NEW LIE TORI FROM NAOI TORI

YOJI YOSHII

Dedicated to Professor Jun Morita on the occasion of his 60th birthday

ABSTRACT. We define general Lie tori which generalize original Lie tori. We show that a Naoi torus is a general Lie torus. We give examples and prove several properties of general Lie tori. We also review isotopies of Lie tori, and prove that a general Lie torus is, in fact, isotopic to an original Lie torus. Finally, we suggest a very simple way of defining a Lie torus corresponding to a locally extended affine root system $\mathfrak{R}$, which we call a Lie $\mathfrak{R}$-torus.

Throughout the paper F is a field of characteristic 0. For a subset S of an abelian group, the subgroup generated by S is denoted by $\langle S \rangle$.

1. INTRODUCTION

Naoi showed in [Na] that Lie tori are not enough to describe the fixed algebras of multi-loop algebras studied in [ABFP]. Thus he defined a modified Lie torus in [Na], which we call a **Naoi torus** (see Definition 5.1). We define a new wider class of Lie tori, called **general Lie tori** by a slight modification of the definition of original Lie tori (see Definition 2.1). We show that any Naoi torus is a general Lie torus in Theorem 5.3.

Next we review the notion of isotopies of Lie tori, introduced in [AF]. Let us call an original Lie torus defined in [Ne] or [Y2] a **normal Lie torus**. As very simple examples, the loop algebra $\mathrm{sl}_2(F[t^{\pm 1}])$ is a normal Lie torus, and the subalgebra P of $\mathrm{sl}_2(F[t^{\pm 1}])$ generated by $e \otimes t$, $f \otimes t^{-1}$ and $h \otimes t^{\pm 2}$, where $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, is not normal but a general Lie torus. We will show that any general Lie torus is isotopic to a normal Lie torus. Thus some important properties of normal Lie tori still hold for general Lie tori. For example, we show that there exists a nonzero symmetric invariant graded form on a general Lie n-torus in Corollary 4.6.

However, the support for the grading of a general Lie torus is quite different from the normal case. We study each support, which is an example of a so-called **reflection space** E of an abelian group G, i.e., a subset E of G satisfies $2x - y \in E$ for all $x, y \in E$. A reflection space S of the normal case has a stronger condition, namely, S satisfies $x - 2y \in S$ for all $x, y \in S$, which is called a **symmetric reflection space**, as in [LN2]. A reflection space and a symmetric reflection space look similar, but they are very different. For example, $m\mathbb{Z} + n$ for any $m, n \in \mathbb{Z}$ is a reflection space of $\mathbb{Z}$, but it is symmetric only when $n = 0$, or $m = 2\ell$ and $n = \ell$ for some $\ell \in \mathbb{Z}$ (see Proposition 6.9). As another interesting example, the solution

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space of a system of linear equations (familiar in elementary linear algebra) is a reflection space in a vector space but not symmetric (see Example 6.2).

The structure of symmetric spaces are simple. Namely, a symmetric space of G is just a union of some cosets, $\bigcup_i (2S + s_i)$ in $S/2S$, for a subgroup S of G. However, reflection spaces are more complicated. The author does not know the classification even for $G = \mathbb{Z}^n$ (or even for $G = \mathbb{Z}^2$). We hope for someone to classify reflection spaces (see Proposition 6.10 and Example 6.13).

We also study the reflection space generated by two elements (see Definition 6.14), and prove some basic properties. As an application, we discuss some subalgebras of a multi-loop algebra (see Example 6.17 and 6.20).

Eventually, we reach a nicer and simpler definition of Lie tori as simply an $\mathfrak{R}$-graded Lie algebra satisfying certain properties (see Definition 7.2), where $\mathfrak{R}$ is a **locally extended affine root system** defined in [MY1]. This new **Lie $\mathfrak{R}$-torus** can be identified with both a general Lie torus and a normal Lie torus. We also show that if G is a torsion-free abelian group, then any Lie G-torus is a Lie $\mathfrak{R}$-torus. Thus there is essentially no difference for them when G is a torsion-free abelian group. One of the major benefits is that no abelian group G is involved in the definition of a Lie $\mathfrak{R}$-torus. So the description of a Lie $\mathfrak{R}$-torus is much shorter, and we see that the definition only depends on the locally extended affine root system $\mathfrak{R}$. It is not difficult to see that if $\mathfrak{R}$ is a finite irreducible root system, then the Lie $\mathfrak{R}$-torus is nothing but a finite-dimensional split simple Lie algebra. If $\mathfrak{R}$ is a locally finite irreducible root system (see [LN1]), then the Lie $\mathfrak{R}$-torus is a locally finite split simple Lie algebra, studied in [NS] and [St].

Let us explain more how nice Lie $\mathfrak{R}$-tori are. If $\mathfrak{R}$ is an affine root system defined in [M], then the Lie $\mathfrak{R}$-torus is a derived affine Lie algebra or a (twisted) loop algebra. If $\mathfrak{R}$ is an extended affine root system defined in [S], then the Lie $\mathfrak{R}$-torus is a central extension of the centerless core of an extended affine Lie algebra, studied in [BGK], [BGKN] and [AABGP]. Actually, Allison and Gao started to research the centerless core as a double graded Lie algebra, which is the origin of a Lie torus (see [AG]). Since then, many people studied Lie tori, for example, in [BY], [AY], [AFY1], [AFY2], [AB], [F], [Y3], etc.

Moreover, J. Morita and the author studied in [MY1] a generalization of locally finite split simple Lie algebras and extended affine Lie algebras. Thus, if $\mathfrak{R}$ is a locally affine root system defined in [Y4], then the Lie $\mathfrak{R}$-torus is the core of a **locally affine Lie algebra** or a **locally (twisted) loop algebra** (see [N] and [MY2]). If $\mathfrak{R}$ is a locally extended affine root system, then the Lie $\mathfrak{R}$-torus is a central extension of the centerless core of a locally extended affine Lie algebra (see [MY1]).

2. Basic concepts

Let $G = (G, +, 0)$ be an arbitrary abelian group. Let Δ be a **locally finite irreducible root system** (see [LN1]), and we denote the Cartan integer by

$$\langle \mu, \nu \rangle := \frac{2(\mu, \nu)}{(\nu, \nu)}$$

for $\mu, \nu \in \Delta$, and also let $\langle 0, \mu \rangle := 0$ for all $\mu \in \Delta$. Recall that Δ is called **reduced** if $2\alpha \notin \Delta$ for all $\alpha \in \Delta$. We define the subset

$$\Delta^{\mathrm{red}} := \{\alpha \in \Delta \mid \tfrac{1}{2}\alpha \notin \Delta\}$$

of Δ, which is a reduced locally finite irreducible root system. Note that $\Delta = \Delta^{\mathrm{red}}$ if Δ is reduced. We review the notion of a Lie G-torus introduced in [Ne], which we call here a normal locally Lie G-torus. (Originally, it is defined for a finite irreducible root system Δ, but it is easily generalized to a locally finite irreducible root system.)

Definition 2.1. Let Δ be a locally finite irreducible root system. A Lie algebra $\mathcal{L}$ is called a **locally Lie G-torus of type Δ** if

(LT1) $\mathcal{L}$ has a decomposition into subspaces

$$\mathcal{L} = \bigoplus_{\mu \in \Delta \cup \{0\}, \ g \in G} \mathcal{L}_\mu^g$$

such that $[\mathcal{L}_\mu^g, \mathcal{L}_\nu^h] \subset \mathcal{L}_{\mu+\nu}^{g+h}$ for $\mu, \nu, \mu+\nu \in \Delta \cup \{0\}$ and $g, h \in G$;

(LT2) For every $g \in G$, $\quad \mathcal{L}_0^g = \sum_{\mu \in \Delta, \ h \in G} [\mathcal{L}_\mu^h, \ \mathcal{L}_{-\mu}^{g-h}]$;

(LT3) For each $0 \neq x \in \mathcal{L}_\mu^g$ ($\mu \in \Delta, \ g \in G$), there exists $y \in \mathcal{L}_{-\mu}^{-g}$ so that $\mu^\vee := [x, y] \in \mathcal{L}_0^0$ satisfies $[\mu^\vee, z] = \langle \nu, \mu \rangle z$ for all $z \in \mathcal{L}_\nu^h$ ($\nu \in \Delta \cup \{0\}, \ h \in G$);

(LT4) $\langle \operatorname{supp}_G \mathcal{L} \rangle = G$, where

$$\operatorname{supp}_G \mathcal{L} := \{g \in G \mid \mathcal{L}_\mu^g \neq 0 \text{ for some } \mu \in \Delta \cup \{0\}\};$$

(LT5) $\dim \mathcal{L}_\mu^g \leq 1$ for all $\mu \in \Delta$ and $g \in G$;

(LT6) $\dim \mathcal{L}_\mu^0 = 1$ for all $\mu \in \Delta^{\mathrm{red}}$.

Remark 2.2. (i) Condition (LT4) is simply a convenience. If it fails to hold, we may replace G by the subgroup generated by $\operatorname{supp}_G \mathcal{L}$.

(ii) It follows from (LT1) and (LT3) that $\mathcal{L}$ admits a grading by the root lattice $Q(\Delta)$: if

$$\mathcal{L}_\lambda := \bigoplus_{g \in G} \mathcal{L}_\lambda^g$$

for $\lambda \in Q(\Delta)$, where $\mathcal{L}_\lambda^g = 0$ if $\lambda \notin \Delta \cup \{0\}$, then $\mathcal{L} = \bigoplus_{\lambda \in Q(\Delta)} \mathcal{L}_\lambda$ and $[\mathcal{L}_\lambda, \mathcal{L}_\mu] \subset \mathcal{L}_{\lambda+\mu}$.

(iii) $\mathcal{L}$ is also graded by the group G. Namely, if

$$\mathcal{L}^g := \bigoplus_{\mu \in \Delta \cup \{0\}} \mathcal{L}_\mu^g,$$

then $\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}^g$ and $[\mathcal{L}^g, \mathcal{L}^h] \subset \mathcal{L}^{g+h}$.

Now, we define a **general locally Lie** G**-torus** as a Lie algebra satisfying (LT1-5) above and instead of (LT6),

$$(\text{LT6})' \qquad \mathcal{L}_\mu \neq 0 \text{ for all } \mu \in \Delta.$$

We often say a Lie torus when 'general', 'locally' or G is clear from the context or no need to specify. If $G \cong \mathbb{Z}^n$, then $\mathcal{L}$ is called a **locally Lie** n**-torus** or simply a **Lie** n**-torus**. We call the rank of Δ the **rank** of $\mathcal{L}$ and the type of Δ the **type** of $\mathcal{L}$. If $\mathcal{L}$ has the trivial center, then $\mathcal{L}$ is called **centerless**.

Finally, we say a **normal locally Lie** G**-torus** when $\mathcal{L}$ satisfies (LT6)$'$ and (LT6), not just (LT6). By this, a Lie torus really has type BC if Δ has type BC. ($\mathcal{L}$ might have reduced type even though Δ is nonreduced if we only assume (LT6).) We may omit the term 'normal' or 'general' unless we compare two concepts.

Before giving examples of general Lie tori, we introduce basic concepts about isomorphisms of graded algebras following [AF].

Definition 2.3. Let $A = \oplus_{g \in G} A_g$ and $A' = \oplus_{g' \in G'} A_{g'}$ be graded algebras, where G and G' are groups. An algebra isomorphism $\varphi : A \longrightarrow A'$ is called an **isograded isomorphism** if there exists a group isomorphism $\psi : G \longrightarrow G'$ such that $\varphi(A_g) \subset A'_{\psi(g)}$. In particular, if $G = G'$ and ψ is the identity map, it is called a **graded isomorphism**. Also, we say that A and A' are **isograded isomorphic** (resp. **graded isomorphic**) if there exists an isograded isomorphism (resp. a graded isomorphism) between them. We sometimes identify two G-graded algebras if they are graded isomorphic.

A Lie G-torus is graded by the abelian group $\langle \Delta \rangle \times G$. Thus an isograded isomorphism between two Lie tori means that they are isograded isomorphic relative to such direct product groups.

For a group homomorphism $s \in \text{hom}(\langle \Delta \rangle, G)$ and a Lie G-torus L, one can change the G-grading as follows, and this new G-graded Lie algebra is called an **isotope** of L by s, denoted by $L^{(s)}$:

$$(L^{(s)})_\alpha^g := L_\alpha^{g + s(\alpha)}$$

for all $\alpha \in \langle \Delta \rangle$ and $g \in G$.

