極小局所アフィンリー環

Minimal Locally Affine Lie Algebras

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平成23年8月22日 (月)

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A Lie algebra $\mathcal{L}$ over a field F of characteristic 0 has a **root decomposition** with respect to an abelian subalgebra $\mathcal{H}$ if

$$\mathcal{L} = \bigoplus_{\xi \in \mathcal{H}^*} \mathcal{L}_\xi,$$

where $\mathcal{H}^*$ is the dual space of $\mathcal{H}$ and

$$\mathcal{L}_\xi = \{x \in \mathcal{L} \mid [h, x] = \xi(h)x \text{ for all } h \in \mathcal{H}\}.$$

An element of the set

$$R = \{\xi \in \mathcal{H}^* \mid \mathcal{L}_\xi \neq 0\}$$

is called a **root**.

Let $\mathcal{B}$ be a symmetric invariant bilinear form of $\mathcal{L}$.

Definition of a LEALA

A triple $(\mathcal{L}, \mathcal{H}, \mathcal{B})$ (or simply $\mathcal{L}$) is called a **locally extended affine Lie algebra** or a **LEALA** for short if it satisfies the following 4 axioms:
(1) $\mathcal{H}$ is self-centralizing, i.e., $\mathcal{L}_0 = \mathcal{H}$;
(2) $\mathcal{B}$ is nondegenerate;
(3) **integrability** for all $\xi \in R^\times$, i.e., for all $x \in \mathcal{L}_\xi$,

$$\mathrm{ad}x \in \mathrm{End}_F\mathcal{L} \text{ is locally nilpotent}$$

(4) $R^\times$ is irreducible, i.e.,

$$R^\times = R_1 \cup R_2 \text{ and } (R_1, R_2) = 0 \implies R_1 = \emptyset \text{ or } R_2 = \emptyset.$$

What is $R^\times$?

By (1) and (2), it is easy to prove that

(i) $\mathcal{B}_{\mathcal{L}_\xi \times \mathcal{L}_{-\xi}}$ is nondegenerate for all $\xi \in R$.

(ii) For each $\xi \in R$, there exists a unique $t_\xi \in \mathcal{H}$ such that

$$\mathcal{B}(h, t_\xi) = \xi(h)$$

for all $h \in \mathcal{H}$.

Thus there is the induced form on the vector space spanned by R over F, simply denoted $(\cdot, \cdot)$. Namely,

$$(\xi, \eta) := \mathcal{B}(t_\xi, t_\eta)$$

for $\xi, \eta \in R$.

We call an element of

$$R^\times := \{\xi \in R \mid (\xi, \xi) \neq 0\}$$

an **anisotropic root**.

Also:

Definition of an EALA

If $\mathcal{H}$ is finite-dimensional, then $\mathcal{L}$ is called an **extended affine Lie algebra** or an **EALA** for short.

Definition of an isotropic LEALA

If $R^\times = \emptyset$, then $(\mathcal{L}, \mathcal{H}, \mathcal{B})$ is called an **isotropic LEALA** (or an **isotropic EALA** if $\mathcal{H}$ is finite-dimensional).

Note that if $R^\times = \emptyset$, then the axioms (3) and (4) are empty statements.

Oscillator algebras are examples of isotropic LEALAs.

Nobody tried to classify isotropic LEALAs nor isotropic EALAs yet!

We assume that $R^\times \neq \emptyset$ in this talk.

We can scale the above form $(\cdot, \cdot)$ by a nonzero scalar so that

$$(\xi, \eta) \in \mathbb{Q}$$

for all $\xi, \eta \in R^\times$.

Let V be the $\mathbb{Q}$-span of R, say

$$V = \mathrm{span}_{\mathbb{Q}} R.$$

Morita and myself showed the Kac conjecture, that is,

the scaled form $(\cdot, \cdot)$ on V is positive semidefinite.

