Canad. Math. Bull. Vol. **XX** (Y), ZZZZ pp. 1–21

Classification of Quantum Tori with Involution

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Abstract. Quantum tori with graded involution appear as coordinate algebras of extended affine Lie algebras of type A_1, C and BC. We classify them in the category of algebras with involution. From this, we obtain precise information on the root systems of extended affine Lie algebras of type C.

Introduction

Let F be a field. A quantum torus $F_{\mathbf{q}}$ is a noncommutative analogue of the algebra of Laurent polynomials over F, determined by a certain $n \times n$ matrix $\mathbf{q}$. Quantum tori appeared in several areas, *e.g.* quantum affine varieties [6], extended affine Lie algebras [5] or quantum physics [7]. In noncommutative geometry or quantum physics, a special type of quantum tori called a *noncommutative torus* is considered (see Remark 1.0).

Our first purpose in this paper is to classify the graded involutions of quantum tori. It is known [1] that the existence of a graded involution of $F_{\mathbf{q}}$ is equivalent to $\mathbf{q}$ being elementary, *i.e.*, all the entries of $\mathbf{q}$ are 1 or -1. We prove that for an elementary $\mathbf{q}$ we have $F_{\mathbf{q}} \cong F_{\mathbf{h}_{l,n}}$, where

$$\mathbf{h}_{l,n} = \overbrace{\mathbf{h} \times \cdots \times \mathbf{h}}^{l\text{-times}} \times \mathbf{1}_{n-2l} \quad \text{and} \quad \mathbf{h} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad \text{(Theorem 1.10)}$$

(see Definition 1.4 for the notation $\times$). Then we classify graded involutions τ of the elementary quantum torus $F_{\mathbf{h}_{l,n}}$. We obtain that the algebra with involution $(F_{\mathbf{h}_{l,n}}, \tau)$ is isomorphic to

$$(F_{\mathbf{h}_{l,n}}, *), \quad (F_{\mathbf{h}_{l,n}}, \tau_1) \quad \text{or} \quad (F_{\mathbf{h}_{l,n}}, \tau_2) \quad \text{(Theorem 2.7)}$$

for three unique involutions $*$, τ_1 and τ_2.

A quantum torus has a natural $\mathbb{Z}^n$-grading. For any graded involution the subset of $\mathbb{Z}^n$, consisting of the degrees in which homogeneous elements are fixed by the involution, is a so-called semilattice, studied in [1]. In Lemma 4.1 we determine the *index*, an invariant of any semilattice [4], for each of the 3 involutions of Theorem 2.7. As a result, the 3 semilattices are pairwise non-similar. Moreover, we introduce a natural similarity invariant of semilattices called *saturation number* (Definition 4.2). Using

Received by the editors January 31, 2001; revised July 23, 2001.
Research supported by a PIMS Postdoctoral Fellowship (2001).
AMS subject classification: 16W50.

this concept, we show that l in the three semilattices above is a similarity invariant. This allows us to complete the classification of semilattices determined by quantum tori with graded involution (Theorem 4.6).

Quantum tori with graded involution appear as coordinate algebras of extended affine Lie algebras of type A_1 in [11], C in [2] and BC in [3]. Isomorphic coordinate algebras give rise to isomorphic extended affine Lie algebras. Thus, our results provide a finer classification of extended affine Lie algebras in the above types. Also, we obtain more precise information on the difference between extended affine root systems and the root systems of extended affine Lie algebras of type C_r for $r \geq 3$ than the one described in [2] (see Corollaries 5.4 and 5.5).

The organization of the paper is as follows. In Section 1 we define elementary quantum tori and classify them. In Section 2 we classify (elementary) quantum tori with involution. In Section 3 we review semilattices. In Section 4 we obtain the classification of semilattices determined by (elementary) quantum tori with involution. In the final section extended affine root systems of type C are reviewed and the difference to the root systems of extended affine Lie algebras of type C is discussed.

This is part of my Ph.D. thesis, written at the University of Ottawa. I would like to thank my supervisor, Professor Erhard Neher, for his encouragement and suggestions.

1 Elementary Quantum Tori

We begin by recalling quantum tori (see [8]). An $n \times n$ matrix $\mathbf{q} = (q_{ij})$ over a field F such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ is called a *quantum data matrix* or simply a *quantum matrix*. (This notion should not be confused with the use of the word "quantum matrix" in quantum algebra, see *e.g.* [9]. But in our argument, no confusion will arise, and so we will simply call the $\mathbf{q}$ a quantum matrix.) The *quantum torus* $F_{\mathbf{q}} = F_{\mathbf{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ *determined by* a quantum matrix $\mathbf{q}$ is defined as the associative algebra over F with $2n$ generators $t_1^{\pm 1}, \dots, t_n^{\pm 1}$, and relations $t_i t_i^{-1} = t_i^{-1} t_i = 1$ and $t_j t_i = q_{ij} t_i t_j$ for all $1 \leq i, j \leq n$. Note that $F_{\mathbf{q}}$ is commutative if and only if $\mathbf{q} = \mathbf{1}$ where all the entries of $\mathbf{1}$ are 1. In this case, the quantum torus $F_{\mathbf{1}}$ becomes the algebra $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of Laurent polynomials.

Remark 1.0 For $F = \mathbb{C}$, if we assume that $|q_{ij}| = 1$ for all i, j, then $\mathbb{C}_{\mathbf{q}}$ is a noncommutative torus [10]. Let $\theta_{ij} \in \mathbb{R}$ be such that $q_{ij} = e^{2\pi i \theta_{ij}}$. Then $\boldsymbol{\theta} = (\theta_{ij})$ is an antisymmetric matrix over $\mathbb{R}$. In noncommutative geometry or quantum physics, one studies the C^*-algebra completion of the quantum torus as defined above (see *e.g.* [10] or [7]).

Let $\Lambda = \Lambda_n$ be the free abelian group of rank n. We give a Λ-grading of the quantum torus $F_{\mathbf{q}} = F_{\mathbf{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ in the following way: For any basis $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\}$ of Λ, we define the degree of

$$t_{\boldsymbol{\alpha}} := t_1^{\alpha_1} \cdots t_n^{\alpha_n} \quad \text{for } \boldsymbol{\alpha} = \alpha_1 \boldsymbol{\sigma}_1 + \cdots + \alpha_n \boldsymbol{\sigma}_n \in \Lambda \text{ as } \boldsymbol{\alpha}.$$

Then $F_{\mathbf{q}} = \bigoplus_{\boldsymbol{\alpha} \in \Lambda} F t_{\boldsymbol{\alpha}}$ becomes a Λ-graded algebra. We call this grading the *toral Λ-grading of $F_{\mathbf{q}}$ determined by* $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$. Sometimes it is referred to as a