Lemma 2.4. *An isotope $L^{(s)}$ of L is in fact isograded isomorphic to L.*

Proof. This follows from a general property for a direct product group. Let M and G are abelian groups, and let $s : M \longrightarrow G$ be a group homorphism. Then the map $f : M \times G \longrightarrow M \times G$ defined by $f(m, g) = (m, g + s(m))$ for $m \in M$ and $g \in G$ is a group automorphism of the product group $M \times G$. In fact, f is clearly a monomorphism. For any $(m, g) \in M \times G$, let $x = (m, g - s(m))$. Then $f(x) = (m, g)$, and so f is onto. $\square$

We will show that any isotope of a general Lie G-torus is again a general Lie G'-torus, where G' is the subgroup of G generated by $\text{supp}_G L^{(s)}$.

Definition 2.5. Let φ be an isograded-isomorphism from a Lie G-torus L onto a Lie G'-torus L', and φ is called a **bi-isomorphism** if the corresponding group isomorphism f from $\langle \Delta \rangle \times G$ onto $\langle \Delta' \rangle \times G'$ decomposes into group isomorphisms of each factor. More precisely, there exist a group isomorphism w from $\langle \Delta \rangle$ onto $\langle \Delta' \rangle$ and a group isomorphism ψ from G onto G' such that $\varphi = w \times \psi$.

If a Lie G-torus L is bi-isomorphic to an isotope of a Lie G'-torus L', we say that L is isotopic to L', denoted by $L \sim L'$.

Remark 2.6. Suppose that φ is an isograded isomorphism from a G-graded algebra A onto a G'-graded algebra. In particular, if φ is the identity map on a G-graded algebra A with a group isomorphism ψ from G onto G', we may say that A is **re-graded by G' through** ψ. For example, an isotope $L^{(s)}$ of a Lie G-torus L is re-graded (through f in the proof of Lemma 2.4). Note that an isograded isomorphism of Lie G-tori is used for $\langle \Delta \rangle \times G$-grading, not just G-grading. Also, an isotope $L^{(s)}$ does not change the degree of the first factor $\langle \Delta \rangle$. However, we note that $\operatorname{supp} L^{(s)} \neq \operatorname{supp} L$ and $\operatorname{supp}_G L^{(s)} \neq \operatorname{supp}_G L$ in general. Thus an isotope $L^{(s)}$ is not necessarily a Lie G-torus since $P := \operatorname{supp}_G L^{(s)}$ might be a proper subgroup of G. But certainly $L^{(s)}$ is a Lie P-torus of the same type. For such a case, **we still say that L is isotopic to a Lie P-torus $L^{(s)}$.**

We will show that any general Lie torus can be re-graded to a normal Lie torus.

3. Examples

Let us give some examples of general Lie tori.

Example 3.1. Let $\{e, f, h\}$ be an sl_2-triple so that $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$, having the root system $\{\pm \alpha\}$ relative to Fh, i.e., α is the linear form of Fh such that $\alpha(h) = 2$.

(1) Let $P_1 := F[t^{\pm p}]$ for an integer $p > 1$ and $L := (e \otimes t^r P_1) \oplus (f \otimes t^{-r} P_1) \oplus (h \otimes P_1)$, where r is a positive integer so that r and p are coprime. One can say that L is the subalgebra of a normal Lie 1-torus $\mathrm{sl}_2(F[t^{\pm 1}]) = \mathrm{sl}_2(F) \otimes F[t^{\pm 1}]$ generated by $e \otimes t^r$, $f \otimes t^{-r}$ and $h \otimes t^{\pm p}$. Then L becomes a general Lie $\mathbb{Z}$-torus by defining $L_\alpha^{pm+r} = Fe \otimes t^{pm+r}$, $L_{-\alpha}^{pm-r} = Ff \otimes t^{pm-r}$, and $L_0^{pm} = Fh \otimes t^{pm}$ for all $m \in \mathbb{Z}$, and all the other homogeneous spaces L_α^k, $L_{-\alpha}^k$ and L_0^k are all 0. Let

$$S_\alpha := \operatorname{supp}_\mathbb{Z} L_\alpha = \{k \in \mathbb{Z} \mid L_\alpha^k \neq 0\} \subset \operatorname{supp}_\mathbb{Z} L.$$

We see that $S_\alpha = p\mathbb{Z} + r$, $S_{-\alpha} = p\mathbb{Z} - r$ (so $S_{-\alpha} = -S_\alpha \neq S_\alpha$), and

$$\operatorname{supp}_\mathbb{Z} L = (p\mathbb{Z} + r) \cup (p\mathbb{Z} - r) \cup p\mathbb{Z},$$

and hence $\langle \operatorname{supp}_\mathbb{Z} L \rangle = \mathbb{Z}$.

Let $s \in \hom(\langle \alpha \rangle, \mathbb{Z})$ define by $s(\alpha) = r$. Then the s-isotope $L^{(s)}$ defined by

$$(L^{(s)})_{\pm\alpha}^n := L_{\pm\alpha}^{n \pm s(\alpha)} = L_{\pm\alpha}^{n \pm r} \quad \text{and} \quad (L^{(s)})_0^n := L_0^{n \pm s(0)} = L_0^n$$

for all $n \in \mathbb{Z}$ can be identified with a normal Lie torus since $(L^{(s)})_{\pm\alpha}^0 \neq 0$. In fact, first note that $\operatorname{supp}_\mathbb{Z}(L^{(s)}) = p\mathbb{Z}$, and then it is clear that $L^{(s)}$ is a centerless normal Lie $p\mathbb{Z}$-torus. Moreover, one can easily check that $L^{(s)}$ is graded isomorphic to $\mathrm{sl}_2(P_1)$. Thus one can say that L is isotopic to $\mathrm{sl}_2(P_1)$.

(2) Let $P_2 := F[t_1^{\pm p_1}, t_2^{\pm p_2}]$ for some integers $p_1, p_2 > 1$, and

$$L = (e \otimes t_1^{r_1} t_2^{r_2} P_2) \oplus (f \otimes t_1^{-r_1} t_2^{-r_2} P_2) \oplus (h \otimes P_2)$$

for some integers $r_1, r_2 > 1$ so that $(p_i, r_i) = 1$ $(i = 1, 2)$. One can say that L is the subalgebra of $\mathrm{sl}_2(F[t_1^{\pm 1}, t_2^{\pm 1}])$ generated by $e \otimes t_1^{r_1} t_2^{r_2}$, $f \otimes t_1^{-r_1} t_2^{-r_2}$, $h \otimes t_1^{\pm p_1}$ and

$h \otimes t_2^{\pm p_2}$. Then L is a general Lie $\mathbb{Z}^2$-torus, and consider the s-isotope $L^{(s)}$, where $s \in \hom(\langle \alpha \rangle, \mathbb{Z}^2)$ is defined as $s(\alpha) = (r_1, r_2)$. Note that

$$\mathrm{supp}_{\mathbb{Z} \times \mathbb{Z}} L_\alpha = (p_1 \mathbb{Z} + r_1) \times (p_2 \mathbb{Z} + r_2) \quad \text{and} \quad \mathrm{supp}_{\mathbb{Z} \times \mathbb{Z}} L_\alpha^{(s)} = p_1 \mathbb{Z} \times p_2 \mathbb{Z},$$

and one can show that $L^{(s)}$ is a normal Lie 2-torus. Thus L is isotopic to a normal Lie 2-torus.

(3) Let $\{e_i, f_i \mid i = 1, 2\}$ be a set of Chevalley generators of $\mathrm{sl}_3(F)$ with a Cartan subalgebra $\mathfrak{h} = \mathrm{span}\, \{[e_i, f_i] \mid i = 1, 2\}$ and the root system $\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$. Let L be the subalgebra of $\mathrm{sl}_3(F[t^{\pm 1}])$ generated by $e_i \otimes t^{r_i}$, $f_i \otimes t^{-r_i}$ and $\mathfrak{h} \otimes t^{\pm p}$. Then L is a general Lie 1-torus. Note that $S_{\alpha_i} = p\mathbb{Z} + r_i$ and $S_{\alpha_1 + \alpha_2} = p\mathbb{Z} + r_1 + r_2$ can be all different sets.

Consider the s-isotope $L^{(s)}$, where $s \in \hom(\langle \alpha_1, \alpha_2 \rangle, \mathbb{Z})$ is defined as $s(\alpha_i) = r_i$. Note that $\mathrm{supp}_{\mathbb{Z}} L = \mathbb{Z} \supset p\mathbb{Z} = \mathrm{supp}_{\mathbb{Z}} \big(\mathrm{sl}_3(P_1)\big)$, and one can show that $L^{(s)}$ is a normal Lie $p\mathbb{Z}$-torus which is graded isomorphic to $\mathrm{sl}_3(P_1)$. Thus L is isotopic to $\mathrm{sl}_3(P_1)$.

(4) Next examples are twisted loop algebras which look different. Let $\mathfrak{J}$ be an arbitrary index set (possibly infinite) with $|\mathfrak{J}| \geq 2$. Let V be a $|\mathfrak{J}|$-dimensional vector space over $\mathbb{Q}$ with a positive definite symmetric bilinear form. Let $\{\epsilon_i \mid i \in \mathfrak{J}\}$ be an orthonormal basis of V. Let

$$\mathrm{D}_{\mathfrak{J}} = \{\pm\epsilon_i \pm \epsilon_j \mid i, j \in \mathfrak{J}, \ i \neq j\} \subset \mathrm{C}_{\mathfrak{J}} = \{\pm\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i \mid i, j \in \mathfrak{J}, \ i \neq j\}$$

be locally finite irreducible root systems of type $\mathrm{D}_{\mathfrak{J}}$ and $\mathrm{C}_{\mathfrak{J}}$ (see [LN1] or [NS]). Let

$$\mathrm{D}_{\mathfrak{J}}' := \{\pm(\epsilon_i - \epsilon_j) \mid i, j \in \mathfrak{J}, \ i \neq j\}$$
$$\mathrm{D}_{\mathfrak{J}}'' := \{\pm(\epsilon_i + \epsilon_j) \mid i, j \in \mathfrak{J}, \ i \neq j\}$$
$$\mathrm{C}_{\mathfrak{J}}' := \{\pm(\epsilon_i + \epsilon_j) \mid i, j \in \mathfrak{J}\} \quad \text{so that}$$
$$\mathrm{D}_{\mathfrak{J}} = \mathrm{D}_{\mathfrak{J}}' \cup \mathrm{D}_{\mathfrak{J}}'' \quad \text{and} \quad \mathrm{C}_{\mathfrak{J}} = \mathrm{D}_{\mathfrak{J}}' \cup \mathrm{C}_{\mathfrak{J}}'.$$

Let

$$s_1 = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}$$

be the matrices of size $2\mathfrak{J}$, where ι is the identity matrix of size $\mathfrak{J}$. Define the automorphisms σ_i $(i = 1, 2)$ of $\mathrm{sl}_{2\mathfrak{J}}(F)$ by

$$\sigma_i(x) = s_i^{-1} x^T s_i$$

for $x \in \mathrm{sl}_{2\mathfrak{J}}(F)$, where x^T is the transpose of x. Then the fixed algebra $\mathrm{sl}_{2\mathfrak{J}}(F)^{\sigma_i}$ of $\mathrm{sl}_{2\mathfrak{J}}(F)$ is a locally finite split simple Lie algebra of type $\mathrm{D}_{\mathfrak{J}}$ or $\mathrm{C}_{\mathfrak{J}}$. Thus, let

$$\mathrm{sl}_{2\mathfrak{J}}(F)^{\sigma_1} = \mathfrak{g}^D = \mathfrak{h} \oplus \bigoplus_{\xi \in \mathrm{D}_{\mathfrak{J}}} \mathfrak{g}_\xi^D \quad \text{and} \quad \mathrm{sl}_{2\mathfrak{J}}(F)^{\sigma_2} = \mathfrak{g}^C = \mathfrak{h} \oplus \bigoplus_{\xi \in \mathrm{C}_{\mathfrak{J}}} \mathfrak{g}_\xi^C.$$

