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Reflection spaces of an abelian group

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We introduce a *reflection space* of an abelian group, which is a generalization of a subgroup.

Let $G = (G, +, 0)$ be an abelian group.

Definition 1 A subset E of G is called a **refection space** of G if

$$2x - y \in E \qquad (\text{or} \quad 2E - E \subset E) \tag{1}$$

for all $x, y \in E$. On the other hand, E is called a **symmetric reflection space** of G if

$$x - 2y \in E \qquad (\text{or} \quad E - 2E \subset E) \tag{2}$$

for all $x, y \in E$. Also, we say that E is **pointed** if $0 \in E$.

(See [LN], [NY], [Y1], [Y2] and [Y3]. In some references a symmetric reflection space is simply called a reflection space.)

For example, if $G = \mathbb{Z}$, then a symmetric reflection space of $\mathbb{Z}$ is just $m\mathbb{Z}$ or $m(2\mathbb{Z}+1)$. On the other hand, $m\mathbb{Z} + e$ for any $m, e \in \mathbb{Z}$ is a reflection space. In particular, any singleton $\{e\}$ is a reflection space.

Lemma 1 *Let E be a symmetric reflection space of G. Then $-E = E$.*

Hence $E + 2E \subset E$ and $2E - E \subset E$. Thus a symmetric reflection space is a reflection space.

Proof) For $x \in E$, we have $x - 2x = -x \in E$. Hence $-E \subset E$. Thus $E \subset -E$. □

Lemma 2 *Let E be a reflection space of G. Then*

$$E \text{ is pointed} \Longrightarrow E \text{ is a symmetric reflection space.}$$

Hence, a pointed reflection space is a pointed symmetric reflection space.

Proof) Since $0 \in E$, we get $-E \subset E$. Hence $E - 2E = -(2E - E) \subset E$. $\Box$

Lemma 3 *Let E be a reflection space of G. Then $E - e$ for any $e \in E$ is a pointed reflection space.*

Proof) We have $2(E - e) - (E - e) = (2E - E) - e \subset E - e$, and so E is a reflection space. It is clear that $0 \in E - e$. $\Box$

Lemma 4 *Let E be a subset of G. For any $e, e' \in E$, we have*

$$\langle E - e \rangle = \langle E - e' \rangle,$$

where the bracket $\langle A \rangle$ means the subgroup generated by a subset A of G.

Proof) For $x \in E$, we have $x - e, e' - e \in E - e$. Hence $x - e' = x - e - (e' - e) \in \langle E - e \rangle$, and so $\langle E - e' \rangle \subset \langle E - e \rangle$. Similarly, we have $\langle E - e \rangle \subset \langle E - e' \rangle$. $\Box$

Lemma 5 *Let E be a pointed reflection space of G, and let $e \in E$. Then $\langle e \rangle \subset E$.*

Proof) Since $0 \in E$ (so E is symmetric), we have $\pm 2e = 0 \pm 2e \in E$ and $\pm 3e = \pm(e + 2e) \in E$. Similarly, we have $2me = 0 \pm (2e + \cdots + 2e) \in E$ and $(2m+1)e = e \pm (2e + \cdots + 2e) \in E$ for all $m \in \mathbb{Z}$. $\Box$

More generally, we have:

Lemma 6 *Let E be a symmetric reflection space of G. Suppose that $\{e_i\}_{i \in \mathfrak{I}} \subset E$, where $\mathfrak{I}$ is any index set. Then $E + 2\langle e_i \rangle_{i \in \mathfrak{I}} \subset E$. Hence, $E + 2\langle E \rangle = E$.*

Proof) Let $x \in E + 2\langle e_i \rangle_{i \in \mathfrak{I}}$. Then $x = e + 2\sum_{j=1}^{n} \epsilon_j e_{i_j}$, where $\epsilon_j = 1$ or -1, and $e_{i_j} \in \{e_i\}_{i \in \mathfrak{I}}$. Thus $x = e + 2\epsilon_1 e_{i_1} + \cdots + 2\epsilon_n e_{i_n} \in E$, inductively. (Note that $-e_{i_j} \in E$ by Lemma 1). $\Box$

Let us classify reflection spaces.