$\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$*-grading*. Also, if we write $F_{\mathbf{q}} = \bigoplus_{\boldsymbol{\alpha} \in \Lambda} (F_{\mathbf{q}})_{\boldsymbol{\alpha}}$ or $F_{\mathbf{q}} = \bigoplus_{\boldsymbol{\alpha} \in \Lambda} F t_{\boldsymbol{\alpha}}$, we are assuming some toral Λ-grading of $F_{\mathbf{q}}$. One can check that the multiplication rule in $F_{\mathbf{q}}$ for $\mathbf{q} = (q_{ij})$ is the following: for $\boldsymbol{\beta} = \beta_1 \boldsymbol{\sigma}_1 + \cdots + \beta_n \boldsymbol{\sigma}_n \in \Lambda$,

$$t_{\boldsymbol{\alpha}} t_{\boldsymbol{\beta}} = \prod_{i<j} q_{ij}^{\alpha_j \beta_i} t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}. \tag{1.1}$$

Lemma 1.2 *If* $\varphi \colon F_{\mathbf{q}} = \bigoplus_{\boldsymbol{\alpha} \in \Lambda} F t_{\boldsymbol{\alpha}} \xrightarrow{\sim} F_{\boldsymbol{\eta}} = \bigoplus_{\boldsymbol{\alpha} \in \Lambda} F t_{\boldsymbol{\alpha}}$ *is an isomorphism of algebras, then there exists the induced group automorphism* p *of* Λ *such that* $\varphi(F t_{\boldsymbol{\alpha}}) = F t_{p(\boldsymbol{\alpha})}$ *for all* $\boldsymbol{\alpha} \in \Lambda$.

Proof It is easily seen that the units of any quantum torus with toral grading are nonzero homogeneous elements. Thus, since $\varphi(t_{\boldsymbol{\alpha}})$ is a unit for any $\boldsymbol{\alpha} \in \Lambda$, there exists $p(\boldsymbol{\alpha}) \in \Lambda$ such that $\varphi(F t_{\boldsymbol{\alpha}}) = F t_{p(\boldsymbol{\alpha})}$, and the map $p \colon \Lambda \longrightarrow \Lambda$ is well-defined. It is straightforward to check that p is an automorphism of Λ. ■

For quantum matrices $\mathbf{q}$ and $\boldsymbol{\eta}$, we say that $\mathbf{q}$ *is equivalent to* $\boldsymbol{\eta}$ and denote this by $\mathbf{q} \cong \boldsymbol{\eta}$ if $F_{\mathbf{q}} \cong F_{\boldsymbol{\eta}}$. This is an equivalence relation. Note that $\mathbf{q} \cong \mathbf{1}$ implies $\mathbf{q} = \mathbf{1}$.

If $F_{\mathbf{q}}$ has a toral Λ-grading, the centre $Z(F_{\mathbf{q}})$ of $F_{\mathbf{q}}$ is graded by some subgroup of Λ which we call the *grading subgroup* of $Z(F_{\mathbf{q}})$. If $F_{\mathbf{q}}$ and $F_{\boldsymbol{\eta}}$ each have toral Λ-gradings, we write $F_{\mathbf{q}} \cong_{\Lambda} F_{\boldsymbol{\eta}}$ to mean that $F_{\mathbf{q}}$ and $F_{\boldsymbol{\eta}}$ are isomorphic as Λ-graded algebras.

Lemma 1.3 *Let* $\mathbf{q}$ *and* $\boldsymbol{\eta} = (\eta_{ij})_{1 \le i,j \le n}$ *be quantum matrices, and let* $F_{\mathbf{q}}$ *respectively* $F_{\boldsymbol{\eta}}$ *be the corresponding quantum tori. Then the following are equivalent:*

- (i) $\mathbf{q} \cong \boldsymbol{\eta}$, *i.e.,* $F_{\mathbf{q}} \cong F_{\boldsymbol{\eta}}$ *as algebras,*
- (ii) *for any toral grading of* $F_{\mathbf{q}}$, *there exists a basis* $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$ *of* Λ *and nonzero homogeneous elements* $x_i \in F_{\mathbf{q}}$ *of degree* $\boldsymbol{\sigma}_i$ *such that* $x_j x_i = \eta_{ij} x_i x_j$ *for all* $1 \le i < j \le n$,
- (iii) *for any toral grading of* $F_{\mathbf{q}}$, *there exists a toral grading of* $F_{\boldsymbol{\eta}}$ *such that* $F_{\mathbf{q}} \cong_{\Lambda} F_{\boldsymbol{\eta}}$. *In that case, the grading subgroups of the centres* $Z(F_{\mathbf{q}})$ *and* $Z(F_{\boldsymbol{\eta}})$ *coincide.*

Proof We prove (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i). Suppose that (i) holds, *i.e.*, there exists an isomorphism φ from $F_{\mathbf{q}}$ onto $F_{\boldsymbol{\eta}}$. Give a toral Λ-grading to $F_{\mathbf{q}}$ so that $F_{\mathbf{q}} = \bigoplus_{\boldsymbol{\alpha} \in \Lambda} (F_{\mathbf{q}})_{\boldsymbol{\alpha}}$ and a toral $\langle \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n \rangle$-grading to $F_{\boldsymbol{\eta}} = F_{\boldsymbol{\eta}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ so that $F_{\boldsymbol{\eta}} = \bigoplus_{\boldsymbol{\alpha} \in \Lambda} F t_{\boldsymbol{\alpha}}$. Then, by Lemma 1.2, there exists the induced automorphism p of Λ such that $\varphi\big((F_{\mathbf{q}})_{\boldsymbol{\alpha}}\big) = F t_{p(\boldsymbol{\alpha})}$ for all $\boldsymbol{\alpha} \in \Lambda$. Let $\boldsymbol{\sigma}_i := p^{-1}(\boldsymbol{\varepsilon}_i)$ and $x_i := \varphi^{-1}(t_i) \in (F_{\mathbf{q}})_{\boldsymbol{\sigma}_i}$ for $i = 1, \ldots, n$. Then $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$ is a basis of Λ, and we have

$$x_j x_i = \varphi^{-1}(t_j) \varphi^{-1}(t_i) = \varphi^{-1}(t_j t_i) = \varphi^{-1}(\eta_{ij} t_i t_j) = \eta_{ij} x_i x_j$$

for all $1 \le i < j \le n$. So (ii) holds. Suppose that (ii) holds. Since $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$ is a basis of Λ, one has $F_{\mathbf{q}} = \bigoplus_{\boldsymbol{\alpha} \in \Lambda} F x_{\boldsymbol{\alpha}}$ where $x_{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\boldsymbol{\alpha} = \alpha_1 \boldsymbol{\sigma}_1 + \cdots + \alpha_n \boldsymbol{\sigma}_n$. Define a map $\varphi \colon F_{\mathbf{q}} \longrightarrow F_{\boldsymbol{\eta}} = F_{\boldsymbol{\eta}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ by $\varphi(x_{\boldsymbol{\alpha}}) = t_{\boldsymbol{\alpha}}$ where $t_{\boldsymbol{\alpha}} = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for all $\boldsymbol{\alpha} \in \Lambda$. Then, since $x_j x_i = \eta_{ij} x_i x_j$, φ is an isomorphism of algebras. Moreover, φ is graded if we give the $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$-grading to $F_{\boldsymbol{\eta}}$. Hence (iii) holds. Finally, (iii) clearly implies (i). ■

For convenience, we use the following notation:

Definition 1.4 For square matrices $A_1, \ldots, A_r$ of sizes l_i, $i = 1, \ldots, r$, we define the square matrix $A_1 \times \cdots \times A_r$ of size $l_1 + \cdots + l_r$ to be

$$A_1 \times \cdots \times A_r = \begin{pmatrix} A_1 & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{1} & A_2 & \mathbf{1} & & \vdots \\ \mathbf{1} & \mathbf{1} & A_3 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{1} \\ \mathbf{1} & \cdots & \cdots & \mathbf{1} & A_r \end{pmatrix},$$

where $\mathbf{1}$'s are matrices of suitable sizes whose entries are all 1. Also, we write $\mathbf{1} = \mathbf{1}_k$ if $\mathbf{1}$ is a square matrix of size k.