Note that one can take the same Cartan subalgebra $\mathfrak{h}$ for the types $\mathrm{D}_{\mathfrak{J}}$ and $\mathrm{C}_{\mathfrak{J}}$, and also $\mathfrak{g}_\xi^D = \mathfrak{g}_\xi^C$ for all $\xi \in D_{\mathfrak{J}}'$. We extend σ_i to the loop algebra $\mathrm{sl}_{2\mathfrak{J}}(F) \otimes F[t^{\pm 1}]$, denoted $\hat{\sigma}_i$, as

$$\hat{\sigma}_i(x \otimes t^m) = (-1)^m \sigma_i(x) \otimes t^m$$

for $x \in \mathrm{sl}_{2\mathfrak{J}}(F)$ and $m \in \mathbb{Z}$. Let

$$T(D) := (\mathrm{sl}_{2\mathfrak{J}}(F) \otimes F[t^{\pm 1}])^{\hat{\sigma}_1} \quad \text{and} \quad T(C) := (\mathrm{sl}_{2\mathfrak{J}}(F) \otimes F[t^{\pm 1}])^{\hat{\sigma}_2}$$

be the fixed algebras, which are usually called the twisted loop algebras by $\hat{\sigma}_i$. Let

$$V_\xi^i = \{x \in \text{sl}_{2\mathfrak{J}}(F) \mid [h, x] = \xi(h)x \text{ for all } h \in \mathfrak{h} \text{ and } \sigma_i(x) = -x\}.$$

Then we can show that

$$V_\xi^1 = V_\xi^2 \quad \text{for all } \xi \in D'_\mathfrak{J} \cup \{0\}$$
$$V_\xi^1 = \mathfrak{g}_\xi^C \quad \text{for all } \xi \in C'_\mathfrak{J}$$
$$V_\xi^2 = \mathfrak{g}_\xi^D \quad \text{for all } \xi \in D''_\mathfrak{J},$$

and moreover, letting $V_\xi := V_\xi^1$ for all $\xi \in D'_\mathfrak{J} \cup \{0\}$, we obtain

$$T(D) = (\mathfrak{h} \oplus \bigoplus_{\xi \in D_\mathfrak{J}} \mathfrak{g}_\xi^D) \otimes F[t^{\pm 2}] \oplus (V_0 \oplus \bigoplus_{\xi \in D'_\mathfrak{J}} V_\xi \oplus \bigoplus_{\xi \in C'_\mathfrak{J}} \mathfrak{g}_\xi^C) \otimes tF[t^{\pm 2}]$$

and

$$T(C) = (\mathfrak{h} \oplus \bigoplus_{\xi \in C_\mathfrak{J}} \mathfrak{g}_\xi^C) \otimes F[t^{\pm 2}] \oplus (V_0 \oplus \bigoplus_{\xi \in D'_\mathfrak{J}} V_\xi \oplus \bigoplus_{\xi \in D''_\mathfrak{J}} \mathfrak{g}_\xi^D) \otimes tF[t^{\pm 2}].$$

The latter algebra $T(C)$ is a so-called twisted loop algebra of type $C_\ell^{(2)}$ (or $A_{2\ell+1}^{(2)}$ in Kac label) when $|\mathfrak{J}| = \ell$ is finite. What is the former algebra $T(D)$ then?

It seems that $T(D)$ dose not appear on the list of Kac-Moody Lie algebras (see e.g. [K]). We can at leaast check that $T(D)$ is a general locally Lie 1-torus of type $C_\mathfrak{J}$ (but not $D_\mathfrak{J}$, see the axiom (LT1) of a Lie G-torus). Of course, $T(C)$ is a normal locally Lie 1-torus of type $C_\mathfrak{J}$.

We can now answer the question (see also [H]).

Proposition 3.2. $T(D)$ *is an isotope of* $T(C)$.

Proof. Define the group homomorphism $s : \langle C_\mathfrak{J} \rangle \longrightarrow \mathbb{Z}$ by $s(\epsilon_{i_0} - \epsilon_i) = 0$ and $s(2\epsilon_{i_0}) = 1$ for a fixed $i_0 \in \mathfrak{J}$ and all $i \in \mathfrak{J}$. Then

$$s(\epsilon_i + \epsilon_{i_0}) = s(\epsilon_i - \epsilon_{i_0} + 2\epsilon_{i_0}) = s(\epsilon_i - \epsilon_{i_0}) + s(2\epsilon_{i_0}) = 0 + 1 = 1, \quad \text{and hence}$$

$$s(2\epsilon_i) = s(\epsilon_i - \epsilon_{i_0} + \epsilon_i + \epsilon_{i_0}) = s(\epsilon_i - \epsilon_{i_0}) + s(\epsilon_i + \epsilon_{i_0}) = 0 + 1 = 1.$$

Also, we have

$$s(\epsilon_i - \epsilon_j) = s(\epsilon_i - \epsilon_{i_0} + \epsilon_{i_0} - \epsilon_j) = s(\epsilon_i - \epsilon_{i_0}) + s(\epsilon_{i_0} - \epsilon_j) = 0 + 0 = 0 \quad \text{and}$$

$$s(\epsilon_i + \epsilon_j) = s(\epsilon_i + \epsilon_{i_0} + \epsilon_j - \epsilon_{i_0}) = s(\epsilon_i + \epsilon_{i_0}) + s(\epsilon_j - \epsilon_{i_0}) = 1 + 0 = 1.$$

Let

$$T(D) = \bigoplus_{(\alpha, m) \in (C_\mathfrak{J} \cup \{0\}) \times \mathbb{Z}} T_\alpha^m, \quad \text{and} \quad P_\alpha^m := T_\alpha^{m+s(\alpha)}.$$

Then we have

$$P^{2m}_{\epsilon_i-\epsilon_j} = T^{2m+s(\epsilon_i-\epsilon_j)}_{\epsilon_i-\epsilon_j} = T^{2m}_{\epsilon_i-\epsilon_j} = \mathfrak{g}^D_{\epsilon_i-\epsilon_j} \otimes t^{2m}$$

$$P^{2m-1}_{\epsilon_i-\epsilon_j} = T^{2m-1+s(\epsilon_i-\epsilon_j)}_{\epsilon_i-\epsilon_j} = T^{2m-1}_{\epsilon_i-\epsilon_j} = V_{\epsilon_i-\epsilon_j} \otimes t^{2m-1}$$

$$P^{2m}_{\epsilon_i+\epsilon_j} = T^{2m+s(\epsilon_i+\epsilon_j)}_{\epsilon_i+\epsilon_j} = T^{2m+1}_{\epsilon_i+\epsilon_j} = \mathfrak{g}^C_{\epsilon_i+\epsilon_j} \otimes t^{2m+1} = (\mathfrak{g}^C_{\epsilon_i+\epsilon_j} \otimes t) \otimes t^{2m}$$

$$P^{2m-1}_{\epsilon_i+\epsilon_j} = T^{2m-1+s(\epsilon_i+\epsilon_j)}_{\epsilon_i+\epsilon_j} = T^{2m}_{\epsilon_i+\epsilon_j} = \mathfrak{g}^D_{\epsilon_i+\epsilon_j} \otimes t^{2m} = (\mathfrak{g}^D_{\epsilon_i+\epsilon_j} \otimes t) \otimes t^{2m-1}$$

$$P^{2m}_{2\epsilon_i} = T^{2m+s(2\epsilon_i)}_{2\epsilon_i} = T^{2m+1}_{2\epsilon_i} = \mathfrak{g}^C_{2\epsilon_i} \otimes t^{2m+1} = (\mathfrak{g}^C_{2\epsilon_i} \otimes t) \otimes t^{2m}$$

$$P^{2m-1}_{2\epsilon_i} = T^{2m-1+s(2\epsilon_i)}_{2\epsilon_i} = T^{2m}_{2\epsilon_i} = 0$$

$$P^{2m}_0 = T^{2m+s(0)}_0 = T^{2m}_0 = \mathfrak{h} \otimes t^{2m}$$

$$P^{2m-1}_0 = T^{2m-1+s(0)}_0 = T^{2m-1}_0 = V_0 \otimes t^{2m-1},$$

and in particular, the subalgebra P^0 of $T(D)$ has the following decomposition:

$$P^0 = \bigoplus_{\xi \in \mathrm{C}_{\mathfrak{J}} \cup \{0\}} P^0_\xi = \bigoplus_{\xi \in \mathrm{D}'_{\mathfrak{J}}} \mathfrak{g}^D_\xi \oplus \bigoplus_{\xi \in \mathrm{C}'^+_{\mathfrak{J}}} \mathfrak{g}^C_{\pm\xi} \otimes t^{\pm 1} \oplus \mathfrak{h},$$

where $\mathrm{C}'^+_{\mathfrak{J}} = \{\epsilon_i + \epsilon_j \mid i, j \in \mathfrak{J}\}$. We can see that P^0 is isomorphic to $\mathfrak{g}^C$ through

$$\mathfrak{g}^C_{\pm\xi} \otimes t^{\pm 1} \ni x_\pm \otimes t^{\pm 1} \;\mapsto\; x_\pm \in \mathfrak{g}^C_{\pm\xi}$$

for $\xi \in \mathrm{C}'^+_{\mathfrak{J}}$ and the identity map for the other root spaces. Moreover,

$$T(D)^{(s)} = \bigoplus_{(\alpha,m) \in (\mathrm{C}_{\mathfrak{J}} \cup \{0\}) \times \mathbb{Z}} P^m_\alpha$$

is graded isomorphic to $T(C)$ through

$$P^{2m}_{\pm(\epsilon_i+\epsilon_j)} = \mathfrak{g}^C_{\pm(\epsilon_i+\epsilon_j)} \otimes t^{2m\pm 1} \ni x_\pm \otimes t^{2m\pm 1} \;\mapsto\; x_\pm \otimes t^{2m} \in T(C)^{2m}_{\pm(\epsilon_i+\epsilon_j)}$$

$$P^{2m\mp 1}_{\pm(\epsilon_i+\epsilon_j)} = \mathfrak{g}^D_{\pm(\epsilon_i+\epsilon_j)} \otimes t^{2m} \ni x_\pm \otimes t^{2m} \;\mapsto\; x_\pm \otimes t^{2m\mp 1} \in T(C)^{2m\mp 1}_{\pm(\epsilon_i+\epsilon_j)}$$

$$P^{2m}_{\pm 2\epsilon_i} = \mathfrak{g}^C_{\pm 2\epsilon_i} \otimes t^{2m\pm 1} \ni x_\pm \otimes t^{2m\pm 1} \;\mapsto\; x_\pm \otimes t^{2m} \in T(C)^{2m}_{\pm 2\epsilon_i}$$

and the identity map for the other homogeneous spaces. Thus $T(D)$ is isotopic to $T(C)$. (In particular, $T(D)^{(s)}$ is a normal locally Lie torus of type $\mathrm{C}_{\mathfrak{J}}$.) $\qquad\square$

4. Relation between general and normal Lie tori

In Example 3.1 we learned how to get a normal Lie torus from a general Lie torus. The process can be generalized. Let us first recall reflectable sections (see [Y4]).

Let (Δ, V) be a locally finite irreducible root system. We define σ_α for $\alpha \in \Delta$ by

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

for $\beta \in \Delta$.

A basis Π of V as a vector space is called a **reflectable base** of Δ if $\Pi \subset \Delta$ and for any $\alpha \in \Delta^{\mathrm{red}}$,

$$\alpha = \sigma_{\alpha_1} \cdots \sigma_{\alpha_k}(\alpha_{k+1})$$

for some $\alpha_1, \ldots, \alpha_{k+1} \in \Pi$. (Any root can be obtained by reflecting a root of Π relative to the hyperplanes determined by Π.) This is a well-known property of a root base in a reduced finite root system. It is known that a locally finite irreducible root system which is countable has a root base, but this is not the case for uncountable ones (see [LN1, §6]). However, it is proved in [LN2, Lem.5.1] that there exists a reflectable base in a reduced locally finite irreducible root system even if it is uncountable.

Definition 4.1. Let $\mathbb{Z}^{(\mathfrak{I})}$ be a free abelian group of rank $\mathfrak{I}$ and let G be an arbitrary abelian group. Let $\mathfrak{B} = \{\mu_i\}_{i \in \mathfrak{I}}$ be a basis of $\mathbb{Z}^{(\mathfrak{I})}$. Fix some $g_i \in G$ for $i \in \mathfrak{I}$. Then the group homomorphism $s \in \hom(\mathbb{Z}^{(\mathfrak{I})}, G)$ defined by $s(\mu_i) = g_i$ for all i is called the **shift** relative to $\mathfrak{B}$ and $\{g_i\}_{i \in \mathfrak{I}}$.

Lemma 4.2. *Let Δ be a locally finite irreducible root system, and let*

$$\mathcal{L} = \bigoplus_{\mu \in \Delta \cup \{0\}, \ g \in G} \mathcal{L}_\mu^g$$

be a general Lie G-torus. Then, for $s \in \hom(\langle \Delta \rangle, G)$, we have

$$L_\alpha^{s(\alpha)} \neq 0 \ \ and \ \ L_\beta^{s(\beta)} \neq 0 \ \implies \ L_{\sigma_\alpha(\beta)}^{s(\sigma_\alpha(\beta))} \neq 0,$$

and moreover,

$$L_{\alpha_1}^{s(\alpha_1)} \neq 0, \ \ldots, \ L_{\alpha_k}^{s(\alpha_k)} \neq 0 \ \ and \ \ L_{\alpha_{k+1}}^{s(\alpha_{k+1})} \neq 0 \ \implies \ L_\alpha^{s(\alpha)} \neq 0,$$

where $\alpha = \sigma_{\alpha_1} \cdots \sigma_{\alpha_k}(\alpha_{k+1})$.

