case (see 4.6). Thus we are left with F_q, F_q^+ and $H(E_\xi, \sigma)$. Since the central grading groups of F_q and F_q^+, and the central grading groups of E_ξ and $H(E_\xi, \sigma)$ coincide, we only need to consider the case $F_q = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. In this case, the result is proven in [**12**, Lemma 3.6]. We include a short proof for the convenience of the reader. Suppose that

$p := \operatorname{ch}. F$ divides $m_k \neq 0$ for some k and let $q = (q_{ij})$. Then $t_k^{m_k} \in Z$, and so $t_k^{m_k} t_j = t_j t_k^{m_k} = q_{kj}^{m_k} t_k^{m_k} t_j$, whence $q_{kj}^{m_k} = 1$ for all j. Since $m_k = pl_k$ for some $l_k \geq 1$ and $\operatorname{ch}. F = p$, we have $q_{kj}^{l_k} = 1$ for all j. Hence $t_k^{l_k}$ commutes with all t_j, and we get $t_k^{l_k} \in Z$, but $l_k \varepsilon_k \notin H$, contradiction.

Thus, (4) holds and ϱ is surjective. Hence (3) is exact and so (2) holds. By 1.10 we then have a decomposition

$$\operatorname{CDer} T = (Z \otimes_F \operatorname{Hom}_{\mathbb{Z}}(H, F)) \ltimes (Z \otimes_F \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n/H, F))$$

keeping in mind that $\mathcal{D} \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, F)$ and hence $\mathcal{D}^H \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n/H, F)$. The structure of $Z \otimes_F \operatorname{Hom}_{\mathbb{Z}}(H, F)$ follows from 3.5(a) and 2.9.

Since $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, F) \cong F^n$, the right kernel of p is trivial, and hence $p \neq 0$ as soon as $r \geq 1$. In particular, by 1.10(c), $\operatorname{CDer} T$ is a perfect Lie algebra in this case. □

We now turn to the description of $\operatorname{IDer} T$. As one can see from [**4**] and [**5**], the knowledge of $\operatorname{IDer} T$ will be useful in the classification of the extended affine Lie algebras with coordinate algebras T. For the convenience of the reader we first give a summary of the results on $\operatorname{IDer} T$ obtained so far.

4.13. Summary of results on $\operatorname{IDer} T$**.** Throughout, we let T be an n-torus over a field F satisfying the assumptions of 4.11. We denote by Z the centre of T and by H the central grading group. We have seen in 4.11.3 that

$$\operatorname{IDer} T = \bigoplus_{\alpha \in \mathbb{Z}^n \setminus H} (\operatorname{IDer}_F T)_\alpha \,. \tag{1}$$

Concerning the Lie algebra structure, we note that by 1.4.2 the Lie algebra $\operatorname{IDer} T$ is a Z-form of $\operatorname{IDer} \overline{T}$ where $\overline{T}$ is the central closure of T.

(a) Specializing 3.5, 3.7 and 1.11.7 we know

$$\dim_F (\operatorname{IDer} T)_\alpha = 1 \quad \text{for all } \alpha \in \mathbb{Z}^n \setminus H \text{ and } T = F_q, F_q^+ \text{ or } H(E_\xi, \sigma). \tag{2}$$

Note that for $T = F_q^+$ and $T = H(E_\xi, \sigma)$, the central grading group H of T coincides with the central grading group of F_q and E_ξ respectively. Of course, (2) for $T = F_q$ is already contained in [**4**, 2.55].

(b) By 3.4 we have a similar result for $T = H(F_\varepsilon, *)$. Denoting by H_ε the central grading group of the quantum torus F_ε we have

$$\dim_F (\operatorname{IDer} H(F_\varepsilon, *))_\alpha = \begin{cases} 1 & \alpha \in \mathbb{Z}^n \setminus (H_\varepsilon \cup \operatorname{supp} H(F_\varepsilon, *)) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

We note that $H = H_\varepsilon \cap \operatorname{supp} H(F_\varepsilon, *)$ by 2.10.1 but $H_\varepsilon \not\subset \operatorname{supp} H(F_\varepsilon, *)$ in general.

(c) For the Cayley torus $\mathbb{O}_t$ where $\operatorname{ch}. F \neq 3$ one knows from [**5**, Theorem 1.40, page 4328]

$$\dim_F (\operatorname{IDer} \mathbb{O}_t)_\alpha = 2 \quad \text{for all } \alpha \in \mathbb{Z}^n \setminus H. \tag{4}$$

(d) The dimensions of $(\operatorname{IDer} T)_\alpha$ in the case of a Clifford torus follows from the general formula 2.14.1. In particular, it is interesting to note that contrary to all other cases, the dimension varies with α and it can also get arbitrarily large. For example, in the setting of 4.5 assume that $I = \{\sigma_1, \sigma_2\}$. Then $\dim(\operatorname{IDer} T)_{\sigma_i} = 0$ while $\dim(\operatorname{IDer} T)_{\sigma_1 + \sigma_2} = 1$ (see [**27**, Lemma 2.4]). Or, if I is maximal, i.e., I represents all non-zero classes in $\mathbb{Z}^n/(2\mathbb{Z}^m \oplus \mathbb{Z}^n) \cong \mathbb{Z}_2^m$, we get $\dim_F (\operatorname{IDer} T)_\alpha = 2^{m-1} - 1$ for all $\alpha \in \mathbb{Z}^n \setminus H$.

It remains to consider the Albert torus. Our analysis will be based on the following general lemma.