Lemma 1.5

(1) *Let* $\mathbf{q} = (q_{ij})$ *be an* $n \times n$ *quantum matrix,* σ *a permutation on* $\{1, \ldots, n\}$, *and put* $\tilde{\mathbf{q}}_\sigma = (\tilde{q}_{ij})$ *where* $\tilde{q}_{ij} = q_{\sigma(i)\sigma(j)}$. *Then* $\mathbf{q} \cong \tilde{\mathbf{q}}_\sigma$. *In particular, for a transposition* $(i\,j) \in S$, *we have* $\mathbf{q} \cong \tilde{\mathbf{q}}_{(ij)}$.

(2) *Let* $\mathbf{r}$, $\mathbf{s}$ *and* $\boldsymbol{\eta}$ *be quantum matrices with* $\mathbf{s} \cong \boldsymbol{\eta}$. *Then:*

   (i) $\mathbf{r} \times \mathbf{s} \cong \mathbf{s} \times \mathbf{r}$,

   (ii) $\mathbf{r} \times \mathbf{s} \cong \mathbf{r} \times \boldsymbol{\eta}$.

Proof For (1), let $F_\mathbf{q} = F_\mathbf{q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, and so we have $t_j t_i = q_{ij} t_i t_j$. Hence the generators $\tilde{t}_i := t_{\sigma(i)}$ satisfy $\tilde{t}_j \tilde{t}_i = t_{\sigma(j)} t_{\sigma(i)} = q_{\sigma(i)\sigma(j)} t_{\sigma(i)} t_{\sigma(j)} = q_{\sigma(i)\sigma(j)} \tilde{t}_i \tilde{t}_j$, and

$$F_\mathbf{q} = F_{\tilde{\mathbf{q}}_\sigma}[\tilde{t}_1^{\pm 1}, \ldots, \tilde{t}_n^{\pm 1}].$$

Thus we get $\mathbf{q} \cong \tilde{\mathbf{q}}_\sigma$.

For (2), let r and s be the sizes of the matrices $\mathbf{r}$ and $\mathbf{s}$, respectively, and let $n := r+s$ and $F_{\mathbf{r}\times\mathbf{s}} = F_{\mathbf{r}\times\mathbf{s}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

(i) follows from (1): Take

$$\sigma = \begin{pmatrix} 1 & \cdots & s & s+1 & \cdots & n \\ r+1 & \cdots & n & 1 & \cdots & r \end{pmatrix}.$$

Then $\mathbf{s} \times \mathbf{r} = (\widetilde{\mathbf{r} \times \mathbf{s}})_\sigma$.

For (ii), we consider a toral $\langle \varepsilon_1, \ldots, \varepsilon_n \rangle$-grading of $F_{\mathbf{r}\times\mathbf{s}}$. Let $\mathbf{r} \times \boldsymbol{\eta} = (a_{ij})$. The subalgebra of $F_{\mathbf{r}\times\mathbf{s}}$ generated by $t_{r+1}^{\pm 1}, \ldots, t_n^{\pm 1}$ can be identified with the quantum torus $F_\mathbf{s}[t_{r+1}^{\pm 1}, \ldots, t_n^{\pm 1}]$ with the $\langle \varepsilon_{r+1}, \ldots, \varepsilon_n \rangle$-grading. By Lemma 1.3, our assumption $\mathbf{s} \cong \boldsymbol{\eta}$ implies that there exists a basis $\langle \boldsymbol{\sigma}_{r+1}, \ldots, \boldsymbol{\sigma}_n \rangle$ of $\mathbb{Z}\varepsilon_{r+1} + \cdots + \mathbb{Z}\varepsilon_n$ in Λ such that $x_j x_i = a_{ij} x_i x_j$ for all $r+1 \leq i, j \leq n$ where x_i is a nonzero element of degree $\boldsymbol{\sigma}_i$. Note that all $x_1 := t_1, \ldots, x_r := t_r$ commute with all $t_{r+1}, \ldots, t_n$, and so all $x_1, \ldots, x_r$ commute with all $x_{r+1}, \ldots, x_n$. Hence we get $x_j x_i = a_{ij} x_i x_j$ for all

$1 \leq i, j \leq n$. Since $\langle \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_r, \boldsymbol{\sigma}_{r+1}, \ldots, \boldsymbol{\sigma}_n \rangle$ is a basis of Λ, we obtain $\mathbf{r} \times \mathbf{s} \cong \mathbf{r} \times \boldsymbol{\eta}$ by Lemma 1.3. ■

Definition 1.6 A quantum matrix $\varepsilon = (\varepsilon_{ij})$ is called *elementary* if $\varepsilon_{ij} = 1$ or -1 for all i, j. Note that ε becomes a symmetric matrix. Also, the quantum torus F_ε determined by an elementary quantum matrix ε is called an *elementary quantum torus*.

Note that any elementary quantum matrix is $\mathbf{1}$ if ch. $F = 2$. Thus our argument will be trivial if ch. $F = 2$, and so for convenience we will assume that ch. $F \neq 2$ from now on.

Example 1.7 Let

$$F_{\mathbf{m}_3} = F_{\mathbf{m}_3}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}] \quad \text{and} \quad F_{\mathbf{m}_4} = F_{\mathbf{m}_4}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]$$

be elementary quantum tori with an $\langle \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3 \rangle$-grading and an $\langle \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_4 \rangle$-grading, respectively, where

$$\mathbf{m}_3 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{m}_4 = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

In $F_{\mathbf{m}_3}$, t_1 commutes with t_2t_3 which has degree $\boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3$, and in $F_{\mathbf{m}_4}$, t_1 commutes with t_2t_3 and t_2t_4 which has degree $\boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3$ and $\boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_4$. Since $\langle \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3 \rangle$ and $\langle \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_4 \rangle$ are bases of Λ_3 and Λ_4, respectively, we have by Lemma 1.3,

$$\mathbf{m}_3 \cong \begin{pmatrix} 1 & -1 & 1 \\ -1 & * & * \\ 1 & * & * \end{pmatrix} \quad \text{and} \quad \mathbf{m}_4 \cong \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & * & * & * \\ 1 & * & * & * \\ 1 & * & * & * \end{pmatrix},$$

and the $*$-parts of both matrices are some elementary matrices. Indeed in both algebras, we have $(t_2t_3)t_2 = -t_2(t_2t_3)$, and in $F_{\mathbf{m}_4}$, $(t_2t_4)t_2 = -t_2(t_2t_4)$ and $(t_2t_3)(t_2t_4) = -(t_2t_4)(t_2t_3)$. So we get

$$\mathbf{m}_3 \cong \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{m}_4 \cong \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

