# JORDAN ANALOGUE OF LAURENT POLYNOMIAL ALGEBRA

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Abstract. Quantum tori or the octonion torus as the non-commutative version or as the alternative but not associative version of the algebra of Laurent polynomials in several variables are considered in Lie theory, especially the theory of extended affine Lie algebras. In this report, we introduce a new way of generalizing the Laurent polynomial algebra in the variety of Jordan algebras. We call these algebras Jordan tori, and we announce the classification of Jordan tori.

## 1. Introduction

Throughout this report, we assume that $F$ is a field of characteristic $\neq 2$. Also, algebras are always assumed to be unital linear algebras.

**Definition 1.** A $\mathbb{Z}^n$-graded Jordan algebra $J = \oplus_{\alpha\in\mathbb{Z}^n} J_\alpha$ over $F$ with the properties:

**(T1):** every nonzero homogeneous element is invertible;
**(T2):** $\dim_F J_\alpha \leq 1$ for all $\alpha \in \mathbb{Z}^n$;
**(T3):** $\mathrm{supp}(J) := \{\alpha \in \mathbb{Z}^n; J_\alpha \neq (0)\}$ generates $\mathbb{Z}^n$

is called a *Jordan $n$-torus* over $F$ or simply a *Jordan torus*.

Clearly, Jordan tori generalize the algebra of Laurent polynomials in $n$-variables. One can check that $S := \mathrm{supp}(J)$ satisfies the following properties:

$$0 \in S \quad \text{and} \quad S - 2S \subset S.$$

Thus $S$ is a *semilattice* in $\mathbb{Z}^n$ (see [1]). As another way of generalizing the Laurent polynomial algebra, we consider a $\mathbb{Z}^n$-graded Jordan algebra $J = \oplus_{\alpha\in\mathbb{Z}^n} J_\alpha$ over $F$ with the property

$$\dim_F J_\alpha = 1 \quad \text{and} \quad J_\alpha J_\beta = J_\beta \quad \text{for all } \alpha, \beta \in \mathbb{Z}^n. \tag{1}$$

One can show that such a $J$ satisfies T1–T3, so $J$ is a Jordan torus called *of strong type*. A Jordan torus $J = \oplus_{\alpha\in\mathbb{Z}^n} J_\alpha$ of strong type satisfies not only $\mathrm{supp}(J) = \mathbb{Z}^n$ but also $xy \neq 0$ for all $0 \neq x, y \in J$. However, none of these properties hold in general Jordan tori.

*Remark* 1. (a) More generally, we call a $\mathbb{Z}^n$-graded algebra over $F$ satisfying T1–T3 a *torus*. We note that alternative tori were classified, namely, they are isomorphic to quantum tori or the octonion torus (see [4] and [9]). Also, one can check that for $\mathbb{Z}^n$-graded alternative algebras, T1–T3 are equivalent to (1). So one can take (1) as the definition of alternative tori as in [4].

(b) The classification of Jordan tori of strong type is announced in [9]. However, for the classification of extended affine Lie algebras of type $A_1$, one needs to consider arbitrary Jordan tori. The classification below shows that there exist five types of Jordan tori, but only three types of them are of strong type.

We start by recalling quantum tori.

**Definition 2.** An $n \times n$ matrix $q = (q_{ij})$ over $F$ such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ is called a *quantum matrix.* The *quantum torus* $F_q = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ *determined by* a quantum matrix $q$ is defined as the associative algebra over $F$ with $2n$ generators $t_1^{\pm 1}, \dots, t_n^{\pm 1}$, and relations $t_i t_i^{-1} = t_i^{-1} t_i = 1$ and $t_j t_i = q_{ij} t_i t_j$ for all $1 \le i, j \le n$. We notice that $F_q$ is commutative if and only if $q = \mathbf{1}$ where $\mathbf{1}$ is the quantum matrix whose entries are all 1. In this case, the quantum torus becomes the Laurent polynomial algebra, $F_{\mathbf{1}} = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

