Theorem 1. *Let $f_1, \dots, f_m, g \in \mathbb{R}[x]$, which are mutually coprime, and* $\deg g < \deg(f_1 \cdots f_m)$. *Then there exist $a_i \in \mathbb{R}[x]$ for $i = 1, \dots, m$ with $\deg a_i < \deg f_i$ so that*

$$\frac{g}{f_1 \cdots f_m} = \frac{a_1}{f_1} + \cdots + \frac{a_m}{f_m}.$$

Proof. If $m = 1$, the statement is clear. Assume $m > 1$ and let $h = f_1 \cdots f_{m-1}$. Then, h, f_m and g are still mutually coprime. By a well-known property of $\mathbb{R}[x]$ (or more generally in a principal ideal domain), the coprimeness of h and f_m implies that there exists some $a, b \in \mathbb{R}[x]$ such that $ah + bf_m = g$. Let a_m be the residue of a by f_m, i.e., $a = f_m q + a_m$ for some $q \in \mathbb{R}[x]$ with $\deg a_m < \deg f_m$, and let b' be the residue of b by h, i.e., $b = hp + b'$ for some $p \in \mathbb{R}[x]$ with $\deg b' < \deg h$. Then we have $(f_m q + a_m)h + (hp + b')f_m = g$, and so $g = f_m q h + a_m h + hp f_m + b' f_m = f_1 \cdots f_m q + a_m h + f_1 \cdots f_m p + b' f_m = a_m h + b' f_m$, comparing the degrees. Dividing by $f_1 \cdots f_m$, we get

$$\frac{g}{f_1 \cdots f_m} = \frac{b'}{f_1 \cdots f_{m-1}} + \frac{a_m}{f_m}.$$

Applying for the induction on m, we get the required expression. □

Theorem 2. *Let $f, g \in \mathbb{R}[x]$ with $\deg g < m \deg f$ for some $m > 0$. Then there exist $a_i \in \mathbb{R}[x]$ for $i = 1, \dots, m$ with $\deg a_i < \deg f$ so that*

$$\frac{g}{f^m} = \frac{a_1}{f} + \frac{a_2}{f^2} + \cdots + \frac{a_m}{f^m}.$$

Proof. Let $n = \deg f$. By Division Theorem, we have

$$\begin{aligned}
g &= f^{m-1} a_1 + r_1, \quad \deg a_1 < n, \quad \deg r_1 < n(m-1) \\
r_1 &= f^{m-2} a_2 + r_2, \quad \deg a_2 < n, \quad \deg r_2 < n(m-2) \\
&\cdots \qquad \cdots \qquad \cdots \\
r_{m-3} &= f^2 a_{m-2} + r_{m-2}, \quad \deg a_{m-2} < n, \quad \deg r_{m-2} < 2n \\
r_{m-2} &= f a_{m-1} + r_{m-1}, \quad \deg a_{m-1} < n, \quad \deg r_{m-1} < n.
\end{aligned}$$

Let $a_m := r_{m-1}$. Then

$$g = f^{m-1} a_1 + f^{m-2} a_2 + \cdots + f^2 a_{m-2} + f a_{m-1} + a_m.$$

Dividing both sides by f^m, we get the required expression. □

Remark 1. If $\deg g = 0$, i.e., g is constant in Theorem 2, the expression does not produce anything, i.e., $a_1 = a_2 = \cdots = a_{m-1} = 0$ and $a_m = 1$. However, there is a nontrivial decomposition into its partial fractions. For example,

$$\frac{1}{(1+x)^2} = \frac{1}{1+x} - \frac{x}{(1+x)^2}.$$

Remark 2. Of course, $\mathbb{R}[x]$ can be $F[x]$ for any field F in the two theorems above, or more generally, it can be a euclidean domain.