In both algebras, t_1 and t_2 commute with $t_1(t_2t_3)$ which has degree $\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3$, and in $F_{\mathbf{m}_4}$, t_1 and t_2 commutes with $t_1(t_2t_4)$ which has degree $\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_4$. Since $\langle \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3 \rangle$ and $\langle \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_4 \rangle$ are bases of Λ_3 and Λ_4, respectively, we have by Lemma 1.3,

$$\mathbf{m}_3 \cong \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{m}_4 \cong \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & * & * \\ 1 & 1 & * & * \end{pmatrix},$$

and the $*$-part is $\mathbf{h}$ by $(t_1t_2t_4)(t_1t_2t_3) = -(t_1t_2t_3)(t_1t_2t_4)$. Thus we have shown

$$\mathbf{m}_3 \cong \mathbf{h}_{1,3} \quad \text{and} \quad \mathbf{m}_4 \cong \mathbf{h}_{2,4} = \mathbf{h} \times \mathbf{h}$$

(see the definition of $\mathbf{h}_{l,n}$ in Theorem 1.10). Note that we also have shown

$$F_{\mathbf{m}_3} \cong_\Lambda F_{\mathbf{h}_{1,3}}[u_1^{\pm1}, u_2^{\pm1}, u_3^{\pm1}] \quad \text{via} \quad t_1 \longmapsto u_1, t_2 \longmapsto u_2, t_1t_2t_3 \longmapsto u_3,$$

$$F_{\mathbf{m}_4} \cong_\Lambda F_{\mathbf{h}\times\mathbf{h}}[u_1^{\pm1}, u_2^{\pm1}, u_3^{\pm1}, u_4^{\pm1}] \quad \text{via}$$

$$t_1 \longmapsto u_1, t_2 \longmapsto u_2, t_1t_2t_3 \longmapsto u_3, t_1t_2t_4 \longmapsto u_4,$$

for the $\langle \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3 \rangle$-grading of $F_{\mathbf{h}_{1,3}}$ and the $\langle \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_4 \rangle$-grading of $F_{\mathbf{h}\times\mathbf{h}}$.

In general, the centre $Z(F_\mathbf{q})$ of a quantum torus $F_\mathbf{q}$ is an algebra of Laurent polynomials, and the grading group is given by

$$\Big\{ \boldsymbol{\alpha} \in \Lambda \;\Big|\; \prod_{i,j} q_{ij}^{\alpha_j\beta_i} = 1 \text{ for all } \boldsymbol{\beta} \in \Lambda \Big\}$$

(see [5] or [8]). For later use, we directly calculate the centre of $F_{\mathbf{h}_{l,n}}$.

Lemma 1.8 *Let $l > 0$ and $F_{\mathbf{h}_{l,n}} = F_{\mathbf{h}_{l,n}}[t_1^{\pm1}, \dots, t_n^{\pm1}]$ be an elementary torus. Then the centre $Z(F_{\mathbf{h}_{l,n}})$ is equal to*

$$F[t_1^{\pm2}, \dots, t_{2l}^{\pm2}, t_{2l+1}^{\pm1}, \dots, t_n^{\pm1}],$$

the algebra of Laurent polynomials in the variables $t_1^2, \dots, t_{2l}^2, t_{2l+1}, \dots, t_n$. Hence for a $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$-grading of $F_{\mathbf{h}_{l,n}}$, the grading group of $Z(F_{\mathbf{h}_{l,n}})$ is equal to

$$2\mathbb{Z}\boldsymbol{\sigma}_1 + \cdots + 2\mathbb{Z}\boldsymbol{\sigma}_{2l} + \mathbb{Z}\boldsymbol{\sigma}_{2l+1} + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n.$$

Proof It is clear that $Z' := F[t_1^{\pm2}, \dots, t_{2l}^{\pm2}, t_{2l+1}^{\pm1}, \dots, t_n^{\pm1}] \subset Z(F_{\mathbf{h}_{l,n}}) =: Z$. For the other inclusion, if $Z \setminus Z' \neq \varnothing$, there exists $x := t_1^{\kappa_1} \cdots t_{2l}^{\kappa_{2l}} \in Z$, where $\kappa_i = 0$ or 1 but not all κ_i are 0. But then, for $\kappa_j \neq 0$, we have $xt_k = -t_kx$ where

$$k = \begin{cases} j+1 & \text{if } j \text{ is odd} \\ j-1 & \text{if } j \text{ is even,} \end{cases}$$

i.e., $x \notin Z$, which is a contradiction. Hence $Z = Z'$. ■

Note that $\mathbf{h}_{0,n} = \mathbf{1}$ and so $Z(F_{\mathbf{h}_{0,n}}) = F[t_1^{\pm1}, \dots, t_n^{\pm1}]$.

Lemma 1.9 *Let $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ be an $n \times n$ elementary quantum matrix for $n \geq 3$. If $\varepsilon_{kp} = \varepsilon_{kq} = -1$ for some distinct $1 \leq k, p, q \leq n$, then there exists an elementary quantum matrix $\boldsymbol{\eta} = (\eta_{ij})$ with*

$$\eta_{ij} = \varepsilon_{ij} \quad \textit{for all } i, j \neq q \quad (\eta_{qq} = \varepsilon_{qq} = 1),$$

$$\eta_{iq} = \varepsilon_{ip}\varepsilon_{iq} \quad \textit{for all } i \neq q$$

such that $\boldsymbol{\varepsilon} \cong \boldsymbol{\eta}$. In particular,

(a) *$\eta_{kq} = 1$ and $\eta_{ki} = \varepsilon_{ki}$ for all $i \neq q$;*
(b) *if $k = 2$ and $p = 1$, then $\eta_{i1} = \varepsilon_{i1}$ for all i, i.e., the first rows of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\eta}$ are the same.*

Proof Let $F_{\boldsymbol{\varepsilon}} = F_{\boldsymbol{\varepsilon}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ with a $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$-grading. Since $\varepsilon_{kp} = \varepsilon_{kq} = -1$, we have $t_p t_k = -t_k t_p$ and $t_q t_k = -t_k t_q$. Hence t_k commutes with $t_p t_q$ which has degree $\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_q$. Let

$$x_1 := t_1, \ldots, x_{q-1} := t_{q-1}, \quad x_q := t_p t_q, \quad x_{q+1} := t_{q+1}, \ldots, x_n := t_n.$$

Then the relations between x_i and x_j for $1 \leq i, j \leq n$ determine an elementary quantum matrix $\boldsymbol{\eta} = (\eta_{ij})$, *i.e.*, $x_j x_i = \eta_{ij} x_i x_j$. It is clear that $\eta_{ij} = \varepsilon_{ij}$ for all $i, j \neq q$. For $i \neq q$, we have $x_q x_i = (t_p t_q) t_i = \varepsilon_{ip} \varepsilon_{iq} t_i (t_p t_q) = \varepsilon_{ip} \varepsilon_{iq} x_i x_q$. Hence $\eta_{iq} = \varepsilon_{ip} \varepsilon_{iq}$. Since

$$\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{q-1}, \boldsymbol{\sigma}_p + \boldsymbol{\sigma}_q, \boldsymbol{\sigma}_{q+1}, \ldots, \boldsymbol{\sigma}_n \rangle$$

is a basis of Λ, we get $\boldsymbol{\varepsilon} \cong \boldsymbol{\eta}$ by Lemma 1.3. (a) and (b) are clear now. ∎