As a corollary,

$(W, R^\times)$ becomes a **reduced locally extended affine root system**,

where

$$W = \mathrm{span}_{\mathbb{Q}} R^\times.$$

Thus $R^\times$ is a natural generalization of

classical finite irreducible root systems,

locally finite irreducible root systems,

affine root systems in the sense of Macdonald,

or extended affine root systems in the sense of Saito.

The dimension of the radical of V is called the **nullity** for a LEALA. If the additive subgroup of V generated by

$$R^0 := \{\xi \in R \mid (\xi, \xi) = 0\},$$

the set of **isotropic roots**, is free, we call the rank the **null rank** of a LEALA.

The **core** of a LEALA $\mathcal{L}$, denoted by $\mathcal{L}_c$, is the subalgebra of $\mathcal{L}$ generated by the root spaces $\mathcal{L}_\alpha$ for all $\alpha \in R^\times$. ($\mathcal{L}_c$ becomes an ideal of $\mathcal{L}$.)

If the centralizer of $\mathcal{L}_c$ in $\mathcal{L}$ is contained in $\mathcal{L}_c$, then $\mathcal{L}$ is called **tame**.

Morita and myself classified LEALAs of nullity 0.
So the next interesting objects are LEALAs of nullity 1.

局所アフィンリー環の定義

We call a tame LEALA of null rank 1 a **locally affine Lie algebra** or a **LALA** for short.

Examples of LEALAs of nullity 0

Let $gl_{\mathbb{N}}(F)$ be the $\mathbb{N} \times \mathbb{N}$ matrix algebra consisting of matrices having only finite nonzero entries.

A subalgebra of $gl_{\mathbb{N}}(F)$

$$sl_{\mathbb{N}}(F) = \{x \in gl_{\mathfrak{I}}(F) \mid \operatorname{tr}(x) = 0\}$$

is an example of a tame LEALA of nullity 0, which is also an example of locally finite split simple Lie algebras classified by Neeb and Stumme.

Let

$$d = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \\ 0 & 0 & 3 & 0 & \\ 0 & 0 & 0 & 4 & \\ \vdots & & & & \ddots \end{pmatrix}$$

be the diagonal matrix of size $\mathbb{N}$ with diagonal entries 1, 2, 3, 4, ..., n, ...

Define a bilinear form $\mathcal{B}$ on the matrix Lie algebra

$$\mathcal{L} = sl_{\mathbb{N}}(F) \oplus Fd$$

by

$$\mathcal{B}(x, y) := \operatorname{tr}(xy)$$

for $x \in sl_{\mathbb{N}}(F)$ and $y \in L$, and $\mathcal{B}(d, d) := 0$.

Let $\mathfrak{h}$ be the subalgebra of $sl_{\mathbb{N}}(F)$ consisting of diagonal matrices, and let

$$\mathcal{H} := \mathfrak{h} \oplus Fd.$$

Then $(\mathcal{L}, \mathcal{H}, \mathcal{B})$ is also an tame LEALA of nullity 0.

Note that you can choose any scalar for $\mathcal{B}(d, d)$ (not necessarily 0).

From the classification, one knows that

any tame LEALA of nullity 0 has centerless.

Examples of LALAs

Let

$$L := sl_{\mathbb{N}}(F) \otimes_F F[t^{\pm 1}]$$

be the Lie algebra with usual bracket.

For $x \otimes t^m,\ y \otimes t^n \in L$, define

$$(x \otimes t^m, y \otimes t^n) := \operatorname{tr}(xy)\delta_{m+n,0}.$$

Then $(\cdot,\cdot)$ is a nondegenerate symmetric invariant bilinear form on L.

Define the new bracket on a 1-dimensional central extension

$$L \oplus Fc$$

by

$$[x \otimes t^m, y \otimes t^n] := [x, y] \otimes t^{m+n} + m(x \otimes t^m, y \otimes t^n)c,$$

Let

$$d_0 = t\frac{d}{dt}$$

be a degree derivation of $F[t^{\pm 1}]$.