Proof. For $(\alpha, g), (\beta, g') \in \Delta \times G$, we define

$$\sigma_{(\alpha, g)}\big((\beta, g')\big) := \big(\sigma_\alpha(\beta), \ g' - \langle \beta, \alpha \rangle g\big).$$

Suppose that there exists $0 \neq x \in L_\alpha^g$. Take $y \in L_{-\alpha}^{-g}$ such that $\{x, y, [x, y]\}$ is an sl_2-triple, and let $\theta_\alpha^g := \exp(\mathrm{ad}\, x) \exp(-\mathrm{ad}\, y) \exp(\mathrm{ad}\, x)$. Then θ_α^g is an automorphism of L, satisfying that $\theta_\alpha^g(L_\beta^{g'}) = L_{\sigma_\alpha(\beta)}^{g' - \langle \beta, \alpha \rangle g}$. (One can prove this by the same way as in [AABGP, Prop.1.27].) In particular, $L_\alpha^g \neq 0$ and $L_\beta^{g'} \neq 0$ implies that $L_{\sigma_\alpha(\beta)}^{g' - \langle \beta, \alpha \rangle g} \neq 0$. Thus we have

$$L_\alpha^{s(\alpha)} \neq 0 \ \ and \ \ L_\beta^{s(\beta)} \neq 0 \ \implies \ L_{\sigma_\alpha(\beta)}^{s(\sigma_\alpha(\beta))} = L_{\sigma_\alpha(\beta)}^{s(\beta) - \langle \beta, \alpha \rangle s(\alpha)} \neq 0.$$

So, for the last assertion, it is true for $k = 1$. But if $L_\beta^{s(\beta)} \neq 0$, where $\beta = \sigma_{\alpha_2} \cdots \sigma_{\alpha_k}(\alpha_{k+1})$, then we have $0 \neq L_{\sigma_{\alpha_1}(\beta)}^{s(\beta) - \langle \beta, \alpha \rangle s(\alpha_1)} = L_{\sigma_{\alpha_1}(\beta)}^{s(\sigma_{\alpha_1}(\beta))} = L_\alpha^{s(\alpha)}$. $\quad\square$

Theorem 4.3. *Any general locally Lie G-torus L of type Δ is isotopic to a normal Lie P-torus, where P is a subgroup of G.*

Proof. Let $\Pi = \{\mu_i\}_{i \in \mathfrak{I}}$ be a reflectable base of Δ. Then there exist $g_i \in G$ for all $i \in \mathfrak{I}$ so that $L_{\mu_i}^{g_i} \neq 0$ for all $i \in \mathfrak{I}$ by the axiom (LT6)′ of a general Lie G-torus. Using the shift $s \in \hom(\mathbb{Z}^{(\mathfrak{I})}, G)$ defined by $s(\mu_i) = g_i$ for all $i \in \mathfrak{I}$, one gets the s-isotope $L^{(s)}$. Now, we have $\Pi \times 0 \subset \mathrm{supp}_{\langle \Delta \rangle \times G} L^{(s)}$ since $(L^{(s)})_{\mu_i}^0 = L_{\mu_i}^{s(\mu_i)} = L_{\mu_i}^{g_i} \neq 0$. Then since Π is reflectable, we get $(L^{(s)})_\alpha^0 = L_\alpha^{s(\alpha)} \neq 0$ for all $\alpha \in \Delta^{\mathrm{red}}$, by Lemma 4.2. Next, for $(\alpha, g) \in \langle \Delta \rangle \times G$, we have $\dim(L^{(s)})_\alpha^g = \dim L_\alpha^{g+s(\alpha)} \leq 1$. Hence we have the 1-dimensionality (LT5). Also, since $s(-\alpha) = -s(\alpha)$, (LT3) holds.

Finally, since L_0 and L_α are unchanged by taking isotopes, the property $(L^{(s)})_0 = \sum_{\alpha \in \Delta}[(L^{(s)})_\alpha, (L^{(s)})_{-\alpha}]$ holds. Thus let P be the subgroup of G generated by $\mathrm{supp}_G L^{(s)}$. Then $L^{(s)}$ is a normal P-torus, and so L is isotopic to the normal P-torus. $\qquad\square$

Remark 4.4. A locally Lie n-torus L is always a free module over the centroid (see [BN]). Thus by Theorem 4.3, a general locally Lie n-torus has the same property. We call the **central rank** of L the rank of the free module L over the centroid.

Definition 4.5. We call a symmetric invariant bilinear form on a Lie algebra L simply a *form*. Here 'invariant' is in the sense that $([x,y],z) = (x,[y,z])$ for all $x,y,z \in L$. Note that if a Δ-graded Lie algebra $L = \oplus_{\mu \in \Delta \cup \{0\}} L_\mu$ has a form $(\cdot,\cdot)$, then $(L_\mu, L_\nu) = 0$ unless $\mu + \nu = 0$ for $\mu, \nu \in \Delta \cup \{0\}$.

Moreover, if L is G-graded and a form satisfies the property that $(L^g, L^k) = 0$ unless $g + k = 0$ for all $g, k \in G$, then the form is called a **graded form**.

The existence of a graded form on a Lie n-torus is shown in [Y3]. Thus we have:

Corollary 4.6. *Any general Lie n-torus admits a nonzero graded form.*

Proof. It follows from the implication $\mu + \nu = 0 \implies s(\mu) + s(\nu) = 0$. $\qquad\square$

We do not know the existence of a nonzero graded form for a general locally Lie G-torus, but if G is torsion-free, it will be affirmative.

5. Naoi tori

We introduce a Naoi torus defined in [Na]. (We slightly modified for our convenience.)

Definition 5.1. A $\mathbb{Z}^n$-graded Lie algebra $L = \oplus_{g \in \mathbb{Z}^n} L^g$ is called a *Naoi torus* if the following conditions are satisfied:
(N1) L is graded simple.
(N2) The central grading group has rank n.
(N3) L admits a nondegenerate graded form $(\ , \ )$.
(N4) There exists an ad-diagonalizable subalgebra $\mathfrak{h} \subset L^0$ ($\mathfrak{h}$ is automatically abelian) such that
$$L^0 \cap L_0 = \mathfrak{h}$$
and the set of roots
$$\Delta := \{0 \neq \mu \in \mathfrak{h}^* \mid L_\mu \neq 0\},$$
where $L_\mu = \{x \in L \mid [h,x] \in \mu(h)x \text{ for all } h \in \mathfrak{h}\}$, forms a finite irreducible root system in the vector space $\mathrm{span}_\mathbb{Q}\Delta$ spanned by Δ over $\mathbb{Q}$ relative to the induced form by scaling the above graded form on L.

More precisely, by the property $L^0 \cap L_0 = \mathfrak{h}$, the restriction of the nondegenerate graded form on $\mathfrak{h}$ is still nondegenerate. Thus for $\mu \in \mathfrak{h}^*$, one can define a unique element t_μ in $\mathfrak{h}$ so that $(t_\mu, h) = \mu(h)$ for all $h \in \mathfrak{h}$. (Note that $t_0 = 0$.) Then one can define a symmetric bilinear form on $\mathrm{span}_\mathbb{Q}\Delta$ as $(\mu, \nu) := (t_\mu, t_\nu)$ for $\mu, \nu \in \mathfrak{h}^*$. Thus the latter part of (N4) says that after scaling the graded form, the symmetric bilinear form on $\mathrm{span}_\mathbb{Q}\Delta$ becomes positive definite on $\mathbb{Q}$, and Δ is a finite irreducible root system in $\mathrm{span}_\mathbb{Q}\Delta$ relative to this positive definite form. In particular, $\mathfrak{h}$ is finite-dimensional.

Any centerless normal Lie n-torus of finite rank is clearly a Naoi torus. Also, any of the Lie algebras in Example 3.1 is a Naoi torus.

Lemma 5.2. *A Naoi torus $L = \bigoplus_{g \in \mathbb{Z}^n} L^g$ has the docompostion $L^g = \oplus_{\mu \in \Delta \cup \{0\}} L^g_\mu$, where $L^g_\mu = L_\mu \cap L^g$. In particular L has the double grading*

$$L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{g \in \mathbb{Z}^n} L^g_\mu,$$

and $L_\mu = \bigoplus_{g \in \mathbb{Z}^n} L^g_\mu$. Moreover, we have

$$[x, y] = (x, y) t_\mu$$

for $x \in L^g_\mu$ and $y \in L^{-g}_{-\mu}$. In particular, $[x, y] = 0$ for $x \in L^g_0$ and $y \in L^{-g}_0$.

Proof. The first assertion is clear since each L^g is an $\mathfrak{h}$-weight module. For the next assertion, we have $([x, y] - (x, y)t_\mu, h) = ([x, y], h) - (x, y)(t_\mu, h) = (x, [y, h]) - \mu(h)(x, y) = 0$ for all $h \in \mathfrak{h}$. Hence $[x, y] = (x, y)t_\mu$. The last assertion follows from $t_0 = 0$. $\qquad \square$

Theorem 5.3. *Any Naoi torus is a centerless general Lie n-torus of finite central rank.*

Conversely, any centerless general Lie n-torus of finite central rank is a Naoi torus.

Proof. Let L be a Naoi torus. Then, by Lemma 5.2,

$$L' = \bigoplus_{\mu \in \Delta} L_\mu \oplus \sum_{\mu \in \Delta} [L_\mu, L_{-\mu}]$$

is a graded ideal of L, and hence we get $L = L'$. In particular, $L_0 = \sum_{\mu \in \Delta} [L_\mu, L_{-\mu}]$. Thus (LT2) holds. Next, for $0 \neq x \in L^g_\mu$, let y be an element in $L^{-g}_{-\mu}$ such that $(x, y) = \frac{2}{(\mu, \mu)}$. Then by Lemma 5.2, letting $\mu^\vee := [x, y] = \frac{2t_\mu}{(\mu, \mu)}$ and for $z \in L^k_\nu$, we have $[\mu^\vee, z] = \nu(\mu^\vee)z = \frac{2\nu(t_\mu)}{(\mu, \mu)}z = \frac{2(\nu, \mu)}{(\mu, \mu)}z = \langle \nu, \mu \rangle z$, and hence (LT3) holds. For the 1-dimesionality (LT5), we first claim that for $0 \neq y \in L^{-g}_{-\mu}$, the linear map

$$\mathrm{ad}\, y : L^g_\mu \longrightarrow Ft_\mu$$

is injective. In fact, suppose $(x', y)t_\mu = 0$ for $x' \in L^g_\mu$. Then we have $(x', y) = 0$, and so $[y, x'] = 0$. But note that $\{x, y, \mu^\vee\}$ is an sl_2-triple, and $[\mu^\vee, x'] = 2x'$. So the identity $[y, x'] = 0$ cannot happen by the sl_2-theory unless $x' = 0$. Hence our claim is settled. Then we have $\dim L^g_\mu \leq \dim Ft_\mu = 1$. Finally, note that the rank of the central grading group of a Naoi torus is equal to the rank of the grading group. Hence the central rank of a Naoi torus is finite.

The converse is clear since a general Lie n-torus is isotopic to a normal Lie torus (see also Corollary 4.6). $\qquad \square$

6. Reflection spaces

Let

$$S_\alpha := \mathrm{supp}_G L_\alpha = \{g \in G \mid L^g_\alpha \neq 0\} \subset \mathrm{supp}_G L$$

for a locally general Lie G-torus L of type Δ. This set for the normal case was classified in [Y2], but the set is quite wild compared with the normal case.