Our first result is the following:

Theorem 1.10 *Let $\boldsymbol{\varepsilon}$ be an $n \times n$ elementary quantum matrix. Then there exists $l \geq 0$ such that $\boldsymbol{\varepsilon} \cong \mathbf{h}_{l,n}$ where*

$$\mathbf{h}_{l,n} = \overbrace{\mathbf{h} \times \cdots \times \mathbf{h}}^{l\text{-times}} \times \mathbf{1}_{n-2l} \quad \text{and} \quad \mathbf{h} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Also, there exists a $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$-grading of $F_{\boldsymbol{\varepsilon}}$ such that the grading group of the centre $Z(F_{\boldsymbol{\varepsilon}})$ is equal to

$$2\mathbb{Z}\boldsymbol{\sigma}_1 + \cdots + 2\mathbb{Z}\boldsymbol{\sigma}_{2l} + \mathbb{Z}\boldsymbol{\sigma}_{2l+1} + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n.$$

Moreover, the number l is an isomorphism invariant of $F_{\boldsymbol{\varepsilon}}$.

Proof We prove this by induction on n. When $n = 1$, $\boldsymbol{\varepsilon}$ has to be (1), and so the statement is clear. Let $n > 1$, $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ and

$$N_k(\boldsymbol{\varepsilon}) := \left| \{ i \mid \varepsilon_{ki} = -1, 1 \leq i \leq n \} \right|$$

where $|\ \ |$ is the number of elements of a set. (We will use this notation only for $k = 1$ and 2.) If $N_1(\boldsymbol{\varepsilon}) = 0$, then $\boldsymbol{\varepsilon} = (1) \times \boldsymbol{\varepsilon}'$ for an elementary quantum matrix $\boldsymbol{\varepsilon}'$ of size $n - 1$. By induction, we have $\boldsymbol{\varepsilon}' \cong \mathbf{h}_{l,n-1}$ for some $l \geq 0$. Then, by Lemma 1.5 (2), we get

$$\boldsymbol{\varepsilon} = (1) \times \boldsymbol{\varepsilon}' \cong (1) \times \mathbf{h}_{l,n-1} \cong \mathbf{h}_{l,n-1} \times (1) = \mathbf{h}_{l,n}.$$

If $N_1(\boldsymbol{\varepsilon}) > 1$, then by Lemma 1.9 (a) for $k = 1$, there exists an elementary quantum matrix $\boldsymbol{\varepsilon}'$ such that $\boldsymbol{\varepsilon} \cong \boldsymbol{\varepsilon}'$ and $N_1(\boldsymbol{\varepsilon}') = N_1(\boldsymbol{\varepsilon}) - 1$. Repeating this, we obtain an elementary quantum matrix $\boldsymbol{\nu}$ such that $\boldsymbol{\varepsilon} \cong \boldsymbol{\nu}$ and $N_1(\boldsymbol{\nu}) = 1$, *i.e.*, only one entry, say the $(1i_0)$-entry, is -1 in the first row of $\boldsymbol{\nu}$. If $N_1(\boldsymbol{\varepsilon}) = 1$, we also put $\boldsymbol{\nu} = \boldsymbol{\varepsilon}$. Then, by Lemma 1.5 (1), we get

$$\boldsymbol{\varepsilon} \cong \boldsymbol{\nu}_{(2i_0)} =: \boldsymbol{\eta} = (\eta_{ij}) = \begin{pmatrix} 1 & -1 & 1 & \cdots & 1 \\ -1 & & & & \\ 1 & & * & & \\ \vdots & & & & \\ 1 & & & & \end{pmatrix},$$

i.e., $\eta_{12} = \eta_{21} = -1$, the other $\eta_{1i} = \eta_{i1} = 1$ and $*$ is some elementary quantum matrix of size $n - 1$.

If $n = 2$, we have $\boldsymbol{\eta} = \mathbf{h}$ and we are done. We assume that $n > 2$. Note that $N_2(\boldsymbol{\eta}) \geq 1$ since $\eta_{21} = -1$. If $N_2(\boldsymbol{\eta}) > 1$, we can apply Lemma 1.9 (b) for any $q > 2$ such that $\eta_{2q} = -1$, and get an elementary quantum matrix $\boldsymbol{\eta}'$ such that $\boldsymbol{\eta} \cong \boldsymbol{\eta}'$, $N_1(\boldsymbol{\eta}') = N_1(\boldsymbol{\eta}) = 1$ and $N_2(\boldsymbol{\eta}') = N_2(\boldsymbol{\eta}) - 1$. Repeating this, we obtain an elementary quantum matrix $\boldsymbol{\mu} = (\mu_{ij})$ such that $\boldsymbol{\eta} \cong \boldsymbol{\mu}$, $N_1(\boldsymbol{\mu}) = N_2(\boldsymbol{\mu}) = 1$ and $\mu_{21} = \mu_{12} = -1$. Also, if $N_2(\boldsymbol{\eta}) = 1$, we put $\boldsymbol{\eta} = \boldsymbol{\mu}$. Thus we have $\boldsymbol{\eta} \cong \boldsymbol{\mu} = \mathbf{h} \times \boldsymbol{\mu}'$ for an elementary quantum matrix $\boldsymbol{\mu}'$ of size $n - 2$. By induction, we have $\boldsymbol{\mu}' \cong \mathbf{h}_{l',n-2}$ for some $l' \geq 0$. Then, by Lemma 1.5 (2) (ii), we get $\boldsymbol{\mu} = \mathbf{h} \times \boldsymbol{\mu}' \cong \mathbf{h} \times \mathbf{h}_{l',n-2} = \mathbf{h}_{l,n}$ where $l = l' + 1$, and hence $\boldsymbol{\varepsilon} \cong \boldsymbol{\eta} \cong \boldsymbol{\mu} \cong \mathbf{h}_{l,n}$.

The description of the centre follows from Lemma 1.3 and Lemma 1.8. For the last statement, suppose that $\mathbf{h}_{l,n} \cong \mathbf{h}_{l',n}$. Then, by Lemma 1.3, $F_{\mathbf{h}_{l,n}} \cong_\Lambda F_{\mathbf{h}_{l',n}}$ for some toral gradings. Hence the grading groups of the centres of $F_{\mathbf{h}_{l,n}}$ and $F_{\mathbf{h}_{l',n}}$ coincide, which implies $l = l'$, by Lemma 1.8. Therefore, l is an isomorphism invariant of $F_{\boldsymbol{\varepsilon}}$. ■

2 Elementary Quantum Tori with Graded Involution

From now on, we always consider a quantum torus as a toral Λ-graded algebra. Let $F_{\mathbf{q}} = F_{\mathbf{q}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the quantum torus determined by $\mathbf{q} = (q_{ij})$, and let τ be a graded involution of $F_{\mathbf{q}}$. Then we have $\tau(t_i) = a_i t_i$ for some $a_i \in F$, $i = 1, \ldots, n$. Since $t_i = \tau^2(t_i) = a_i^2 t_i$, one gets $a_i = \pm 1$ for all $1 \leq i \leq n$. Moreover, one has

$$a_i a_j q_{ij} t_j t_i = \tau(q_{ij} t_i t_j) = \tau(t_j t_i) = a_i a_j t_i t_j = a_i a_j q_{ji} t_j t_i,$$

and hence $q_{ij}^{-1} = q_{ji}$, *i.e.*, $q_{ij} = \pm 1$ for all $1 \leq i, j \leq n$. Thus $\mathbf{q}$ has to be elementary.