Let

$$\mathcal{L} = L \oplus Fc \oplus Fd_0,$$

extending the bracket on $L \oplus Fc$ by

$$[d_0, x \otimes t^m] = mx \otimes t^m = -[x \otimes t^m, d_0].$$

Extend the form $(\cdot,\cdot)$ to a form $\mathcal{B}(\cdot,\cdot)$ on $\mathcal{L}$ by

$$\mathcal{B}(c, c) = \mathcal{B}(d_0, d_0) = \mathcal{B}(c, L) = \mathcal{B}(d_0, L) = 0 \quad \text{and} \quad \mathcal{B}(c, d_0) = 1.$$

Then $\mathcal{B}(\cdot,\cdot)$ is a nondegenerate symmetric invariant bilinear form $\mathcal{L}$.

Let

$$\mathcal{H} = \mathfrak{h} \oplus Fc \oplus Fd_0,$$

where $\mathfrak{h}$ is the abelian subalgebra of $sl_{\mathbb{N}}(F)$ consisting of diagonal matrices.

This $(\mathcal{L}, \mathcal{H}, \mathcal{B})$ is a LALA, which is an affine Lie algebra if we take $sl_n(F)$ instead of $sl_{\mathbb{N}}(F)$.

Also, if we take $sl_{\mathbb{N}}(F) \oplus Fd$ instead of $sl_{\mathbb{N}}(F)$ mentioned above with

$$\tilde{\mathcal{H}} = \mathfrak{h} \oplus Fd \oplus Fc \oplus Fd_0,$$

then $(\tilde{\mathcal{L}}, \tilde{\mathcal{H}}, \tilde{\mathcal{B}})$ is still a LALA.

From the classification, one knows that

any LALA has 1-dimensional center.

A LALA $\mathcal{L}$ is called **minimal** if the core is a hyperplane in $\mathcal{L}$.
Two examples $\mathcal{L}$ and $\tilde{\mathcal{L}}$ above have the same core $sl_{\mathbb{N}}(F) \oplus Fc$.
Thus $\mathcal{L}$ is minimal, but $\tilde{\mathcal{L}}$ is not.
There exists a minimal subalgebra $\mathcal{L}^{min}$ of any LALA so that $\mathcal{L}^{min}$ is a LALA.
Note that $\mathcal{L}^{min}$ is is not unique.
In fact, $\mathcal{L}$ is a subalgebra of $\tilde{\mathcal{L}}$, but

$$\mathcal{L}' := \big(sl_{\mathbb{N}}(F) \otimes_F F[t^{\pm 1}]\big) \oplus Fc \oplus F(d + d_0)$$

with

$$\mathcal{H}' := \mathfrak{h} \oplus Fc \oplus F(d + d_0)$$

is also a subalgebra of $\tilde{\mathcal{L}}$ which is a minimal LALA.

A LALA $\mathcal{L}$ is called **standard** if $\mathcal{L}$ contains the degree derivation of the core.
Thus the example $\mathcal{L}$ above is a standard minimal LALA.
Moreover, one can show that the example $\mathcal{L}'$ above is also a standard minimal LALA.

There exists an example of **non-standard minimal LALAs**.

In fact, if we take

$$d = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \\ 0 & 0 & \frac{1}{3} & 0 & \\ 0 & 0 & 0 & \frac{1}{4} & \\ \vdots & & & & \ddots \end{pmatrix}$$

be the diagonal matrix of size $\mathbb{N}$ with diagonal entries $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$, instead of d above.

Then one can show that $\mathcal{L}'$ constructed above using this d is a non-standard minimal LALA.

LALA の分類方法

First, classify **reduced locally affine root systems**.

Let $\mathcal{L}$ be a LALA.

The core of $\mathcal{L}$ is a 1-dimensional **universal central extension** of a **locally loop algebra** or a **twisted locally loop algebra**.

There always exists a unique **maximal LALA** $\mathcal{L}^{max}$ containing $\mathcal{L}$, which is automatically standard.