First since the reflections by the locally extended affine root system via $(\alpha, s) \leftrightarrow \alpha + s$, acts on supp L, we have, for $s \in S_\alpha$,

$$\sigma_{\alpha+s}(\alpha + s) = \alpha + s - \langle \alpha + s, \alpha + s \rangle(\alpha + s) = -\alpha - s \in \text{supp } L.$$

Thus $-s \in S_{-\alpha}$, and so $-S_\alpha \subset S_{-\alpha}$. Similarly, we have $-S_{-\alpha} \subset S_\alpha$, and hence

$$-S_\alpha = S_{-\alpha}. \tag{1}$$

Also, we have

$$\sigma_{\alpha+t}(\alpha + s) = \alpha + s - \langle \alpha + s, \alpha + t \rangle(\alpha + t) = -\alpha + s - 2t \in \text{supp } L,$$

and hence $s - 2t \in S_{-\alpha}$ for all $s, t \in S_\alpha$. Thus $S_\alpha - 2S_\alpha \subset S_{-\alpha}$, and by (1), we get

$$2S_\alpha - S_\alpha \subset S_\alpha \tag{2}$$

for all $\alpha \in \Delta$. We note that (2) is quite different from

$$S_\alpha - 2S_\alpha \subset S_\alpha. \tag{3}$$

We call a subset satisfying (2) a **reflection space** of G, and satisfying (3) a **symmetric reflection space** of G (though we used (3) to be the definition of a **reflection space** of G in [Y2] and [NY]). Also, we call **full** if such a subset generates G. A reflection space E is called **pointed** if $0 \in E$.

We will not classify the family $\{S_\alpha \mid \alpha \in \Delta\}$ of supports, which is maybe difficult to classify. We only concentrate on each reflection space S_α.

Example 6.1. If $G = \mathbb{Z}$, then $p\mathbb{Z} + e$ for any $p, e \in \mathbb{Z}$ is a reflection space, and it is full if and only if $(p, e) = 1$. Note that any singleton $\{e\}$ is a reflection space. On the other hand, a symmetric reflection space of $\mathbb{Z}$ is just $p\mathbb{Z}$ or $p(2\mathbb{Z}+1)$, and $2\mathbb{Z}+1$ and $\mathbb{Z}$ are the only full symmetric reflection spaces of $\mathbb{Z}$. (We will summarize these in Proposition 6.9.)

Example 6.2. Let A be a matrix. Then the solution space to the system $A\mathbf{x} = \mathbf{b}$ of equations is a reflection space. In fact, if $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b}$, then $A(2\mathbf{x} - \mathbf{y}) = 2A\mathbf{x} - A\mathbf{y} = 2\mathbf{b} - \mathbf{b} = \mathbf{b}$. Hence $2\mathbf{x} - \mathbf{y}$ is also a solution.

The following lemmas are basic.

Lemma 6.3. *If E is a reflection space of G, then $-E$ and $E + g$ are also reflection spaces for any $g \in G$. In particular, if $e \in E$, then $E - e$ for any $e \in E$ is a pointed reflection space.*

Proof. For $x, y \in E$, we have $2(-x) - (-y) = -(2x - y) \in -E$. We also have $2(x + g) - (y + g) = 2x - y + g \in E + g$. $\square$

Lemma 6.4. *Let E be a symmetric reflection space of G. Then $-E = E$. Hence $E + 2E \subset E$ and $2E - E \subset E$. Thus a symmetric reflection space is a reflection space.*

Proof. For $x \in E$, we have $x - 2x = -x \in E$. Hence $-E \subset E$. Thus $E \subset -E$. $\square$

Lemma 6.5. *Let E be a reflection space of G. Then*

$$E \text{ is pointed} \implies E \text{ is a symmetric reflection space.}$$

Hence, a pointed reflection space is a pointed symmetric reflection space.

Proof. Since $0 \in E$, we get $-E \subset E$. Hence $E - 2E = -(2E - E) \subset E$. $\square$

Lemma 6.6. *Let E be a subset of G. For any $e, e' \in E$, we have*

$$\langle E - e \rangle = \langle E - e' \rangle,$$

where the bracket $\langle A \rangle$ denotes the subgroup generated by a subset A of G.

Proof. For $x \in E$, we have $x - e, e' - e \in E - e$. Hence $x - e' = x - e - (e' - e) \in \langle E - e \rangle$, and so $\langle E - e' \rangle \subset \langle E - e \rangle$. Similarly, we have $\langle E - e \rangle \subset \langle E - e' \rangle$. $\square$

Lemma 6.7. *Let E be a pointed reflection space of G, and let $e \in E$. Then $\langle e \rangle \subset E$.*

Proof. Since $0 \in E$ (so E is symmetric), we have $\pm 2e = 0 \pm 2e \in E$ and $\pm 3e = \pm(e + 2e) \in E$. Similarly, we have $2me = 0 \pm (2e + \cdots + 2e) \in E$ and $(2m + 1)e = e \pm (2e + \cdots + 2e) \in E$ for all $m \in \mathbb{Z}$. $\square$

More generally, we have:

Lemma 6.8. *Let E be a symmetric reflection space of G. Suppose that $\{e_i\}_{i \in \mathfrak{I}} \subset E$, where $\mathfrak{I}$ is any index set. Then $E + 2\langle e_i \rangle_{i \in \mathfrak{I}} \subset E$. Hence, $E + 2\langle E \rangle = E$.*

Proof. Let $x \in E + 2\langle e_i \rangle_{i \in \mathfrak{I}}$. Then $x = e + 2\sum_{j=1}^n \epsilon_j e_{i_j}$, where $\epsilon_j = 1$ or -1, and $e_{i_j} \in \{e_i\}_{i \in \mathfrak{I}}$. Thus $x = e + 2\epsilon_1 e_{i_1} + \cdots + 2\epsilon_n e_{i_n} \in E$, inductively. (Note that $-e_{i_j} \in E$ by Lemma 6.4). $\square$

Now we classify reflection spaces of $\mathbb{Z}$.

Proposition 6.9. *Let E be a subset of $\mathbb{Z}$. Then*

$$E \text{ is a pointed reflection space} \implies E = p\mathbb{Z} \tag{4}$$

for some $p \in \mathbb{Z}_{\geq 0}$. So a pointed reflection space of $\mathbb{Z}$ is just a subgroup of $\mathbb{Z}$.
Moreover,

$$E \text{ is a reflection space} \implies E = p\mathbb{Z} + e \tag{5}$$

for any $e \in E$ and some $p \in \mathbb{Z}_{\geq 0}$, and

$$E \text{ is a symmetric reflection space} \implies E = p\mathbb{Z} \text{ or } p(2\mathbb{Z} + 1). \tag{6}$$

Proof. Since $\langle E \rangle$ is a subgroup of $\mathbb{Z}$, we have $\langle E \rangle = p\mathbb{Z}$ for some $p \in \mathbb{Z}_{\geq 0}$. Thus it is enough to show that $p \in E$, by Lemma 6.7. Let $p = \sum_i e_i$ for $e_i \in E \subset p\mathbb{Z}$. If all $e_i = 2pk_i$ for $k_i \in \mathbb{Z}$, then $p = 2p\sum_i k_i$, and so $1 = 2\sum_i k_i$, which is absurd. Hence for some j, we have $e_j = p(2k_j + 1) \in E$. Since $pk_j \in \langle E \rangle$, we have, by Lemma 6.8, $p = p(2k_j + 1) - 2pk_j \in E$.

For (5), note that $E - e$ for $e \in E$ is a pointed reflection space $\mathbb{Z}$, by Lemma 6.3. Hence by (4), we have $E - e = p\mathbb{Z}$ for some $p \in \mathbb{Z}_{\geq 0}$. Thus $E = p\mathbb{Z} + e$.

For (6), we have $E = p\mathbb{Z} + e$, by the same reason above. But if $p = 0$, then e has to be 0, and if $p = 1$, then $E = \mathbb{Z}$. Thus, we may assume $p > 1$ and also $e > 0$. Let $d = (p, e)$ (the greatest common divisor). Then one can write $E = d(p'\mathbb{Z} + e')$ so that $(p', e') = 1$ for some $p' > 1$. Since $-E \subset E$, we have $de' \equiv -de' \pmod{dp' = p}$. Hence $p' \mid 2e'$, and we obtain $p' = 2$. Then one can take e' to be 1. Therefore, $E = d(2\mathbb{Z} + 1)$. $\square$

In general, we have the following.

Proposition 6.10. *Let E be a subset of G. Then*

$$E \text{ is a symmetric reflection space} \implies E = \bigcup_{i=1}^m (2\langle E \rangle + e_i) \tag{7}$$

for some $1 \leq m \leq |\langle E \rangle / 2 \langle E \rangle|$ (possibly infinite) and some $e_i \in E$, and if E is pointed, then some $e_i = 0$.

Moreover,

$$E \text{ is a reflection space} \implies E = \bigcup_{i=1}^{m} (2\langle E - e \rangle + e_i) \qquad (8)$$

for some $1 \leq m \leq \infty$ and any $e \in E$ (see Lemma 6.6), and some $e_i \in E$ (possibly $e_i \notin \langle E - e \rangle$).

Conversely,

(i) *$E = \bigcup_{i=1}^{m} (2S + s_i)$ for any subgroup S of G, $s_i \in S$, and any $1 \leq m \leq |S/2S|$ (possibly infinite) is a symmetric reflection space.*

(ii) *Let $E = \bigcup_{i=1}^{m} (S + x_i)$ for any subgroup S of G and some $x_i \in G$. Suppose also that for all $1 \leq i, j \leq m$ ($1 \leq i, j < \infty$ if $m = \infty$), there exists some $1 \leq k \leq m$ ($1 \leq k < \infty$ if $m = \infty$) such that $S + 2x_i - x_j = S + x_k$. Then E is a reflection space. (If $m = 1$, then E is always a reflection space by Lemma 6.3.)*

Proof. For (7), it follows from Lemma 6.8.

For (8), since $E - e$ is a (pointed) symmetric reflection space, we have, by (7),

$$E - e = \bigcup_{i=1}^{m} (2\langle E - e \rangle + g_i)$$

for some $g_i \in E - e$. So letting $e_i := g_i + e \in E$, we obtain (8).

Conversely, for (i), let $2s + s_i$, $2s' + s_j \in E$. Then $2s + s_i - 2(2s' + s_j) \in 2S + s_i$. Hence E is a symmetric reflection space.

For (ii), let $s + x_i$, $s' + x_j \in E$. Then we have

$$2(s + x_i) - (s' + x_j) = 2s - s' + 2x_i - x_j \in S + x_k,$$

by our assumption. Hence E is a reflection space. $\qquad \square$

Definition 6.11. Let E and E' be reflection spaces of abelian groups G and G', respectively. We say that E is **isomorphic** to E', denoted $E \cong E'$, if there exists a bijection $f : E \longrightarrow E'$ such that $f(2x - y) = 2f(x) - f(y)$ for all $x, y \in E$. In particular, if there exists a group isomorphism $\varphi : G \longrightarrow G'$ such that $\varphi(E) = E'$, then $E \cong E'$.

Lemma 6.12. *Let E be a reflection space of an abelian group G. Then $E + g \cong E$ for any $g \in G$.*

Proof. Let $f : E \longrightarrow E + g$ be the bijection defined by $f(x) = x + g$ for $x \in E$. Then we have $2f(x) - f(y) = 2(x + g) - (y + g) = 2x - y + g = f(2x - y)$, and hence f is an isomorphism. $\qquad \square$

(Note that if $g \neq 0$, the translation f cannot be the restriction of an automorphism of G since $f(0) \neq 0$.)

Example 6.13. (1) For any $(p_1, p_2), (e_1, e_2) \in \mathbb{Z}^2$,

$$E = (p_1 \mathbb{Z} + e_1) \times (p_2 \mathbb{Z} + e_2)$$

is a reflection space of $\mathbb{Z}^2$, which is isomorphic to $p_1 \mathbb{Z} \times p_2 \mathbb{Z}$.

(2) Taking $S = 3\mathbb{Z}^2$ in Proposition 6.10(ii),

$$E_1 = \left(3\mathbb{Z}^2 + (1, 0)\right) \cup \left(3\mathbb{Z}^2 + (0, 1)\right) \cup \left(3\mathbb{Z}^2 + (2, 2)\right) \quad \text{or}$$
$$E_2 = \left(3\mathbb{Z}^2 + (2, 0)\right) \cup \left(3\mathbb{Z}^2 + (0, 1)\right) \cup \left(3\mathbb{Z}^2 + (1, 2)\right)$$

is a reflection space of $\mathbb{Z}^2$.

(3) Taking $S = 6\mathbb{Z}^2$ in Proposition 6.10(ii),

$$E = \left(6\mathbb{Z}^2 + (1,0)\right) \cup \left(6\mathbb{Z}^2 + (0,1)\right) \cup \left(6\mathbb{Z}^2 + (2,5)\right)$$
$$\cup \left(6\mathbb{Z}^2 + (5,2)\right) \cup \left(6\mathbb{Z}^2 + (3,4)\right) \cup \left(6\mathbb{Z}^2 + (4,3)\right)$$

is a reflection space of $\mathbb{Z}^2$.