It is easily seen that $F_q$ has a $\mathbb{Z}^n$-grading, i.e., $F_q = \oplus_{\alpha \in \mathbb{Z}^n} Ft_\alpha$ where $t_\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, and that $F_q$ is an associative torus. Moreover, any associative torus is isomorphic to some quantum torus $F_q$, see [3]. The multiplication rule for $F_q$ is the following: for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$,

$$t_\alpha t_\beta = \prod_{i<j} q_{ij}^{\alpha_i \beta_j} t_{\alpha+\beta}. \tag{2}$$

**Construction 1.** (i) Let $F_q = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be a quantum torus. Then the plus algebra $F_q^+$ is a Jordan torus. In fact, $F_q^+ = \oplus_{\alpha \in \mathbb{Z}^n} Ft_\alpha$ as $F$-vector spaces. By (2), we have

$$\begin{aligned} t_\alpha \cdot t_\beta &= \frac{1}{2}\left[\prod_{i<j} q_{ij}^{\alpha_i \beta_j} + \prod_{i<j} q_{ij}^{\beta_i \alpha_j}\right] t_1^{\alpha_1+\beta_1} \cdots t_n^{\alpha_n+\beta_n} \\ &= \frac{1}{2} \prod_{i<j} q_{ij}^{\alpha_i \beta_j} \left[1 + \prod_{i,j} q_{ij}^{\alpha_i \beta_j}\right] t_{\alpha+\beta} \in Ft_{\alpha+\beta}. \end{aligned} \tag{3}$$

Also, the invertible elements in $F_q$ and in $F_q^+$ coincide (see e.g. [5]), and we have $\operatorname{supp}(F_q^+) = \mathbb{Z}^n$. Therefore, $F_q^+$ is a Jordan torus. By (3), $F_q^+$ is a Jordan torus of strong type if and only if

$$\prod_{i,j} q_{ij}^{\alpha_i \beta_j} \neq -1 \quad \text{for all} \quad \alpha, \beta \in \mathbb{Z}^n. \tag{4}$$

(ii) Let $E$ be a field extension of $F$ with $[E : F] \le 2$, say $E := F(\sqrt{a}\,)$ for some $a \in F$ if $[E : F] = 2$ and $E = F$ if $[E : F] = 1$. Let $^-$ be the non-trivial Galois automorphism of $E$ over $F$ if $[E : F] = 2$ and the identity map of $F$ if $E = F$. Let $\varepsilon = (\varepsilon_{ij})$ be a quantum matrix over $E$ satisfying

$$\overline{\varepsilon}_{ij} \varepsilon_{ij} = 1 \quad (\Longleftrightarrow \overline{\varepsilon}_{ij} = \varepsilon_{ji}) \quad \text{for all} \quad i, j. \tag{5}$$

Note that $\varepsilon_{ij} = 1$ or $-1$ if $E = F$. For the quantum torus $E_\varepsilon = E_\varepsilon[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over $E$ determined by $\varepsilon$, there exists the unique involution $* : E_\varepsilon \longrightarrow E_\varepsilon$ over $F$ such that $x^* = \overline{x}$ for all $x \in E$ and $t_i^* = t_i$ for all $i = 1, \dots, n$. In fact, $*$ is given by

$$(xt_\alpha)^* := \overline{x} t_n^{\alpha_n} \cdots t_1^{\alpha_1} = \overline{x} \prod_{i<j} \varepsilon_{ij}^{\alpha_i \alpha_j} t_\alpha. \tag{6}$$

