ノルム空間のある幾何的定数

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1. Absolute normalized norm & convex function

- $\mathbb{K}$ 上のベクトル空間 X がノルム空間 (with norm $\|\cdot\|$) :
  (N1) $\|\boldsymbol{x}\| \geq 0$ かつ $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{o}$
  (N2) $\|k\boldsymbol{x}\| = |k|\,\|\boldsymbol{x}\| \quad (k \in \mathbb{K})$
  (N3) $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

- ノルム $\|\cdot\|$ on $\mathbb{K}^n$ が absolute normalized ノルム :
  (AN1) $\|(x_1, x_2, \cdots, x_n)\| = \|(|x_1|, |x_2|, \cdots, |x_n|)\| \quad (x_i \in \mathbb{K})$
  (AN2) $\|\boldsymbol{e}_1\| = \|\boldsymbol{e}_2\| = \cdots = \|\boldsymbol{e}_n\| = 1$
  - absolute normalized ノルムの全体を $\mathcal{AN}_n$ で表す.

- 例 : ℓ^p ノルム $\|\cdot\|_p \quad (1 \leq p \leq \infty)$

$$\|(x_1, \cdots, x_n)\|_p = \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} & (1 \leq p < \infty) \\ \max\{|x_1|, \cdots, |x_n|\} & (p = \infty) \end{cases}$$

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- $\Delta_n := \left\{ \boldsymbol{s} = (s_1, \cdots, s_{n-1}) \in \mathbb{R}_{\geq 0}^{n-1} \ : \ \sum_{j=1}^{n-1} s_j \leq 1 \right\}$

  Ψ_n : 次を満たす Δ_n 上の continuous convex functions ψ 全体

  (A$_0$) $\psi(\boldsymbol{o}) = \psi(\boldsymbol{e}_1) = \cdots = \psi(\boldsymbol{e}_{n-1}) = 1$

  (A$_j$) $\psi(\boldsymbol{s}) \geq (1 - s_j)\,\psi\left(\dfrac{1}{1 - s_j} \left(s_1, \cdots, \underset{j}{0}, \cdots, s_{n-1} \right) \right)$

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  ▸ Fig

- $n = 2$ のとき $\Delta_2 = [0,\ 1]$ であり, Ψ_2 の convex functions ψ の条件は次のようになる.

  $\psi(0) = \psi(1) = 1 \qquad \& \qquad \psi(t) \geq \max\{1 - t,\ t\}$

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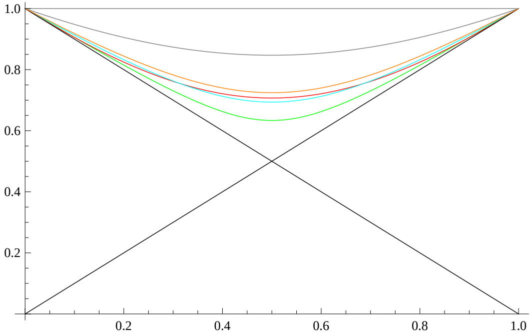

例： Ψ_2 の双曲線（赤は ℓ^2 ノルムに対応の双曲線 $\psi_2(t)$）

- $\psi \in \Psi_n$ に対し，$\mathbb{K}^n$ 上のノルム $\|\cdot\|_\psi$ を定義：
  $\|(x_1, \cdots, x_n)\|_\psi$

$$\underset{\text{def}}{=} \begin{cases} \left(\sum_{j=1}^n |x_j|\right) \psi\left(\dfrac{1}{\sum_{j=1}^n |x_j|} (|x_2|, \cdots, |x_n|)\right) & (\boldsymbol{x} \neq \boldsymbol{o}) \\ 0 & (\boldsymbol{x} = \boldsymbol{o}) \end{cases}$$

- $\|\cdot\|_\psi$ は absolute normalized ノルムになる．この対応
  $\Psi_n \ni \psi \longleftrightarrow \|\cdot\|_\psi \in \mathcal{AN}_n$ は 1 対 1 対応
  (F. Bonsall, J. Duncan, K. Saito, M. Kato et al., 1973, 2000)

- ℓ^2 ノルム $\|\cdot\|_2$ に対応の
  $\psi_2(x_1, \cdots, x_{n-1}) = \sqrt{2\sum_{j=1}^{n-1}(x_j^2 - x_j) + 2\sum_{i<j} x_i x_j + 1}$
  $n = 2$ のとき，$\psi_2(t) = \sqrt{2t^2 - 2t + 1}$

check: $\|(x_1, x_2)\|_{\psi_2} = (|x_1| + |x_2|)\sqrt{2|x_2|^2/(|x_1|+|x_2|)^2 - 2|x_2|/(|x_1|+|x_2|) + 1}$
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2. von Neumann Jordan constant

- ノルム空間 X において

$$C_{NJ}(X) \underset{\text{def}}{=} \sup\left\{ \frac{\|\boldsymbol{x}+\boldsymbol{y}\|^2+\|\boldsymbol{x}-\boldsymbol{y}\|^2}{2\left(\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2\right)} : \boldsymbol{x},\boldsymbol{y}\in X, \text{not both } \boldsymbol{o} \right\}$$

  - von Neumann Jordan (簡単に NJ) constant という.
  - $1 \le C_{NJ}(X) \le 2$ であり，$C_{NJ} = 1 \iff X$: Hirbert space

  ▸ 説明

  - $C_{NJ}(X) < 2 \iff X$: uniformly non-square
  - $C_{NJ}(\|\cdot\|_p) = 2^{|(2-p)/p|}$ for $1 \le p \le \infty$
    (by Hölder's and Clarkson's inequalities)

- 目標：$\psi \in \Psi_n$ に対応の $\|\cdot\|_\psi \in \mathcal{AN}_n$ に対し，C_{NJ} を求める.

