DIVERGENCE THEOREM. *Let A be a vector field on E in xyz-space. Let V be a compact subset of E, and S its piecewise smooth boundary. Let $r(u,v)$ be a parametric presentation of S on D in uv-plane by outward orientation. Then*

$$\int_S A \cdot dS = \int_V divA\ dV,$$

i.e.,

$$\iint_D A(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right)\ dudv = \iiint_V divA\ dxdydz.$$

We prove this for a special case where V is the rectangular parallelepiped determined by the vectors $a\boldsymbol{i}$, $b\boldsymbol{j}$ and $c\boldsymbol{k}$ ($\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are the fundamental vectors in xyz-space and $a, b, c > 0$). Let

$$D = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$$

be the square region, and define

$$\begin{aligned} S_1 &: r_1(u,v) = (au, bv, c)\\ S_2 &: r_2(u,v) = (av, bu, 0)\\ S_3 &: r_3(u,v) = (av, b, cu)\\ S_4 &: r_4(u,v) = (au, b, cv)\\ S_5 &: r_5(u,v) = (a, bu, cv)\\ S_6 &: r_6(u,v) = (a, bv, cu) \end{aligned}$$

on D. Then the boundary S of V is the union of S_1, S_2, $\dots$, and S_6. Moreover,

$$\begin{aligned} \frac{\partial r_1}{\partial u} \times \frac{\partial r_1}{\partial v} &= (a,0,0) \times (0,b,0) = a\boldsymbol{i} \times b\boldsymbol{j} = ab\boldsymbol{k}\\ \frac{\partial r_2}{\partial u} \times \frac{\partial r_2}{\partial v} &= (0,b,0) \times (a,0,0) = b\boldsymbol{j} \times a\boldsymbol{i} = -ab\boldsymbol{k}\\ \frac{\partial r_3}{\partial u} \times \frac{\partial r_3}{\partial v} &= (0,0,c) \times (a,0,0) = c\boldsymbol{k} \times a\boldsymbol{i} = ac\boldsymbol{j}\\ \frac{\partial r_4}{\partial u} \times \frac{\partial r_4}{\partial v} &= (a,0,0) \times (0,0,c) = a\boldsymbol{i} \times c\boldsymbol{k} = -ab\boldsymbol{j}\\ \frac{\partial r_5}{\partial u} \times \frac{\partial r_5}{\partial v} &= (0,b,0) \times (0,0,c) = b\boldsymbol{j} \times = c\boldsymbol{k} = bc\boldsymbol{i}\\ \frac{\partial r_6}{\partial u} \times \frac{\partial r_6}{\partial v} &= (0,0,c) \times (0,b,0) = c\boldsymbol{k} \times b\boldsymbol{j} = -bc\boldsymbol{i} \end{aligned}$$

and thus all $r_1(u,v)$, $r_2(u,v)$ $\dots$, and $r_6(u,v)$ have the outward orientation.

Now, we have

$$\int_S A \cdot dS = \int_{S_1} A \cdot dS + \int_{S_2} A \cdot dS + \cdots + \int_{S_6} A \cdot dS.$$

Let

$$A = (A_1, A_2, A_3).$$

We first compute $\int_{S_1} A \cdot dS + \int_{S_2} A \cdot dS$. Using the information above, we have

$$\begin{aligned}
\int_{S_1} A \cdot dS + \int_{S_2} A \cdot dS &= ab \iint_D A(r_1(u,v)) \cdot \boldsymbol{k}\ dudv - ab \iint_D A(r_2(u,v)) \cdot \boldsymbol{k}\ dudv \\
&= ab \iint_D A_3(r_1(u,v))\ dudv - ab \iint_D A_3(r_2(u,v))\ dudv \\
&= ab \int_0^1 \int_0^1 A_3(au, bv, c)\ dudv - ab \int_0^1 \int_0^1 A_3(av, bu, 0)\ dudv \\
&= \int_0^b \int_0^a A_3(x,y,c)\ dxdy - \int_0^b \int_0^a A_3(x,y,0)\ dxdy
\end{aligned}$$

$x := au$, $y := bv$ in the first term, and $x := av$, $y := bu$ in the second term

$$\begin{aligned}
&= \int_0^b \int_0^a \big(A_3(x,y,c) - A_3(x,y,0)\big)\ dxdy \\
&= \int_0^b \int_0^a \int_0^c \frac{\partial A_3}{\partial z}\ dz\ dxdy \quad \text{(Fundamental Theorem of Calculus)}.
\end{aligned}$$

In the same way, we get

$$\begin{aligned}
\int_{S_3} A \cdot dS + \int_{S_4} A \cdot dS &= \int_0^c \int_0^a \int_0^b \frac{\partial A_2}{\partial y}\ dy\ dxdz \\
\int_{S_5} A \cdot dS + \int_{S_6} A \cdot dS &= \int_0^c \int_0^b \int_0^a \frac{\partial A_1}{\partial x}\ dx\ dydz.
\end{aligned}$$

Therefore, we obtain

$$\int_S A \cdot dS = \int_0^c \int_0^b \int_0^a \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dxdydz = \iiint_V divA\ dxdydz. \quad \square$$