4.14. Lemma. *Let $G = \mathbb{Z}_3 \oplus \cdots \oplus \mathbb{Z}_3$ with $r \geq 1$ factors and let $J = \oplus_{g\in G} J_g$ be a Jordan G-torus over F which is of strong type. For an F-vector space V we consider an F-bilinear map $\{\cdot,\cdot\} : J \times J \to V$ satisfying*

(i) $\{xy, z\} + \{yz, x\} + \{zx, y\} = 0$ *and*

(ii) $\{x, y\} + \{y, x\} = 0$.

For $g \in G$ denote $\{J, J\}_g = \sum_{g=e+f} \{J_e, J_f\}$. Then

$$\dim_F\{J, J\}_g \leq r - 1 \quad \text{for } g \neq 0. \tag{1}$$

Proof. We choose $0 \neq t_g \in J_g$ and note that then $t_g t_h = st_{g+h}$ for some $0 \neq s \in F$. It will be convenient to use the abbreviation $\{x : y\}$ in case $\{x : y\} = s\{x, y\}$ for some $0 \neq s \in F$. Then

$$\{t_g^n, t_h\} = n\{t_g : t_{(n-1)g+h}\} \quad \text{for } 0 \leq n \leq 2. \tag{2}$$

Indeed, $\{1, J\} = 0$ by (i) and for $n = 2$ we have $\{t_g^2, t_h\} = 2\{t_g, t_g t_h\} = 2\{t_g : t_{g+h}\}$. Next, we let $\varepsilon_i = 0 \oplus \cdots \oplus 1 \oplus \cdots \oplus 0$ with $1 \in \mathbb{Z}_3$ at the ith position and put $t_i := t_{\varepsilon_i}$. We claim that for $g = e + f$, $e = \sum_i e_i\varepsilon_i$ with $0 \leq e_i < 3$ we have

$$\{t_e, t_f\} = \textstyle\sum_{i=1}^r \{t_i^{e_i} : t_{g-e_i\varepsilon_i}\} \tag{3}$$

To prove (3) we observe that in general by (i), (ii) and commutativity of J we hvae $\{xy, z\} = \{x, yz\} + \{y, xz\}$. Hence $\{t_e, t_f\} = \{t_1^{e_1}(t_2^{e_2}(\cdots t_r^{e_r})\cdots) : t_r\} = \{t_1^{e_1} : (t_2^{e_2}(\cdots t_r^{e_r})\cdots)t_r\} + \{(t_2^{e_2}(\cdots t_r^{e_r})\cdots) : t_1^{e_1}t_r\} = \{t_1^{e_1} : t_{e-e_1\varepsilon_1}\} + \{(t_2^{e_2}(\cdots t_r^{e_r})\cdots) : t_{e-e_1\varepsilon_1}\}$. Continuing similarly with the second term proves (3). From (3) and (2) we then obtain

$$\{t_e, t_f\} = \textstyle\sum_{i=1}^r e_i\{t_i : t_{g-\varepsilon_i}\}. \tag{4}$$

Now consider $g = \sum_{i=1}^r g_i\varepsilon_i \neq 0$. Applying (4) yields

$$\begin{aligned}\{t_1, t_{g-\varepsilon_1}\} &= -\{t_{g-\varepsilon_1}, t_1\}\\ &= -(g_1 - 1)\{t_1 : t_{g-\varepsilon_1}\} - g_2\{t_2 : t_{g-\varepsilon_2}\} - g_3\{t_3 : t_{g-\varepsilon_3}\} - \cdots,\end{aligned}$$

whence $0 = \sum_{i=1}^r g_i\{t_i : t_{g-\varepsilon_i}\}$ which gives a nontrivial relation among the $\{t_i, t_{g-\varepsilon_i}\}$, $i = 1, \ldots, r$. □

4.15. Theorem. *For the Albert torus $\mathbb{A}_t$ we have*

$$\dim_F(\mathrm{IDer}\,\mathbb{A}_t)_\alpha = 2 \quad \text{for all } \alpha \in \mathbb{Z}^n \setminus H.$$

Proof. We apply 4.14 to the central closure $J = \overline{\mathbb{A}_t}$ of $\mathbb{A}_t$. By [**31**] we know that $J = \oplus_{\overline{\alpha}\in\mathbb{Z}_3^3} J_{\overline{\alpha}}$ is a $\mathbb{Z}_3^3$-torus with $J_{\overline{\alpha}} = \overline{Z} \otimes_Z Z(\mathbb{A}_t)_\alpha$. Also, we let $\{x, y\} := [L_x, L_y]$ for $x, y \in J$. Since $\{J, J\}_{\overline{\alpha}} = (\mathrm{IDer}\,J)_{\overline{\alpha}}$, we get $\dim_{\overline{Z}}(\mathrm{IDer}\,J)_{\overline{\alpha}} \leq 2$ for $\overline{\alpha} \neq 0$ by 4.14 while $(\mathrm{IDer}\,J)_{\overline{0}} = Z \otimes (\mathrm{IDer}\,\mathbb{A}_t)_H = 0$ by 1.4.2. On the other side, one knows that $\mathrm{Der}_{\overline{Z}} J = \mathrm{IDer}\,J$ is a simple Lie algebra of type F_4 (see [**15**, page 21]), and so $\dim_{\overline{Z}} \mathrm{Der}\,J = 52$. Hence

$$52 = \dim_{\overline{Z}} \mathrm{IDer}\,J = \sum_{\overline{\alpha}\neq 0} \dim_{\overline{Z}}(\mathrm{IDer}\,J)_{\overline{\alpha}} \leq 26 \cdot 2 = 52,$$

and so $\dim_F(\mathrm{IDer}\,\mathbb{A}_t)_\alpha = \dim_{\overline{Z}}(\mathrm{IDer}\,J)_{\overline{\alpha}} = 2$. □

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