ここでは $n = 2$ とする.（$n \ge 3$ のときの計算法は，ほとんど何も知られていない.）

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- ノルム空間 X において

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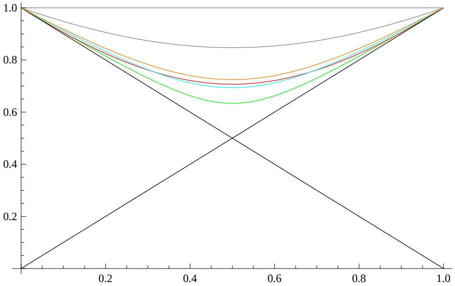
3. Examples: C_{NJ} for the norms ass. with conics

(Ik, Kai, Kato)

- Hyperbola norm

$$0 \underset{(-)}{<} a \underset{(-)}{<} 4c,\ a \leq 2\sqrt{c},\ \ \psi_{hyp(a,c)}(t) = \sqrt{at^2 - at + c} + 1 - \sqrt{c}$$



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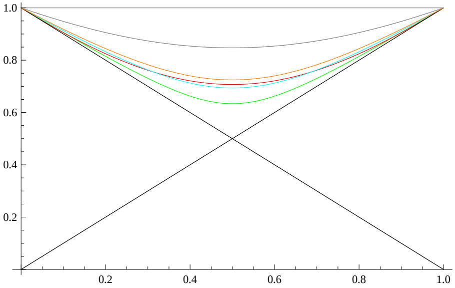

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$$\begin{cases} \frac{1}{2}(\sqrt{4c-a}+2-2\sqrt{c})^2 & \text{if} \quad c \geq 1,\ \ 0 < a \leq 2\sqrt{c} \\ \text{(on gray domain)} & \quad \text{or} \ \ 0 < c < 1,\ \ 0 < a \leq \rho(c), \\ \frac{a(a+2-4\sqrt{c})}{2(a-2c)} & \text{if} \quad 0 < c \leq \frac{3-2\sqrt{2}}{2},\ \ \rho(c) < a < 4c \\ \text{(on orange domain)} & \quad \text{or} \ \ \frac{3-2\sqrt{2}}{2} < c < 1,\ \ \rho(c) < a \leq \sigma(c), \\ \frac{a(a+2-4\sqrt{c})}{(a-2c)(\sqrt{4c-a}+2-2\sqrt{c})^2} & \text{if} \quad \frac{3-2\sqrt{2}}{2} < c < \frac{1}{4},\ \ \sigma(c) < a < 4c \\ \text{(on green domain)} & \quad \text{or} \ \ \frac{1}{4} \leq c < 1,\ \ \sigma(c) < a < 2\sqrt{c}, \end{cases}$$

where $\rho(c) = \frac{1}{2}\left[3c+2\sqrt{c}-1+(1-\sqrt{c})\sqrt{9c-2\sqrt{c}+1}\right]$,

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- ノルム空間 X において

$$J(X) \underset{\text{def}}{=} \sup\{\min(\|x+y\|,\ \|x-y\|)\ :\ x, y \in S_X\}$$

- James constant という.
- $J(X)^2/2 \le C_{NJ}(X) \le J(X)$
- $\psi \in \Psi_2$, $M_1 = \max_{0\le t\le 1} \psi(t)/\psi_2(t)$, $M_2 = \max_{0\le t\le 1} \psi_2(t)/\psi(t)$ とすると

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Theorem 2

$\psi \in \Psi_2, \ \ \psi(t) = \psi(1-t)$ for $0 \le t \le 1, \ (\mathbb{K} = \mathbb{R})$

$M_1 = \max_{0 \le t \le 1/2} \dfrac{\psi(t)}{\psi_2(t)}, \ M_2 = \max_{0 \le t \le 1/2} \dfrac{\psi_2(t)}{\psi(t)}$ とするとき

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$\Longleftrightarrow$

$$\exists t_1, t_2 \in [0, 1/2] \text{ s.t.}$$

$$M_1 = \frac{\psi(t_1)}{\psi_2(t_1)}, \ M_2 = \frac{\psi_2(t_2)}{\psi(t_2)} \ \ \& \ \ (1-t_1)(1-t_2) = \frac{1}{2}$$

Corollary 2

Theorem 2 の条件のもとで

$$C_{NJ}(\|\cdot\|_\psi) = C'_{NJ}(\|\cdot\|_\psi) = C_Z(\|\cdot\|_\psi) = C'_Z(\|\cdot\|_\psi) = \frac{1}{2} J(\|\cdot\|_\psi)^2 = M_1^2 M_2^2$$

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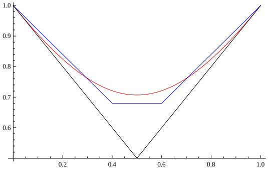
- Example

$$0 \le c \le 1, \quad \psi(t) = \max\left\{1 - ct,\ 1 - c + ct,\ 1 - \frac{c^2}{2}\right\}$$



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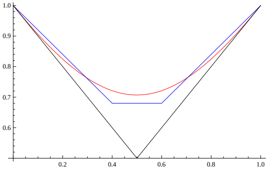

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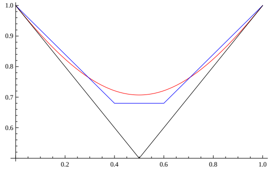

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