Proposition 1 *Let E be a subset of G. Then*

$$E \text{ is a symmetric reflection space} \iff E = \bigcup_{i=1}^{m} (2\langle E \rangle + e_i) \tag{3}$$

for some $1 \leq m \leq |\langle E \rangle / 2\langle E \rangle|$ and some $e_i \in E$, and if E is pointed, then some $e_i = 0$.

Moreover,

$$E \text{ is a reflection space} \iff E = \bigcup_{i=1}^{m} (2\langle E - e \rangle + x_i) \tag{4}$$

for any $e \in G$ (see Lemma 4), and some $x_i \in E$ and $1 \leq m \leq |\langle E - e \rangle / 2\langle E - e \rangle|$.

Proof) For (3), ($\Longleftarrow$) is clear. For the other implication, E contains $2\langle E\rangle$ by Lemma 6. Thus E is a union of cosets in $\langle E\rangle/2\langle E\rangle$.

For (4), ($\Longleftarrow$) is clear. For the other implication, note that $E-e$ for $e\in E$ is a pointed reflection space, by Lemma 3. Hence by (3), we have

$$E-e=\bigcup_{i=1}^{m}(2\langle E-e\rangle+g_i),$$

where $g_i\in E-e$. So letting $x_i:=g_i+e$, we obtain (4). $\Box$

Example 1 A union of cosets in $(m_1\mathbb{Z}\times m_2\mathbb{Z})/(2m_1\mathbb{Z}\times 2m_2\mathbb{Z})$ plus some $(e_1,e_2)\in\mathbb{Z}^2$ for $m_1,m_2\in\mathbb{Z}$ is an example of reflection spaces of $\mathbb{Z}^2$. In particular, $(m_1\mathbb{Z}+e_1)\times(m_2\mathbb{Z}+e_2)$ is a reflection space of $\mathbb{Z}^2$.

Where can we find reflection spaces?

Let us recall *extended affine root systems.*

Definition 2 Let V be a finite-dimensional vector space over $\mathbb{Q}$ with a positive semidefinite symmetric bilinear form $(\cdot,\cdot)$. A subset $\mathfrak{R}$ of V is called an **extended affine root system** if $\mathfrak{R}$ satisfies the following:

(A1) $(\alpha,\alpha)\neq 0$ for all $\alpha\in\mathfrak{R}$, and $\mathfrak{R}$ spans V;

(A2) $\langle\alpha,\beta\rangle\in\mathbb{Z}$ for all $\alpha,\beta\in\mathfrak{R}$, where $\langle\alpha,\beta\rangle=\frac{2(\alpha,\beta)}{(\beta,\beta)}$;

(A3) $\sigma_\alpha(\beta)\in\mathfrak{R}$ for all $\alpha,\beta\in\mathfrak{R}$, where $\sigma_\alpha(\beta)=\beta-\langle\beta,\alpha\rangle\alpha$;

(A4) $\mathfrak{R}=\mathfrak{R}_1\cup\mathfrak{R}_2$ and $(\mathfrak{R}_1,\mathfrak{R}_2)=0$ imply $\mathfrak{R}_1=\emptyset$ or $\mathfrak{R}_2=\emptyset$ (irreducibillity)

(see [MY] and [Y2]).

One can show that if $(\cdot,\cdot)$ is positive definite, then $\mathfrak{R}$ is a **finite irreducible root system** (see [MY]).

Let

$$V^0:=\{x\in V\mid (x,y)=0 \text{ for all } y\in V\}$$

be the radical of the form, and

$$^{-}: V\longrightarrow V/V^0$$

the canonical surjection. Note that $\overline{\mathfrak{R}}$ is a finite irreducible root system.