Definition 6.14. For a subset $A \subset G$, the reflection space generated by A is denoted by $[A]$, and the symmetric reflection space generated by A is denoted by $[A\rangle$. Also, the pointed reflection space generated by A is denoted by $[A\rangle_0$.

More precisely, we have

$$[A] = \{g_1 \cdot (g_2 \cdot \cdots (g_r \cdot g_{r+1})) \cdots) \mid g_i \in A\},$$
$$[A\rangle = \{g_1 \circ (g_2 \cdots \circ (g_r \circ g_{r+1})) \cdots) \mid g_i \in A\},$$
$$\text{and} \quad [A\rangle_0 = [A \cup \{0\}] \ (= [A \cup \{0\}\rangle \text{ by Lemma 6.5}),$$

where $g_i \cdot g_j = 2g_i - g_j$ and $g_i \circ g_j = g_i - 2g_j$.

Example 6.15. (1) Let $E = [6,15]$ be the reflection space generated by 6 and 15 in $\mathbb{Z}$. Since $E = p\mathbb{Z} + e$ for some $p, e \in \mathbb{Z}$, we have $15 - 6 = 9 = pm$ for some $m \in \mathbb{Z}$. So p can be 1, 3, or 9. But E should be the smallest one containing 6 and 15, and hence $E = 9\mathbb{Z} + 6$. Similarly, for $a, b \in \mathbb{Z}$, one can show that

$$[a,b] = (b-a)\mathbb{Z} + a. \tag{9}$$

(2) Let $S = [6,15\rangle$ be the symmetric reflection space generated by 6 and 15 in $\mathbb{Z}$. Since $\langle S \rangle = 3\mathbb{Z}$, we have $S = 3\mathbb{Z}$ or $6\mathbb{Z} + 3$. But $6 \notin 6\mathbb{Z} + 3$, we get $S = 3\mathbb{Z}$.

(3) Let $S = [15,27\rangle$ be the symmetric reflection space generated by 15 and 27 in $\mathbb{Z}$. Since $\langle S \rangle = 3\mathbb{Z}$, we have $S = 3\mathbb{Z}$ or $6\mathbb{Z} + 3$. Since S should be the smallest one, we get $S = 6\mathbb{Z} + 3$. Note that $[6,15\rangle_0 = 3\mathbb{Z}$.

We generalize the formula (9).

Proposition 6.16. *Let $x, y \in G$ for an abelian group G. Then we have*

$$[x,y] = \langle y - x \rangle + x \tag{10}$$

and so $[x,y] - x = \langle x - y \rangle$. Also, we have

$$[x,y] = \langle x - y \rangle + x = \langle x - y \rangle + y. \tag{11}$$

Proof. The right-hand side is a reflection space containing x and y (see Lemma 6.3), and so it is enough to show that $[x,y] \supset \langle y - x \rangle + x$. We show that $m(y-x) + x \in [x,y]$ for all $m \in \mathbb{Z}$. It is clear that $m = 0, \pm 1$. Assume that $|m| > 1$ and we use the induction on m. If m is even, then $|m/2| < |m|$, and so we have $\frac{m}{2}(y-x) + x \in [x,y]$. Hence $m(y-x) + 2x - x = m(y-x) + x \in [x,y]$. If m is odd, then $|\frac{m+1}{2}| < |m|$, and so we have $\frac{m+1}{2}(y-x) + x \in [x,y]$. Hence $(m+1)(y-x) + 2x - y = m(y-x) + y - x + 2x - y = m(y-x) + x \in [x,y]$. Therefore, (10) holds, and (11) is clear since $[x,y] = [y,x]$. $\square$

Example 6.17. Let $L = \mathfrak{g} \otimes F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the multi-loop algebra, where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra. Let e and f be nonzero root vectors of $\mathfrak{g}$ for roots α and $-\alpha$, respectively. Let

$$A := \{x, y\} \subset \mathbb{Z}^n \quad \text{and} \quad U := \{e \otimes t^x, \quad f \otimes t^{-x}, \quad e \otimes t^y, \quad f \otimes t^{-y}\},$$

where t^v means $t_1^{v_1} \cdots t_n^{v_n}$ for $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$.

Let M be the subalgebra of L generated by U. Then M is a general Lie torus of type $A_1 = \{\pm\alpha\}$. Since $\text{supp}_{\mathbb{Z}^n} M_\alpha$ is a reflection space containing A, we have $[A] \subset \text{supp}_{\mathbb{Z}^n} M_\alpha$. For the other inclusion, let $z \in \text{supp}_{\mathbb{Z}^n} M_\alpha$. Since M is generated by U, one can see that $z = (m+1)x - mx = x$, $(m+1)x - my$, $(m+1)y - mx$ or $(m+1)y - my = y$ for some $m \in \mathbb{Z}_{\geq 0}$. But each of them is in $\mathbb{Z}(x-y) + x = [A]$. Hence we have $\text{supp}_{\mathbb{Z}^n} M_\alpha = [A]$. Similarly, we have $\text{supp}_{\mathbb{Z}^n} M_{-\alpha} = -[A]$. Note that $\text{supp}_{\mathbb{Z}^n} M_0 = \text{supp}_{\mathbb{Z}^n} M_\alpha + \text{supp}_{\mathbb{Z}^n} M_{-\alpha} = [A] - [A] = \mathbb{Z}(x-y)$.

Let $s \in \hom(\langle\alpha\rangle, \mathbb{Z}^n)$, defined by $s(\alpha) = x$. Then the s-isotope $M^{(s)}$ is a normal Lie 1-torus. In fact, we have

$$Fe \otimes t^x = M_\alpha^x = (M^{(s)})_\alpha^0 \quad \text{and} \quad Ff \otimes t^{-x} = M_{-\alpha}^{-x} = (M^{(s)})_{-\alpha}^0,$$

and letting $p := y - x$,

$$Fe \otimes t^y = M_\alpha^y = M_\alpha^{p+x} = (M^{(s)})_\alpha^p \quad \text{and} \quad Ff \otimes t^{-y} = M_{-\alpha}^{-y} = M_{-\alpha}^{-p-x} = (M^{(s)})_{-\alpha}^{-p},$$

and so $M^{(s)}$ is isograded isomorphic to the loop algebra $\text{sl}_2(F[X^{\pm 1}])$, where $X = t^p$.

Lemma 6.18. *Let $x, y \in G$ for an abelian group G. Then we have*

$$[x, y\rangle = (2\langle x, y\rangle + x) \cup (2\langle x, y\rangle + y) \tag{12}$$

$$\text{and} \quad [x, y\rangle_0 = 2\langle x, y\rangle \cup [x, y\rangle. \tag{13}$$

Proof. It follows from Proposition 6.10. □

Lemma 6.19. *We have the formula*

$$[x, y\rangle = \langle x - y\rangle + \langle x + y\rangle + x = \langle x - y\rangle + \langle x + y\rangle + y = [x, y] + \langle x + y\rangle. \tag{14}$$

Proof. By (11), it is enough to show the first identity. From (12) and the inclusion

$$2\langle x, y\rangle \subset \langle x - y\rangle + \langle x + y\rangle$$

(since $2mx + 2ny = (m-n)(x-y) + (m+n)(x+y)$ for $m, n \in \mathbb{Z}$), we have $[x, y\rangle \subset \langle x - y\rangle + \langle x + y\rangle + x$. We show the other inclusion. For $X := m(x-y) + n(x+y) + x$, if $m + n$ is even, then $-m + n$ is also even, and so $X = (m+n)x + (-m+n)y + x \in [x, y\rangle$. If $m + n$ is odd, then $m + n + 1$ and $-m + n - 1$ are even, and so

$$X = (m + n + 1)x + (-m + n - 1)y + y \in [x, y\rangle.$$

□

Example 6.20. In the notations in Example 6.17, Let

$$T := U \cup \{e \otimes t^{-x}, \quad f \otimes t^x, \quad e \otimes t^{-y}, \quad f \otimes t^y\}.$$

Let N be the subalgebra of L generated by T. Then N is a general Lie torus of type $A_1 = \{\pm\alpha\}$. Since $\text{supp}_{\mathbb{Z}^n} N_\alpha$ is a reflection space containing $\pm A$, we have $[A\rangle \subset \text{supp}_{\mathbb{Z}^n} N_\alpha$. For the other inclusion, let $z \in \text{supp}_{\mathbb{Z}^n} N_\alpha$. Since N is generated by T, we have $z \in \epsilon_1 + \cdots + \epsilon_{2m+1}$ for some $m \in \mathbb{Z}_{\geq 0}$, where all $\epsilon_i = \epsilon = \{\pm x, \pm y\}$. We show that $\epsilon_1 + \cdots + \epsilon_{2m+1} \subset [A\rangle$, by induction on m. It is clear for $m = 0$. Suppose that the statement is true for m. Let $p \in \epsilon_1 + \cdots + \epsilon_{2m+1}$. Then we have

$$p + \epsilon + \epsilon = \{p, p + 2x, p - 2x, p + 2y, p - 2y, p + x + y, p + x - y, p - x + y, p - x - y\}.$$

The first 5 elements in the right hand side are clearly in $[A\rangle$. The last 4 elements are also in $[A\rangle$, by (14). Hence we have shown $\text{supp}_{\mathbb{Z}^n} N_\alpha = [A\rangle$. Similarly, we have $\text{supp}_{\mathbb{Z}^n} N_{-\alpha} = [A\rangle$. Note that $\text{supp}_{\mathbb{Z}^n} N_0 = \text{supp}_{\mathbb{Z}^n} N_\alpha + \text{supp}_{\mathbb{Z}^n} N_{-\alpha} = [A\rangle + [A\rangle = \langle x + y, \ x - y\rangle \neq \langle A\rangle$.

Let $s \in \hom(\langle \alpha \rangle, \mathbb{Z}^n)$, defined by $s(\alpha) = x$. Then the s-isotope $N^{(s)}$ is a normal Lie 2-torus. In fact, we have

$$Fe \otimes t^x = N_\alpha^x = (N^{(s)})_\alpha^0 \quad \text{and} \quad Ff \otimes t^{-x} = N_{-\alpha}^{-x} = (N^{(s)})_{-\alpha}^0,$$

and letting $p := y - x$ and $q := x + y$,

$$Fe \otimes t^y = N_\alpha^y = N_\alpha^{p+x} = (N^{(s)})_\alpha^p \quad \text{and} \quad Ff \otimes t^{-y} = N_{-\alpha}^{-y} = N_{-\alpha}^{-p-x} = (N^{(s)})_{-\alpha}^{-p},$$

$$Fe \otimes t^{-y} = N_\alpha^{-y} = N_\alpha^{-q+x} = (N^{(s)})_\alpha^{-q} \quad \text{and} \quad Ff \otimes t^y = N_{-\alpha}^y = N_{-\alpha}^{q-x} = (N^{(s)})_{-\alpha}^q,$$

$$Fe \otimes t^{-x} = N_\alpha^{-x} = N_\alpha^{p-q+x} = (N^{(s)})_\alpha^{p-q} \quad \text{and} \quad Ff \otimes t^x = N_{-\alpha}^x = N_{-\alpha}^{-p+q-x} = (N^{(s)})_{-\alpha}^{q-p}.$$

So we have $\operatorname{supp}_{\mathbb{Z}^n} N_\alpha^{(s)} = [0, p, -q, p-q] = \langle p, q \rangle = [0, -p, q, q-p] = \operatorname{supp}_{\mathbb{Z}^n} N_{-\alpha}^{(s)}$. Thus $N^{(s)}$ is isograded isomorphic to the double loop algebra $\mathrm{sl}_2(F[X^{\pm 1}, Y^{\pm 1}])$, where $X = t^p$ and $Y = t^q$.

Remark 6.21. We note that $[A, 0] = [A]_0$ (see Lemma 6.5). However, for the subalgebra N' of L generated by $U \cup \{e, f\}$, which is a normal Lie 2-torus, we have $\operatorname{supp}_{\mathbb{Z}^n} N_\alpha' \neq [A]_0$. In fact, $\operatorname{supp}_{\mathbb{Z}^n} N_\alpha'$ contains $\epsilon_1 + \cdots + \epsilon_m$ for any $m \in \mathbb{Z}_{\geq 0}$, where ϵ_i is defined above. Hence $\operatorname{supp}_{\mathbb{Z}^n} N_\alpha'$ contains $\langle A \rangle$, and therefore, $\operatorname{supp}_{\mathbb{Z}^n} N_\alpha' = \langle A \rangle$.