The symmetric elements $H := H(E_\varepsilon, *)$ form a Jordan algebra over $F$. By (6), $*$ is graded on $E_\varepsilon = \oplus_{\alpha \in \mathbb{Z}^n} Et_\alpha$, i.e., $(Et_\alpha)^* = Et_\alpha$ for all $\alpha \in \mathbb{Z}^n$, so we get $H = \oplus_{\alpha \in \mathbb{Z}^n} (Et_\alpha \cap H)$ as $F$-vector spaces. Since $E_\varepsilon^+ = \oplus_{\alpha \in \mathbb{Z}^n} Et_\alpha$ is a Jordan torus over $E$, $H$ is a $\mathbb{Z}^n$-graded Jordan algebra over $F$. If $E = F$, then $\dim_F(Ft_\alpha \cap H) \le 1$. Otherwise, $Et_\alpha = (F + F\sqrt{a}\,)t_\alpha$, so $\dim_F Et_\alpha = 2$. If $xt_\alpha \in H$ for some $x \in E$, then $\sqrt{a} x t_\alpha \notin H$. Hence $\dim_F(Et_\alpha \cap H) \le 1$. In general, the inverse of a symmetric element is also symmetric. Since $t_1, \dots, t_n \in H$, $\mathrm{supp}(J)$ generates $\mathbb{Z}^n$. Thus $H = H(E_\varepsilon, *)$ is a Jordan torus over $F$.

When $E = F$, we have $\varepsilon_{ij} = 1$ or $-1$ for all $1 \le i, j \le n$. So if $\varepsilon \ne \mathbf{1}$, then $\varepsilon_{ij} = -1$ for some $i, j$, and hence $(t_i t_j)^* = -t_i t_j$. Therefore, $\mathrm{supp}(H) = \mathbb{Z}^n$ if and only if $\varepsilon = \mathbf{1}$, i.e., $H = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. In particular, $H = H(F_\varepsilon, *)$ is never of strong type unless $\varepsilon = \mathbf{1}$.

Let $E \ne F$. Assume that $Et_\alpha \cap H = (0)$ for some $\alpha \in \mathbb{Z}^n$. Then for $0 \ne x \in E$, we have $xt_\alpha + (xt_\alpha)^* = 0$, so $(xt_\alpha)^* = -xt_\alpha$ and $(\sqrt{a}xt_\alpha)^* = \sqrt{a}xt_\alpha$. Hence $0 \ne \sqrt{a}xt_\alpha \in Et_\alpha \cap H$, which is a contradiction. Therefore, we get $\mathrm{supp}(H) = \mathbb{Z}^n$ if $E \ne F$. Also, $H = H(E_\varepsilon, *)$ is a Jordan torus over $F$ of strong type if and only if $\varepsilon$ satisfies (4).

**Example 1.** Let $\zeta_r \in \mathbb{C}$ (the field of complex numbers) be a primitive $r$-th root of unity. Consider the following quantum matrix

$$\zeta = \zeta(r) = \begin{pmatrix} 1 & \zeta_r & 1 & \cdots & 1 \\ \zeta_r^{-1} & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}.$$

Then $\zeta$ satisfies (5) for the complex conjugate $^-$ over $\mathbb{R}$ (the field of real numbers), so $H(\mathbb{C}_\zeta, *)$ is a Jordan torus over $\mathbb{R}$. Note that $H(\mathbb{C}_\zeta, *)$ is of strong type if and only if $r$ is odd. Also, $r = 2$ is the case $E = F$, and $r > 2$ is the case $E \ne F$.

**Construction 2.** Let $2 \le m \le n$ and let $S = S_m$ be a semilattice in $\mathbb{Z}^m$. Choose a generating set $\{\sigma_1, \dots, \sigma_m\}$ of $S$ which is a basis of $\mathbb{Z}^m$ (this is possible by [1]). Note that any element of $S$ can be written as

$$2(\alpha_1\sigma_1 + \cdots + \alpha_m\sigma_m) + \epsilon_1\sigma_1 + \cdots + \epsilon_m\sigma_m$$

for $\alpha_i \in \mathbb{Z}$ and $\epsilon_i \in \{0, 1\}$, $i = 1, \dots, m$. We put

$$I = I_S := \{\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{0,1\}^m \ ; \ \epsilon_1\sigma_1 + \cdots + \epsilon_m\sigma_m \in S\}.$$