Conversely, it is straightforward to check that for an elementary quantum tours $F_{\boldsymbol{\varepsilon}} = F_{\boldsymbol{\varepsilon}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ and each $(a_1, \ldots, a_n)$, $a_i = \pm 1$, there exists a unique involution of $F_{\boldsymbol{\varepsilon}}$ such that $\tau(t_i) = a_i t_i$ for all $1 \leq i \leq n$. We call this τ *of type* $(a_1, \ldots, a_n)$, denoted $\tau = (a_1, \ldots, a_n)$. The graded involution of type $(1, \ldots, 1)$ is called the *main involution*, denoted $*$. Thus we have the following proposition, which is stated in [1]:

Proposition 2.1 *Let $F_{\mathbf{q}} = F_{\mathbf{q}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be a quantum torus over F. Then there exists a graded involution τ of $F_{\mathbf{q}}$ if and only if $\mathbf{q}$ is elementary. In this case, τ has type $(a_1, \ldots, a_n)$, i.e., $\tau(t_i) = a_i t_i$ where $a_i = 1$ or -1 for all $1 \leq i \leq n$.* ■

Recall the notion of isomorphism in the class of algebras with involution. Namely, for algebras with involution (A, τ) and (B, ρ), an *isomorphism of algebras with involution* from (A, τ) onto (B, ρ) is an isomorphism f from A onto B satisfying $f\tau = \rho f$, and in this case we denote this by $(A, \tau) \cong (B, \rho)$. Moreover, if A and B are Λ-graded algebras, τ and ρ are graded involutions and the f happens to be a graded isomorphism, we write $(A, \tau) \cong_\Lambda (B, \rho)$. Finally, the *centre* $Z(A, \tau)$ of (A, τ) is defined as

$$Z(A, \tau) = Z(A) \cap \{a \in A \mid \tau(a) = a\},$$

where $Z(A)$ is the centre of the algebra A.

One can prove the following lemmas similar to Lemmas 1.3 and 1.5. Since the proofs can be done in the same manner, they will be left to the reader.

Lemma 2.2 *Let (F_ε, τ) and (F_η, ρ) be elementary quantum tori with graded involution. Let $\boldsymbol{\eta} = (\eta_{ij})_{1\le i,j\le n}$ and $\rho = (a_1, \ldots, a_n)$. Then the following are equivalent:*

(i) $(F_\varepsilon, \tau) \cong (F_\eta, \rho)$,
(ii) *for any toral grading of F_ε, there exists a basis $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$ of Λ and nonzero homogeneous elements $x_i \in F_\varepsilon$ of degree $\boldsymbol{\sigma}_i$ such that $x_j x_i = \eta_{ij} x_i x_j$ and $\tau(x_i) = a_i x_i$ for all $1 \le i < j \le n$,*
(iii) *for any toral grading of F_ε, there exists a toral grading of F_η such that $(F_\varepsilon, \tau) \cong_\Lambda (F_\eta, \rho)$. In that case, the grading subgroups of the centres $Z(F_\varepsilon, \tau)$ and $Z(F_\eta, \rho)$ coincide.* ■

For graded involutions τ and ρ of type $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$, respectively, we denote the graded involution of type $(a_1, \ldots, a_r, b_1, \ldots, b_s)$ by $\tau \times \rho$.

Lemma 2.3 *Let $(F_{\mathbf{r}}, \tau)$, $(F_{\mathbf{s}}, \rho)$ and (F_η, ρ_1) be elementary quantum tori with graded involution. Assume that $(F_{\mathbf{s}}, \rho) \cong (F_\eta, \rho_1)$. Then:*

(i) $(F_{\mathbf{r}\times\mathbf{s}}, \tau \times \rho) \cong (F_{\mathbf{s}\times\mathbf{r}}, \rho \times \tau)$,
(ii) $(F_{\mathbf{r}\times\mathbf{s}}, \tau \times \rho) \cong (F_{\mathbf{r}\times\eta}, \tau \times \rho_1)$. ■

We start to classify elementary tori with graded involution. Let τ be a graded involution of an elementary quantum torus F_ε. Then, by Theorem 1.10 and Lemma 1.3, we have $F_\varepsilon \cong_\Lambda F_{\mathbf{h}_{l,n}}$ for some $l \ge 0$ and toral gradings, and hence $(F_\varepsilon, \tau) \cong_\Lambda (F_{\mathbf{h}_{l,n}}, \rho)$ for some graded involution ρ of $F_{\mathbf{h}_{l,n}}$. Thus it is enough to classify $F_{\mathbf{h}_{l,n}}$ with graded involutions. Besides the main involution $* = (1, \ldots, 1)$, we define two specific graded involutions of $F_{\mathbf{h}_{l,n}}$, namely,

$$\tau_1 = (1, \ldots, 1, -1, 1, \ldots, 1),$$
where only the $2l + 1$ position is -1, if $n - 2l \ge 1$,

$$\tau_2 = (1, \ldots, 1, -1, -1, 1, \ldots, 1),$$
where only the $2l - 1$ and $2l$ positions are -1, if $l \ge 1$.

Remark By Lemma 1.8, $*$ and τ_2 fix the centre Z of $F_{\mathbf{h}_{l,n}}$ but τ_1 does not. It is easily seen that the *central closure* $\overline{F}_{\mathbf{h}_{l,n}} = \overline{Z} \otimes_Z F_{\mathbf{h}_{l,n}}$ is a simple algebra over $\overline{Z}$, where $\overline{Z}$ is the field of fractions of Z. Let $\tau = *$, τ_1 or τ_2. By the universal property of the central closure $\overline{F}_{\mathbf{h}_l^{(n)}}$, the natural extension $\overline{\tau}$ of τ defined by $\overline{\tau}(z \otimes x) = \tau(z) \otimes \tau(x)$ is an involution of $\overline{F}_{\mathbf{h}_l^{(n)}}$. Since $\overline{*}$ and $\overline{\tau_2}$ fix $\overline{Z}$, they are involutions of first kind, while $\overline{\tau_1}$ does not, and so it is an involution of second kind.