For $\bar\alpha\in\overline{\mathfrak{R}}$, let $\dot\alpha\in V$ be an inverse image of $\bar\alpha$, i.e., $\overline{\dot\alpha}=\bar\alpha$. Let

$$S_{\dot\alpha}:=\{s\in V^0\mid \dot\alpha+s\in\mathfrak{R}\}.$$

Then we have

$$\sigma_{\dot\alpha+s}(\dot\alpha+s)=\dot\alpha+s-\langle\dot\alpha+s,\dot\alpha+s\rangle(\dot\alpha+s)=-\dot\alpha-s\in\mathfrak{R}.$$

Thus $-s \in S_{-\dot\alpha}$, and so $-S_{\dot\alpha} \subset S_{-\dot\alpha}$. Similarly, we have $-S_{-\dot\alpha} \subset S_{\dot\alpha}$, and hence

$$-S_{\dot\alpha} = S_{-\dot\alpha}. \tag{5}$$

Also, we have

$$\sigma_{\dot\alpha+t}(\dot\alpha + s) = \dot\alpha + s - \langle \dot\alpha + s, \dot\alpha + t\rangle(\dot\alpha + t) = -\dot\alpha + s - 2t \in \mathfrak{R},$$

and hence $s - 2t \in S_{-\dot\alpha}$ for all $s, t \in S_{\dot\alpha}$. Thus $S_{\dot\alpha} - 2S_{\dot\alpha} \subset S_{-\dot\alpha}$, and by (5), we get

$$2S_{\dot\alpha} - S_{\dot\alpha} \subset S_{\dot\alpha},$$

for all $\alpha \in \mathfrak{R}$. Thus, $S_{\dot\alpha}$ is a reflection space of V^0. We note that if we take $\dot\alpha \in \mathfrak{R}$, e.g. $\dot\alpha = \alpha$, then $0 \in S_{\dot\alpha}$, and so $S_{\dot\alpha}$ is a pointed reflection space (see Lemma 2).

Reflection spaces are important not only for root systems but also for Lie algebras. Let us give one simple example. Let F be a field of characteristic $\neq 2$.

Let $\{e, f, h\}$ be a standard basis of the Lie algebra $\mathrm{sl}_2(F)$ so that $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$, having the root system $\{\pm\alpha\}$ relative to Fh, i.e., α is the linear form of Fh such that $\alpha(h) = 2$. Let

$$L := \mathrm{sl}_2(F[t^{\pm 1}]) = \mathrm{sl}_2(F) \otimes F[t^{\pm 1}]$$

be the loop algebra, which is a $(\mathbb{Z}\alpha \times \mathbb{Z})$-graded Lie algebra, defining

$$L^n_\alpha = Fe \otimes t^n, \quad L^n_{-\alpha} = Ff \otimes t^n \quad \text{and} \quad L^n_0 = Fh \otimes t^n$$

for all $n \in \mathbb{Z}$, and all the other homogeneous spaces are 0, i.e., $L^n_{k\alpha} = 0$ for $k \neq \pm 1, 0$. Let

$$M := (e \otimes t^r F[t^{\pm p}]) \oplus (f \otimes t^{-r} F[t^{\pm p}]) \oplus (h \otimes F[t^{\pm p}])$$

be the homogeneous subalgebra of L generated by $e \otimes t^r$, $f \otimes t^{-r}$ and $h \otimes t^{\pm p}$ for $p, r \in \mathbb{Z}$. Let

$$S_{\pm\alpha} := \mathrm{supp}_{\pm\alpha} M = \{n \in \mathbb{Z} \mid M \cap L^n_{\pm\alpha} \neq 0\}$$

be subsets of $\mathbb{Z}$. For $m, k \in S_\alpha$, since

$$[e \otimes t^m, [e \otimes t^m, f \otimes t^{-k}]] \neq 0,$$

we have $2m - k \in S_\alpha$. Thus S_α is a reflection space of $\mathbb{Z}$. Similarly, $S_{-\alpha}$ is a reflection space of $\mathbb{Z}$. Moreover, one can easily see that

$$S_\alpha = p\mathbb{Z} + r \quad \text{and} \quad S_{-\alpha} = p\mathbb{Z} - r.$$

Thus reflection spaces naturally appear in supports of graded subalgebras of a loop algebra (see [Y3] for more general examples).

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