We note that $I$ always contains $\mathbf{0} = (0, \dots, 0)$. Let

$$Z := F[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \oplus_{\alpha \in \mathbb{Z}^n} \ F z_\alpha$$

be the Laurent polynomial algebra, and $V$ a free $Z$-module with basis

$$\{t_\epsilon \ ; \ \epsilon \in I \setminus \{\mathbf{0}\}\}.$$

Define a $Z$-bilinear form $f : V \times V \longrightarrow Z$ by

$$f(t_\epsilon, t_\eta) = \begin{cases} a_\epsilon z_\epsilon & \text{if } \epsilon = \eta \\ 0 & \text{otherwise} \end{cases}$$

for all $t_\epsilon, t_\eta$, where $0 \ne a_\epsilon \in F$ and $z_\epsilon = z_1^{\epsilon_1} \cdots z_m^{\epsilon_m}$ for $\epsilon = (\epsilon_1, \dots, \epsilon_m)$. Let

$$J =: Z \oplus V$$

be the Jordan algebra over $Z$ determined by the symmetric bilinear form $f$, i.e., the multiplication on $J$ is defined by

$$(x + v)(y + w) = xy + f(v, w) + xw + yv \tag{7}$$

for $x, y \in Z$ and $v, w \in V$. We put $t_{\mathbf{0}} := 1$ so that $\{t_\epsilon\ ;\ \epsilon \in I\}$ is a $Z$-basis of $J$, and $\{z_\alpha t_\epsilon\ ;\ \alpha \in \mathbb{Z}^n,\ \epsilon \in I\}$ is an $F$-basis of $J$. Let $\mathbb{Z}^n := \mathbb{Z}^m \oplus \mathbb{Z}^{n-m}$ and extend $\{\sigma_1, \dots, \sigma_m\}$ to a basis $\{\sigma_1, \dots, \sigma_m, \sigma_{m+1}, \dots, \sigma_n\}$ of $\mathbb{Z}^n$. For $\alpha = \alpha_1\sigma_1 + \cdots + \alpha_n\sigma_n \in \mathbb{Z}^n$, there exist unique $(\alpha'_1, \dots, \alpha'_m) \in \mathbb{Z}^m$ and $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{0,1\}^m$ such that

$$\alpha = 2(\alpha'_1\sigma_1 + \cdots + \alpha'_m\sigma_m) + \epsilon_1\sigma_1 + \cdots + \epsilon_m\sigma_m + \alpha_{m+1}\sigma_{m+1} + \cdots + \alpha_n\sigma_n.$$

Put $\alpha' := (\alpha'_1, \dots, \alpha'_m, \alpha_{m+1}, \dots, \alpha_n)$, $t_\epsilon := 0$ for $\epsilon \in \{0,1\}^m \setminus I$, and $t_\alpha := z_{\alpha'} t_\epsilon$. Since $\mathbb{Z}^n \ni \alpha \longrightarrow (\alpha', \epsilon) \in \mathbb{Z}^n \times \{0,1\}^m$ is a bijective map, we get

$$J = \bigoplus_{(\alpha',\epsilon) \in \mathbb{Z}^n \times \{0,1\}^m} F z_{\alpha'} t_\epsilon = \oplus_{\alpha \in \mathbb{Z}^n}\ F t_\alpha$$

as $F$-vector spaces. By (7), we have

$$t_\alpha t_\beta = (z_{\alpha'} t_\epsilon)(z_{\beta'} t_\eta) = \begin{cases} a_\epsilon z_{\alpha'+\beta'+\epsilon} & \text{if } \epsilon = \eta \neq \mathbf{0} \\ z_{\alpha'+\beta'} t_\epsilon & \text{if } \eta = \mathbf{0} \\ z_{\alpha'+\beta'} t_\eta & \text{if } \epsilon = \mathbf{0} \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