There are examples that a normal Lie torus W of type A_1 satisfies $\operatorname{supp}_{\mathbb{Z}^n} W_\alpha = [A]_0$. For example, let $W = \mathrm{TKK}\big(H(F_h[t_1^{\pm 1}, t_2^{\pm 1}], *)\big)$ be the Tit-Koecher-Kantor Lie algebra constructed from $H(F_h[t_1^{\pm 1}, t_2^{\pm 1}], *)$, which is the Jordan algebra of the fixed points in the quantum torus $F_h[t_1^{\pm 1}, t_2^{\pm 1}]$ defined by $t_2 t_1 = -t_1 t_2$, by the involution $*$ determined by $t_1^* = t_1$ and $t_2^* = t_2$. Let $x = (1, 0)$, $y = (0, 1)$ and $A = \{x, y\}$. Then $\operatorname{supp}_{\mathbb{Z}^2} W_\alpha = [A]_0$.

7. A NEW DEFINITION OF A LIE TORUS

Let us recall the definition of a locally extended affine root system.

Definition 7.1. Let V be a vector space over $\mathbb{Q}$ with a positive semidefinite symmetric bilinear form $(\cdot, \cdot)$. A subset $\mathfrak{R}$ of V is called a *locally extended affine root system* or a *LEARS* for short if $\mathfrak{R}$ satisfies the following:

(A1) $(\alpha, \alpha) \neq 0$ for all $\alpha \in \mathfrak{R}$, and $\mathfrak{R}$ spans V;

(A2) $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \mathfrak{R}$, where $\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$;

(A3) $\sigma_\alpha(\beta) \in \mathfrak{R}$ for all $\alpha, \beta \in \mathfrak{R}$, where $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$;

(A4) $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2$ and $(\mathfrak{R}_1, \mathfrak{R}_2) = 0$ imply $\mathfrak{R}_1 = \emptyset$ or $\mathfrak{R}_2 = \emptyset$ (irreducibility).

A LEARS $\mathfrak{R}$ is called *reduced* if $2\alpha \notin \mathfrak{R}$ for all $\alpha \in \mathfrak{R}$.

Note that if V is finite-dimensional and $(\cdot, \cdot)$ is positive definite, then $\mathfrak{R}$ is exactly a finite irreducible root system (see [MY1, Prop. 4.2]).

Now we define a new Lie torus determined by a LEARS.

Definition 7.2. Let $(\mathfrak{R}, V)$ be a LEARS, and let $Q := \langle \mathfrak{R} \rangle$ be the root lattice, i.e., the subgroup of V generated by $\mathfrak{R}$. Let

$$L = \bigoplus_{\xi \in Q} L_\xi$$

be a Q-graded Lie algebra over F generated by

$$\bigcup_{\alpha \in \mathfrak{R}} L_\alpha \quad \text{so that} \quad (\operatorname{supp} L)^\times = \mathfrak{R},$$

where $(\operatorname{supp} L)^\times = \{\xi \in \operatorname{supp} L \mid (\xi, \xi) \neq 0\}$.

(1) L is called **division graded** if for each $\alpha \in \mathfrak{R}$ and $0 \neq x \in L_\alpha$, there exists $y \in L_{-\alpha}$ such that

$$\alpha^\vee := [x, y] \in L_0 \quad \text{satisfies} \quad [\alpha^\vee, z] = \langle \xi, \alpha \rangle z$$

for all $z \in L_\xi$ for $\xi \in Q$.

(2) A division graded Lie algebra $L = \oplus_{\xi \in Q} L_\xi$ is called a **Lie $\mathfrak{R}$-torus** if

$$\dim_F L_\alpha = 1$$

for all $\alpha \in \mathfrak{R}$.

Lemma 7.3. *A Lie $\mathfrak{R}$-torus L only exists for a reduced LEARS $\mathfrak{R}$.*

Proof. Suppose $\alpha, 2\alpha \in \mathfrak{R}$. Then we have $\dim L_\alpha = \dim L_{2\alpha} = 1$. Let $A := L_{-\alpha} \oplus L_0 \oplus L_\alpha \cong \mathrm{sl}_2(F)$, and $M := L_{-2\alpha} \oplus L_{-\alpha} \oplus L_0 \oplus L_\alpha \oplus L_{2\alpha}$ the A-submodule of L. Then by the complete reducibility of M, we get $\dim L_\alpha = 2$ since M decomposes the direct sum of two irreducible submodules. This is a contradiction. $\square$

Let

$$V^0 := \{x \in V \mid (x, y) = 0 \text{ for all } y \in V\}$$

be the radical of the form. Note that

$$V^0 = \{x \in V \mid (x, x) = 0\}.$$

We call $\dim_{\mathbb{Q}} V^0$ the *null dimension* of $\mathfrak{R}$, which can be any cardinality. We call a LEARS $(\mathfrak{R}, V)$ an *extended affine root system* or an *EARS* for short if

$$\dim_{\mathbb{Q}} V/V^0 < \infty \text{ and } \langle \mathfrak{R} \rangle \text{ is free.}$$

This coincides with the concept, which was first introduced by Saito in 1985 [S]. The notion of an EARS was also used in a different sense in [AABGP], but Azam showed that there is a natural correspondence between the two notions in [A]. We use here the Saito's one since he is the first person who defined it and his root systems naturally generalize Macdonald's *affine root systems* in [M].

Let $(\mathfrak{R}, V)$ be a LEARS, and $(\bar{\mathfrak{R}}, \bar{V})$ the canonical image onto V/V^0. Then $\bar{V}$ admits the induced positive definite form, and thus

$$(\bar{\mathfrak{R}}, \bar{V}) \text{ is a locally finite irreducible root system.}$$

Now we show the new Lie $\mathfrak{R}$-torus $L = \bigoplus_{\xi \in Q} L_\xi$ is a general Lie torus. Let V' be a section of $\bar{V}$, i.e., $V = V' \oplus V^0$, and let

$$\Delta := \{\dot{\alpha} \in V' \mid \bar{\alpha} \in \bar{\mathfrak{R}}\}.$$

Then (Δ, V') is a locally finite irreducible root system isomorphic to $\bar{\mathfrak{R}}$. For $\dot{\alpha} \in \Delta$, let

$$S_{\dot{\alpha}} := \{v \in V^0 \mid \dot{\alpha} + v \in \mathfrak{R}\},$$

and let

$$G := \left\langle \bigcup_{\dot{\alpha} \in \Delta} S_{\dot{\alpha}} \right\rangle.$$

So for $\xi \in Q$, we have

$$\xi = \sum_{\alpha \in \mathfrak{R}} a_\alpha \alpha = \sum_{\alpha \in \mathfrak{R}} a_\alpha(\dot{\alpha} + g_\alpha) = \sum_{\dot{\alpha} \in \Delta} b_{\dot{\alpha}} \dot{\alpha} + \sum_{\dot{\alpha} \in \Delta} k_{\dot{\alpha}}$$

for $a_\alpha \in \mathbb{Z}$, $g_\alpha \in S_{\dot{\alpha}}$, $b_{\dot{\alpha}} = \sum_{\alpha \in \bar{\dot{\alpha}}} a_\alpha$ and $k_{\dot{\alpha}} = \sum_{\alpha \in \bar{\dot{\alpha}}} a_\alpha g_\alpha$. Note that

$$(\xi, \xi) \neq 0 \Leftrightarrow \text{ the first term } \sum_{\dot{\alpha} \in \Delta} b_{\dot{\alpha}} \dot{\alpha} \neq 0.$$

Note also that

$$\sum_{\dot{\alpha} \in \Delta} b_{\dot{\alpha}} \dot{\alpha} \in \Delta \Leftrightarrow \overline{\sum_{\dot{\alpha} \in \Delta} b_{\dot{\alpha}} \dot{\alpha}} \in \bar{\mathfrak{R}}$$

since $\sum_{\dot{\alpha} \in \Delta} b_{\dot{\alpha}} \dot{\alpha} \in V'$. Therefore, since $(\operatorname{supp} L)^\times = \mathfrak{R}$, we have

$$\xi \in \operatorname{supp} L \Leftrightarrow \sum_{\dot{\alpha} \in \Delta} b_{\dot{\alpha}} \dot{\alpha} \in \Delta \cup \{0\}.$$

Let $\mu := \sum_{\dot{\alpha} \in \Delta} b_{\dot{\alpha}} \dot{\alpha}$ and $g := \sum_{\dot{\alpha} \in \Delta} k_{\dot{\alpha}}$, which are unique for ξ. Then through

$$\xi \in \operatorname{supp} L \longrightarrow (\mu, g) \in (\Delta \cup \{0\}) \times G,$$

we have

$$L = \bigoplus_{\xi \in Q} L_\xi = \bigoplus_{(\mu, g) \in (\Delta \cup \{0\}) \times G} L_\mu^g,$$

where $L_\mu^g = L_{\mu+g}$ if $\mu + g \in Q$ and $L_\mu^g = 0$ otherwise. Then it is clear that L satisfies the axioms of a general Lie G-torus, except (LT2).

We show (LT2). For every $g \in G$, we need to show that $L_0^g \subset \sum_{\mu \in \Delta, \ h \in G} [L_\mu^h, L_{-\mu}^{g-h}]$. (The other inclusion is clear.) Since $L_0^g = L_{0+g}$ is contained in the subalgebra generated by $\bigcup_{\alpha \in \mathfrak{R}} L_\alpha$ and in the degree $(0+g)$-space, we have

$$L_0^g \subset \left(\sum_{\alpha, \beta \in \mathfrak{R}} [L_\alpha, L_\beta] \right)_{0+g} = \sum_{\dot{\alpha} \in \mathfrak{R}, \ g_\alpha \in G} [L_{\dot{\alpha}+g_\alpha}, L_{-\dot{\alpha}+g-g_\alpha}] = \sum_{\mu \in \Delta, \ h \in G} [L_\mu^h, L_{-\mu}^{g-h}].$$

Therefore:

Theorem 7.4. *A Lie $\mathfrak{R}$-torus can be identified with a general Lie G-torus, where G is determined by a section V' above.*

Let Π be a reflectable base of $\bar{\mathfrak{R}}$. For each $\alpha \in \Pi$, let $\dot{\alpha}$ be an element of $\mathfrak{R}$ so that $\bar{\dot{\alpha}} = \alpha$. Let

$$V' := \operatorname{span} \{\dot{\alpha} \in \mathfrak{R} \mid \alpha \in \Pi\}.$$

Since Π is a basis of $\bar{V}$, we have $V = V' \oplus V^0$. We call this V' a **reflectable section** relative to a reflectable base Π (and a choice of $\{\dot{\alpha} \in \mathfrak{R} \mid \alpha \in \Pi\}$). We set $\dot{\alpha} \in V'$ for other $\alpha \in \bar{\mathfrak{R}}$ so that $\bar{\dot{\alpha}} = \alpha$.

Claim 7.5. *If $\bar{\dot{\alpha}} \in \bar{\mathfrak{R}}^{red}$, then $\dot{\alpha} \in \mathfrak{R}$. Hence $0 \in S_{\dot{\alpha}}$.*

Proof. We have $\bar{\dot{\alpha}} = \sigma_{\alpha_1} \cdots \sigma_{\alpha_k}(\alpha_{k+1})$ for some $\alpha_1, \ldots, \alpha_{k+1} \in \Pi$. Then we have

$$\sigma_{\dot{\alpha}_1} \cdots \sigma_{\dot{\alpha}_k}(\dot{\alpha}_{k+1}) \in V' \quad \text{and} \quad \overline{\sigma_{\dot{\alpha}_1} \cdots \sigma_{\dot{\alpha}_k}(\dot{\alpha}_{k+1})} = \bar{\dot{\alpha}}.$$

Hence $\dot{\alpha} = \sigma_{\dot{\alpha}_1} \cdots \sigma_{\dot{\alpha}_k}(\dot{\alpha}_{k+1}) \in \mathfrak{R}$ since each $\dot{\alpha}_i \in \mathfrak{R}$. $\square$

Thus L satisfies (LT6), and hence:

Theorem 7.6. *A Lie $\mathfrak{R}$-torus can be identified with a normal Lie G-torus, where G is determined by a reflectable section.*

Suppose that G is a torsion-free abelian group. Then $\langle \Delta \rangle \times G$ is also a torsion-free abelian group for a locally Lie G-torus $L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{g \in G} L_\mu^g$. Hence $\langle \Delta \rangle \times G$ can embed into the vector space $V := (\langle \Delta \rangle \times G) \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. Then it is easily seen that

$$\mathfrak{R} := \bigcup_{\alpha \in \Delta} (\alpha, \operatorname{supp}_G L_\alpha)$$

is a reduced locally extended affine root system in V, extending the symmetric bilinear form having the radical $(0, G) \otimes_{\mathbb{Z}} \mathbb{Q}$. One can now easily check that L is a Lie $\mathfrak{R}$-torus. Thus we obtain:

Theorem 7.7. *Any Lie G-torus for a torsion-free abelian group G is a Lie $\mathfrak{R}$-torus.*

Remark 7.8. Let G be a free abelian group of rank ∞. Then one can construct a Lie G-torus by TKK construction from a Jordan G-torus, and this can be considered as a Lie A_1^∞-torus, where A_1^∞ is an extended affine root system of type A_1 with nullity ∞ (in the sense of [MY1]). We note that Jordan G-tori for any torsion-free abelian group G have been recently classified in [AYY].