Example 2.4 Recall the two elementary quantum matrices $\mathbf{m}_3$ and $\mathbf{m}_4$ defined in Example 1.7. The isomorphisms $\mathbf{m}_3 \cong \mathbf{h}_{1,3}$ and $\mathbf{m}_4 \cong \mathbf{h}_{2,4}$ there give isomorphisms of algebras with involution, namely,

$$(F_{\mathbf{m}_3}, *) \cong (F_{\mathbf{h}_{1,3}}, \tau_1) \quad \text{and} \quad (F_{\mathbf{m}_4}, *) \cong (F_{\mathbf{h}_{2,4}}, \tau_2).$$

Like Lemma 1.8, we have the following lemma about the centres:

Lemma 2.5 *Let $F_{\mathbf{h}_{l,n}} = F_{\mathbf{h}_{l,n}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be an elementary torus. Then*

$$Z(F_{\mathbf{h}_{l,n}}, *) = Z(F_{\mathbf{h}_{l,n}}, \tau_2) = F[t_1^{\pm 2}, \ldots, t_{2l}^{\pm 2}, t_{2l+1}^{\pm 1}, \ldots, t_n^{\pm 1}]$$
$$Z(F_{\mathbf{h}_{l,n}}, \tau_1) = F[t_1^{\pm 2}, \ldots, t_{2l+1}^{\pm 2}, t_{2l+2}^{\pm 1}, \ldots, t_n^{\pm 1}].$$

(For $(F_{\mathbf{h}_{l,n}}, \tau_2)$, we are always assuming $l \geq 1$, but for the others, l can be 0.)

Hence for a $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$-grading of $F_{\mathbf{h}_{l,n}}$, the grading groups of $Z(F_{\mathbf{h}_{l,n}}, *)$ and $Z(F_{\mathbf{h}_{l,n}}, \tau_2)$ are equal to

$$2\mathbb{Z}\boldsymbol{\sigma}_1 + \cdots + 2\mathbb{Z}\boldsymbol{\sigma}_{2l} + \mathbb{Z}\boldsymbol{\sigma}_{2l+1} + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n,$$

and the grading group of $Z(F_{\mathbf{h}_{l,n}}, \tau_1)$ is equal to

$$2\mathbb{Z}\boldsymbol{\sigma}_1 + \cdots + 2\mathbb{Z}\boldsymbol{\sigma}_{2l+1} + \mathbb{Z}\boldsymbol{\sigma}_{2l+2} + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n.$$

Proof From Lemma 1.8, we already knows the description of the centre $Z(F_{\mathbf{h}_{l,n}})$ of $F_{\mathbf{h}_{l,n}}$. So only the fixed elements of $Z(F_{\mathbf{h}_{l,n}})$ under each $*$, τ_1 and τ_2 have to be calculated. This easy exercise is left to the reader. ■

For the classification of elementary tori with graded involution, we use the following:

Lemma 2.6 *Let $*$ be the main involution and τ_1 the graded involution of $F_{\mathbf{h}_{l,n}}$ defined above. Then:*

(i) $\big(F_{\mathbf{h}}, (1, -1)\big) \cong \big(F_{\mathbf{h}}, (-1, 1)\big) \cong (F_{\mathbf{h}}, *)$,
(ii) $\big(F_{\mathbf{1}_2}, (-1, -1)\big) \cong (F_{\mathbf{1}_2}, \tau_1)$,
(iii) $\big(F_{\mathbf{h}_{1,3}}, (-1, -1, -1)\big) \cong (F_{\mathbf{h}_{1,3}}, \tau_1)$,
(iv) $\big(F_{\mathbf{h}_{2,4}}, (-1, -1, -1, -1)\big) \cong (F_{\mathbf{h}_{2,4}}, *)$.

Proof Let $F_{\mathbf{h}_{l,n}} = F_{\mathbf{h}_{l,n}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ with an $\langle \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n \rangle$-grading. Then we note that $t_{i_1} \cdots t_{i_r}$ has degree $\boldsymbol{\varepsilon}_{i_1} + \cdots + \boldsymbol{\varepsilon}_{i_r}$.

For (i), we have $n = 2$ and $l = 1$. Let $\tau = (1, -1)$. Then we have $\tau(t_1) = t_1$ and $\tau(t_2) = -t_2$. Since $(t_1t_2)t_1 = -t_1(t_1t_2)$ and $\tau(t_1t_2) = t_1t_2$, and since $\langle \varepsilon_1, \varepsilon_1 + \varepsilon_2 \rangle$ is a basis of Λ_2, we get $(F_{\mathbf{h}}, \tau) \cong (F_{\mathbf{h}}, *)$ by Lemma 2.2. The case $(-1, 1)$ can be proven in the same way.

For (ii), we have $n = 2$ and $l = 0$. Let $\tau = (-1, -1)$. Then we have $\tau(t_1) = -t_1$ and $\tau(t_2) = -t_2$. Since $(t_1t_2) = t_1(t_1t_2)$ and $\tau(t_1t_2) = t_1t_2$, and since $\langle \varepsilon_1, \varepsilon_1 + \varepsilon_2 \rangle$ is a basis of Λ_2, we get $(F_{\mathbf{1}_2}, \tau) \cong (F_{\mathbf{1}_2}, \tau_1)$ by Lemma 2.2.

For (iii), we have $n = 3$ and $l = 1$. Let $\tau = (-1, -1, -1)$. Then we have $\tau(t_1) = -t_1$, $\tau(t_2) = -t_2$ and $\tau(t_3) = -t_3$. Since $(t_2t_3)(t_1t_2t_3) = -(t_1t_2t_3)(t_2t_3)$, $t_3(t_1t_2t_3) = (t_1t_2t_3)t_3$, $t_3(t_2t_3) = (t_2t_3)t_3$, $\tau(t_1t_2t_3) = t_1t_2t_3$ and $\tau(t_2t_3) = t_2t_3$, and since $\langle \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_2 + \varepsilon_3, \varepsilon_3 \rangle$ is a basis of Λ_3, we get $(F_{\mathbf{h}_{1,3}}, \tau) \cong (F_{\mathbf{h}_{1,3}}, \tau_1)$ by Lemma 2.2.

For (iv), we have $n = 4$ and $l = 2$. Let $\tau = (-1, -1, -1, -1)$. Then we have $\tau(t_1) = -t_1$, $\tau(t_2) = -t_2$, $\tau(t_3) = -t_3$ and $\tau(t_4) = -t_4$. Put $x_1 := t_1t_2t_4$, $x_2 := t_2t_4$, $x_3 := t_1t_3$ and $x_4 := t_1t_3t_4$. Then one can check that $x_jx_i = a_{ij}x_ix_j$ where $(a_{ij}) = \mathbf{h}_{2,4}$ and $\tau(x_i) = x_i$ for $1 \leq i, j \leq 4$. Also, one can check that

$$\langle \varepsilon_1 + \varepsilon_2 + \varepsilon_4, \varepsilon_2 + \varepsilon_4, \varepsilon_1 + \varepsilon_3, \varepsilon_1 + \varepsilon_3 + \varepsilon_4 \rangle$$

is a basis of Λ_4. Hence by Lemma 2.2, we get $(F_{\mathbf{h}_{2,4}}, \tau) \cong (F_{\mathbf{h}_{2,4}}, *)$. ■

Now we state one of our main theorems.