so we obtain $t_\alpha t_\beta \subset F t_{\alpha+\beta}$. For $\alpha = (\alpha', \epsilon) \in \mathbb{Z}^n \times I$, since $t_\alpha^2 = a_\epsilon z_{2\alpha'+\epsilon}$ is invertible, $t_\alpha$ is invertible (see e.g. [5]). Since $\mathrm{supp}(J)$ contains the basis $\{\sigma_1, \dots, \sigma_n\}$ of $\mathbb{Z}^n$, $J = \oplus_{\alpha \in \mathbb{Z}^n}\ F t_\alpha$ is a Jordan torus over $F$. We call $J = J(S_m, \{a_\epsilon\})$ the *Clifford torus determined by $S_m$ and $\{a_\epsilon\}$*. Note that Clifford tori are never of strong type by (8), even if we take $S_m = \mathbb{Z}^m$. Also, clearly we have $\mathrm{supp}(J) = S_m + \mathbb{Z}^{n-m}$ which is a semilattice in $\mathbb{Z}^n$.

*Remark* 2. When $m = n$ and all $a_\epsilon = 1$, this algebra $J$ appeared in [1] as the first example of an extended affine Lie algebra of type $A_1$ graded by an arbitrary semilattice.

For the final construction, we consider so-called first Tits constructions (see e.g. [5]).

**Definition 3.** We say that a prime associative (or Jordan) algebra $A$ over $F$ has *central degree* 3 if the central closure $\overline{A} = \overline{Z} \otimes_Z A$ is a finite dimensional central simple algebra over $\overline{Z}$ of degree 3 where $Z$ is the centre of $A$ and $\overline{Z}$ is the field of fractions of $Z$.

For example, a strongly prime exceptional Jordan algebra has central degree 3 by Zelmanov's Prime Structure Theorem [7]. Since $\overline{A}$ is a finite dimensional power associative algebra over $\overline{Z}$, there exists the generic trace tr of $\overline{A}$ over $\overline{Z}$ (see e.g. [5]). Note that $A$ embeds into $\overline{A}$ via $a \mapsto 1 \otimes a$ for $a \in A$ (see e.g. [6]), so we identify $A$ with the subring of $\overline{A}$. Considering this, we have the following lemma:

**Lemma 1.** *Let $A$ be a prime associative algebra over $F$ of central degree 3, and $\mu \in Z$ a unit where $Z$ is the centre of $A$. Assume that $\mathrm{tr}(A) \subset Z$. Then the subset $(A, \mu) := A \oplus A \oplus A$ of the first Tits construction $(\overline{A}, \mu) = \overline{A} \oplus \overline{A} \oplus \overline{A}$ is a $Z$-subalgebra such that $\overline{(A, \mu)} \cong (\overline{A}, \mu)$ over $\overline{Z}$ where $\overline{(A, \mu)} = \overline{Z} \otimes_Z (A, \mu)$.* □

We call the $(A, \mu)$ in Lemma 1 a *first Tits construction over $Z$*. It is a special type of the general first Tits construction studied in [8].

**Construction 3.** Assume that $F$ contains a primitive 3rd root of unity (hence $\operatorname{char} F \neq 3$ in particular). Let $\omega := \zeta(3)$ in Example 1, $F_\omega = F_\omega[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ the quantum torus determined by $\omega$, and $Z$ the centre of $F_\omega$. Then one finds that

$$Z = F[t_1^{\pm 3}, t_2^{\pm 3}, t_3^{\pm 1}, \dots, t_n^{\pm 1}],$$

the Laurent polynomial algebra in $n$-variables $t_1^3, t_2^3, t_3, \dots, t_n$, and that $F_\omega$ has central degree 3 with $\operatorname{tr}(F_\omega) \subset Z$ where tr is the generic trace of the central closure $\overline{F}_\omega$ of $F_\omega$ over $\overline{Z}$. Let

$$\mathbb{A}_t = (F_\omega, t_3) = F_\omega \oplus F_\omega \oplus F_\omega$$

be the first Tits construction over $Z$ in Lemma 1. It is shown in [9] that $\mathbb{A}_t$ is a Jordan torus of strong type called the *Albert torus*.