8. Appendix

We compare the original definition of Lie tori in [Y2]. We recall Δ-graded Lie algebras introduced in [BM]. The original definition of a Lie torus is simply a generalization of a Δ-graded Lie algebra.

Definition 8.1. Let Δ be a **finite irreducible root system** and let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta^{\mathrm{red}}} \mathfrak{g}_\mu$ be a finite-dimensional split simple Lie algebra over F of type Δ^{red} with a split Cartan subalgebra $\mathfrak{h}$ and the finite irreducible reduced root system Δ^{red}.

(1) A Δ-graded Lie algebra L over F with grading pair $(\mathfrak{g}, \mathfrak{h})$ is defined as
    (i) L contains $\mathfrak{g}$ as a subalgebra;
    (ii) $L = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu$, where $L_\mu = \{x \in L \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$;
    (iii) $L_0 = \sum_{\mu \in \Delta} [L_\mu, L_{-\mu}]$.
We also assume that

$$L_\mu \neq 0 \quad \text{for all } \mu \in \Delta. \tag{15}$$

(This is automatically true when Δ is reduced.)
(2) A Δ-graded Lie algebra $L = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu$ is called (Δ, G)-*graded* if $L = \bigoplus_{g \in G} L^g$ is a G-graded Lie algebra such that

$$\operatorname{supp} L := \{g \in G \mid L^g \neq 0\} \text{ generates } G, \quad \text{and} \quad \mathfrak{g} \subset L^0.$$

Then we have

$$L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{g \in G} L_\mu^g,$$

where $L_\mu^g = L_\mu \cap L^g$ since L^g is an $\mathfrak{h}$-submodule of L. Note that if $G = \{0\}$, then L is just a Δ-graded Lie algebra.
(3) Let $Z(L)$ be the centre of L and let $\mu^\vee \in \mathfrak{h}$ for $\mu \in \Delta$ be the coroot of μ. Then L is called a *division (Δ, G)-graded Lie algebra* if for any $\mu \in \Delta$ and any $0 \neq x \in L_\mu^g$,

    there exists $y \in L_{-\mu}^{-g}$ such that $[x, y] \equiv \mu^\vee$ modulo $Z(L)$.   (division property)

(4) A division (Δ, G)-graded Lie algebra $L = \oplus_{\mu \in \Delta \cup \{0\}} \ \oplus_{g \in G} \ L_\mu^g$ is called a *Lie G-torus of type* Δ if

$$\dim_F L_\mu^g \leq 1 \text{ for all } g \in G \text{ and } \mu \in \Delta. \quad \text{(1-dimensionality)}$$

If $G = \mathbb{Z}^n$, it is called a *Lie n-torus* or simply a *Lie torus*.

Remark 8.2. The assumption that supp L generates G in (1) is not essential because if supp L of a G-graded algebra L is a proper subset of G, then the subalgebra L' of L generated by the homogeneous spaces of degree in G', where $G' = \langle \text{supp } L \rangle$, is a G'-graded alegebra. Moreover, L' can be identified with L.

What happens if we change the condition $\mathfrak{g} \subset L^0$ in (2) into the condition $\mathfrak{h} \subset L^0$?

For comparison, we call the Lie G-torus a **normal** Lie G-torus, and the Lie G-torus under the assumption $\mathfrak{h} \subset L^0$ a **general Lie G-torus**.

Also, one can easily generalize both concepts based on a finite irreducible root system Δ to the concepts based on a locally finite irreducible root system Δ. Only difference is that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta^{\text{red}}} \mathfrak{g}_\mu$ is a locally finite split simple Lie algebra introduced in [St] if Δ is infinite. When we emphasize that Δ is a locally finite irreducible root system, we say a general **locally** Lie G-torus or a normal **locally** Lie G-torus. But we omit the term 'locally' if there is no confusion, as in Section 2.

Proposition 8.3. *Two definitions in Section 2 and this section of a general (or normal) locally Lie G-torus are equivalent.*

Proof. Let L be a general locally Lie G-torus defined in this section. Then L clearly satisfies (LT1-5) in Section 2. Since $0 \neq \mathfrak{g}_\mu \subset L_\mu$ for all $\mu \in \Delta^{\text{red}}$, (L6)$'$ holds (see (15)). If $\mathfrak{g} \subset L^0$, i.e., L is normal, then $L_\mu^0 = \mathfrak{g}_\mu$ for all $\mu \in \Delta^{\text{red}}$, and so (L6) holds.

Next, suppose that $\mathcal{L}$ is a normal locally Lie G-torus defined in Section 2. From (LT3) and (LT6) we see for $\mu \in \Delta^{\text{red}}$ that there exist elements $e_\mu \in \mathcal{L}_\mu^0$, $f_\mu \in \mathcal{L}_{-\mu}^0$, and $\mu^\vee := [e_\mu, f_\mu]$ so that $[\mu^\vee, z] = \langle \nu, \mu \rangle z$ for all $z \in \mathcal{L}_\nu^h$, $\nu \in \Delta$ and $h \in G$. Thus, the elements $e_\mu, f_\mu, \mu^\vee$ determine a canonical basis for a copy of the Lie algebra $\mathfrak{sl}_2(F)$. The subalgebra $\mathfrak{g}$ of $\mathcal{L}$ generated by the subspaces $\mathcal{L}_\mu^0$ for $\mu \in \Delta^{\text{red}}$ is a locally finite split simple Lie algebra with split Cartan subalgebra

$$\mathfrak{h} := \sum_{\mu \in \Delta^{\text{red}}} [\mathcal{L}_\mu^0, \mathcal{L}_{-\mu}^0]$$

and $\mu^\vee$ are the coroots in $\mathfrak{h}$. (One can show this in the same way as the proof of [MY1, Prop.8.3], or see [St, Sec.III]). Thus $\mathcal{L}$ is a normal locally Lie G-torus defined in this section. (Note that if Δ is finite, then $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra.) Suppose that $\mathcal{L}$ is a general locally Lie G-torus defined in Section 2. By Theorem 4.3, $\mathcal{L}$ is isotopic to a normal locally Lie G-torus, say $\mathcal{L}^{(s)}$, and so $\mathfrak{h} \oplus \bigoplus_{\mu \in \Delta^{\text{red}}} \mathcal{L}_\mu^{s(\mu)}$ is a locally finite split simple Lie algebra, which is a graded subalgebra of $\mathcal{L} = \bigoplus_{\mu \in \Delta \cup \{0\}} \mathcal{L}_\mu$ such that $\mathfrak{h} \subset \mathcal{L}^0$. Thus $\mathcal{L}$ is a general locally Lie G-torus defined in this section. $\square$

References

[A]   S. Azam, *Extended affine root systems*, J. Lie Theory, no. 2, **12** (2002), 515–527.

[AABGP]   B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. 126 (1997), no. 603.

[AB]   B. Allison, G. Benkart, *Unitary Lie algebras and Lie tori of type BC_r, $r \geq 3$*, to appear.

[ABFP]   B. Allison, S. Berman, J. Faulkner, A. Pianzola, *Multiloop realization of extended affine Lie algebras and Lie tori*, Trans. Amer. Math. Soc., **361** (2009), 4807-4842.

[AFY1]   B. Allison, J. Faulkner, Y. Yoshii, *Lie tori of rank 1*, RESENHAS IME-USP, Vol. 6, No. 2/3 (2004), 99–109.

[AFY2]   B. Allison, J. Faulkner, Y. Yoshii, *Structurable tori*, Comm. Algebra, **36**(6) (2008), 2265–2332.

[AG]   B. Allison, Y. Gao, *The root system and the core of an extended affine Lie algebra*, Selecta Mathematica, New Series, **7** (2001), 149–212.

[AF]   B. Allison and J. Faulkner, *Isotopy for extended affine Lie algebras and Lie tori*, Prog. Math., **288** (2008), 3–43.

[AY]   B. Allison, Y. Yoshii, *Structurable tori and extended affine Lie algebras of type BC_1*, Pure Appl. Algebra, no. 2-3, **184** (2003), 105–138.

[AYY]   S. Azam, Y. Yoshii, M. Yousofzadeh *Jordan tori for a torsion free abelian group*, Front. Math. China, Vol.10(3) (2014), 477–509.

[BGK]   S. Berman, Y. Gao, Y. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras*, J. Funct. Anal., **135** (1996), 339–389.

[BGKN]   S. Berman, Y. Gao, Y. Krylyuk, E. Neher, *The alternative tori and the structure of elliptic quasi-simple Lie algebras of type A_2*, Trans. Amer. Math. Soc., **347** (1995), 4315–4363.

[BM]   S. Beerman, R. Moody, *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy*, Invent. Math. 108 (1992), 323–347.

[BN]   G. Benkart, E. Neher, *The centroid of extended affine and root graded Lie algebras*, J. Pure Appl. Algebra, 205 (2006), 117–145.

[BY]   G. Benkart, Y. Yoshii, *Lie G-tori of symplectic type*, Quarterly J. Math Vol.57, no.4 (2006), 425–448.

[F]   J. Faulkner, *Lie tori of type BC_2 and structurable quasitori*, Comm. Algebra, Vol 36(7) (2008), 2593–2618.

[H]   S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, American Mathematical Society, (2001).

[K]   V. Kac *Infinite dimensional Lie algebras*, Cambridge University Press, third edition (1990).

[LN1]   O. Loos, E. Neher, *Locally finite root systems*, Memoirs Amer. Math. Soc., **811** vol.171 (2004).

[LN2]   O. Loos, E. Neher, *Reflections systems and partial root systems*, Forum Math., Vol 23(2) (2011), 349–411.

[M]   M. Macdonald, *Affine root systems and Dedekind's η-functions*, J. Invent. Math., **15** (1972), 91–143.

[MY1]   J. Morita, Y. Yoshii, *Locally extended affine Lie algebras*, J. Algebra 301 (2006), 59–81.

[MY2]   J. Morita, Y. Yoshii, *Locally loop algebras and locally affine Lie algebras*, J. Algebra, to appear.

[Na]   K. Naoi, *Multiloop Lie algebras and the construction of extended affine Lie algebras*, J. Algebra 323 (2010), 2103–2129.

[N]   K.-H. Neeb, *Unitary highest weight modules of locally affine Lie algebras*, Quantum affine algebras, extended affine Lie algebras, and their applications,. Contemp. Math., 506, Amer. Math. Soc., Providence, RI (2010), 227–262.

[Ne]   E. Neher, *Lie tori*, C. R. Math. Rep. Acad. Sci. Canada, **26(3)** (2004), 84–89.

[NS]   K.-H. Neeb, N. Stumme, *The classification of locally finite split simple Lie algebras*, J. reine angew. Math., **533** (2001), 25–53.

[NY]   E. Neher, Y. Yoshii, *Derivations and invariant forms of alternative or Jordan G-tori*, Trans. Amer. Math. Soc. 355 (3) (2002), 1079–1108.

[S]   K. Saito, *Extended affine root systems 1 (Coxeter transformations)*, RIMS., Kyoto Uviv. **21** (1985), 75–179.

[St]   N. Stumme, *The structure of locally finite split Lie algebras*, J. Algebra **220** (1999), 664–693.

[Y1]   Y. Yoshii, *Coordinate algebras of extended affine Lie algebras of type A_1*, J. Algebra, **234** (2000), 128–168.

[Y2]   Y. Yoshii, *Root systems extended by an abelian group and their Lie algebras*, J. Lie Theory **14(2)** (2004), 371–394.

[Y3]   Y. Yoshii, *Lie tori — a simple characterization of extended affine Lie algebras*, RIMS., Kyoto Uviv. **42** (2006), 739–762.

[Y4]   Y. Yoshii, *Locally extended affine root systems*, Proc. on Quantum Affine Algebras, Extended Affine Lie Algebras and Applications, Contemp. Math., **508** (2010), 285–302.

(Yoji Yoshii) Iwate University, 3-18-33 Ueda, Morioka, Iwate, Japan 020-8550

E-mail address: `yoshii@iwate-u.ac.jp`