*Remark* 3. The Albert torus appeared as a coordinate algebra of extended affine Lie algebras of type $G_2$ in [1] and [2].

The classification of Jordan tori proceeds as follows. First one can easily show that a Jordan torus $J$ is a Jordan domain, i.e.,

$$U_x y = 0 \implies x = 0 \quad \text{or} \quad y = 0 \quad \text{for all } x, y \in J,$$

where $U$ is the $U$-operator. So $J$ is in particular strongly prime. Thus by Zelmanov's Prime Structure Theorem [7], $J$ is one of Hermitian, Clifford or Albert type. One can easily see that $F_q^+$, $H(F_\varepsilon, *)$ and $H(E_\varepsilon, *)$ are of Hermitian type, that $J(S_m, \{a_\epsilon\})$ is of Clifford type, and that $\mathbb{A}_t$ is of Albert type.

Before we state the result, we mention a strong property of isomorphisms in the class of Jordan tori.

**Definition 4.** Let $J = \oplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ and $J' = \oplus_{\alpha \in \mathbb{Z}^n} J'_\alpha$ be two Jordan tori. Then $J$ and $J'$ are called *isomorphic as graded algebras* and denoted by $J \cong_{\mathbb{Z}^n} J'$ if there exists an isomorphism $\varphi$ (as algebras) from $J$ onto $J'$ such that $\varphi(J_\alpha) = J_\alpha$ for all $\alpha \in \mathbb{Z}^n$.

If we have an automorphism $f$ of $\mathbb{Z}^n$, we always obtain a new Jordan tori by renaming the degrees. More precisely, for $J = \oplus_{\alpha \in \mathbb{Z}^n} J_\alpha$, we put $\tilde{J}_\alpha := J_{f(\alpha)}$. Then a new Jordan algebra $\tilde{J} := \oplus_{\alpha \in \mathbb{Z}^n} \tilde{J}_\alpha$ defined by the same multiplication of $J$, is a Jordan torus. Note that $J = \tilde{J}$ as algebras.

Using the above terminology, we have the following proposition saying that isomorphisms between Jordan tori become 'almost graded isomorphisms':

**Proposition 1.** *Let* $\varphi : J = \oplus_{\alpha \in \mathbb{Z}^n} J_\alpha \xrightarrow{\sim} J' = \oplus_{\alpha \in \mathbb{Z}^n} J'_\alpha$ *be an isomorphism between Jordan tori. Then there exists an automorphism* $f$ *of* $\mathbb{Z}^n$ *such that*

$$\varphi(J_{f(\alpha)}) = J'_\alpha \quad \textit{for all } \alpha \in \mathbb{Z}^n.$$

*Hence* $J \cong J'$ *if and only if* $\tilde{J} \cong_{\mathbb{Z}^n} J'$. □

Finally, we state the classification result:

**Theorem 1.** *Let* $J$ *be a Jordan torus over* $F$*. Then* $J$ *is isomorphic to one of the five types of Jordan tori*

$$F_q^+, \quad H(F_\varepsilon, *), \quad H(E_\varepsilon, *), \quad J(S_m, \{a_\epsilon\}) \quad \textit{or} \quad \mathbb{A}_t. \quad \square$$

*Remark* 4. (a) If $F$ is algebraically closed, then $H(E_\varepsilon, *)$, $E \neq F$, does not exist, and all $a_\epsilon$ of $J(S_m, \{a_\epsilon\})$ can be chosen to be 1.

(b) If $F$ does not contain a primitive 3rd root of unity, then $\mathbb{A}_t$ does not exist.

(c) There does not exist a Jordan torus obtained from a second Tits construction.

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