Theorem 2.7 *Let τ be an arbitrary graded involution of an elementary quantum torus F_ε. Let $*$ be the main involution, and τ_1 and τ_2 the graded involutions of $F_{\mathbf{h}_{l,n}}$ defined above. Then (F_ε, τ) is graded isomorphic to exactly one of*

$$\begin{cases} (F_{\mathbf{h}_{l,n}}, *), & \text{or} \\ (F_{\mathbf{h}_{l,n}}, \tau_1) & \text{or} \\ (F_{\mathbf{h}_{l,n}}, \tau_2), \end{cases}$$

and for each of these l is an invariant of the isomorphism class. Moreover, we have

(i) $(F_\varepsilon, *) \cong (F_{\mathbf{h}_{l,n}}, \tau_1) \Longrightarrow l \geq 1$;
(ii) $(F_\varepsilon, *) \cong (F_{\mathbf{h}_{l,n}}, \tau_2) \Longrightarrow l \geq 2$;
(iii) $(F_{\mathbf{h}_{l,n}}, \tau_1) \cong_\Lambda (F_{\mathbf{h}_{l-1,n-3} \times \mathbf{m}_3}, *)$ *for* $l \geq 1$;
(iv) $(F_{\mathbf{h}_{l,n}}, \tau_2) \cong_\Lambda (F_{\mathbf{h}_{l-2,n-4} \times \mathbf{m}_4}, *)$ *for $l \geq 2$, where $\mathbf{m}_3$ and $\mathbf{m}_4$ are the elementary quantum matrices defined in Example 1.7.*

In particular, (F_ε, τ) is graded isomorphic to exactly one of $(F_{\mathbf{h}_{0,n}}, \tau_1)$, $(F_{\mathbf{h}_{1,n}}, \tau_2)$ or $(F_{\boldsymbol{\eta}}, *)$ for some elementary quantum matrix $\boldsymbol{\eta}$.

Proof We have $(F_\varepsilon, \tau) \cong_\Lambda (F_{\mathbf{h}_{l,n}}, \rho)$ for some graded involution ρ of $F_{\mathbf{h}_{l,n}}$ as mentioned above. So we classify $(F_{\mathbf{h}_{l,n}}, \rho)$ for $\rho = (a_1, \ldots, a_n)$. Note that $\mathbf{h}_{l,n} = \mathbf{h}_{l,2l} \times$

$\mathbf{1}_{n-2l}$. We consider $\left(F_{\mathbf{h}_{l,2l}}, (a_1, \ldots, a_{2l})\right)$ and $\left(F_{\mathbf{1}_{n-2l}}, (a_{2l+1}, \ldots, a_n)\right)$ separately. By Lemma 2.3 and Lemma 2.6 (i) and (iv), we have

$$\left(F_{\mathbf{h}_{l,2l}}, (a_1, \ldots, a_{2l})\right) \cong \begin{cases} (F_{\mathbf{h}_{l,2l}}, *) & \text{or} \\ (F_{\mathbf{h}_{l,2l}}, \tau_2), \end{cases}$$

and by Lemma 2.6 (ii),

$$\left(F_{\mathbf{1}_{n-2l}}, (a_{2l+1}, \ldots, a_n)\right) \cong \begin{cases} (F_{\mathbf{1}_{n-2l}}, *) & \text{or} \\ (F_{\mathbf{1}_{n-2l}}, \tau_1). \end{cases}$$

Hence by Lemma 2.3, we get

$$(F_{\mathbf{h}_{l,n}}, \rho) \cong \begin{cases} (F_{\mathbf{h}_{l,n}}, *), & \text{or} \\ (F_{\mathbf{h}_{l,n}}, \tau_1), & \text{or} \\ (F_{\mathbf{h}_{l,n}}, \tau_2), & \text{or} \\ \left(F_{\mathbf{h}_{l,n}}, (1, \ldots, 1, -1, -1, -1, 1, \ldots, 1)\right), \end{cases}$$

and the last one is isomorphic to $(F_{\mathbf{h}_{l,n}}, \tau_1)$ by Lemma 2.6 (iii). Hence, by Lemma 2.2, we have obtained $(F_{\boldsymbol{\varepsilon}}, \tau) \cong_\Lambda (F_{\mathbf{h}_{l,n}}, *)$, $(F_{\mathbf{h}_{l,n}}, \tau_1)$ or $(F_{\mathbf{h}_{l,n}}, \tau_2)$.

By Lemma 2.5, we know the grading groups of the centres $Z(F_{\mathbf{h}_{l,n}}, *)$, $Z(F_{\mathbf{h}_{l,n}}, \tau_1)$ and $Z(F_{\mathbf{h}_{l,n}}, \tau_2)$, and hence by Lemma 2.2, l is an invariant of the isomorphism classes. Moreover, the grading groups of $Z(F_{\mathbf{h}_{l,n}}, *)$ and $Z(F_{\mathbf{h}_{l,n}}, \tau_2)$, are the same but different from the one of $Z(F_{\mathbf{h}_{l,n}}, \tau_1)$. Thus, by Lemma 2.2, we get $(F_{\mathbf{h}_{l,n}}, *) \not\cong (F_{\mathbf{h}_{l,n}}, \tau_1)$ and $(F_{\mathbf{h}_{l,n}}, \tau_1) \not\cong (F_{\mathbf{h}_{l,n}}, \tau_2)$. We postpone the proof of $(F_{\mathbf{h}_{l,n}}, *) \not\cong (F_{\mathbf{h}_{l,n}}, \tau_2)$ until Section 4 (right after the proof of Lemma 4.1).

(i) Suppose that $(F_{\mathbf{h}_{0,n}}, \tau_1) \cong (F_{\boldsymbol{\varepsilon}}, *)$. We have $\mathbf{h}_{0,n} = \mathbf{1}$, which forces $\boldsymbol{\varepsilon} = \mathbf{1}$, and hence $*$ is the identity map. This is a contradiction since τ_1 is not the identity map. Therefore, we get $(F_{\mathbf{h}_{0,n}}, \tau_1) \not\cong (F_{\boldsymbol{\varepsilon}}, *)$.

(ii) Suppose that $(F_{\mathbf{h}_{1,n}}, \tau_2) \cong (F_{\boldsymbol{\varepsilon}}, *)$. Let $F_{\mathbf{h}_{1,n}} = F_{\mathbf{h}_{1,n}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ with an $\langle \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n \rangle$-grading. By Lemma 2.2, there exists a basis $\langle \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n \rangle$ of Λ such that a nonzero element $x_i \in F_{\mathbf{h}_{1,n}}$ of degree $\boldsymbol{\rho}_i$ are fixed by τ_2 for all $i = 1, \ldots, n$. Let $\boldsymbol{\rho}_i = \alpha_{i1}\boldsymbol{\varepsilon}_1 + \cdots + \alpha_{in}\boldsymbol{\varepsilon}_n$ for $\alpha_{ij} \in \mathbb{Z}$. Then one can take $x_i = t_1^{\alpha_{i1}} \cdots t_n^{\alpha_{in}}$. Since $\tau_2 = (-1, -1, 1, \ldots, 1)$, we have, by the multiplication rule (1.1) of a quantum torus,

$$\tau_2(x_i) = (-1)^{\alpha_{i1}+\alpha_{i2}} t_1^{\alpha_{in}} \cdots t_n^{\alpha_{i1}} = (-1)^{\alpha_{i1}+\alpha_{i2}+\alpha_{i1}\alpha_{i2}} x_i = x_i.$$

Hence α_{i1} and α_{i2} are both even for all $i = 1, \ldots, n$. This implies that the determinant of the matrix (α_{ij}) is even. This is absurd since $\langle \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n \rangle$ is a basis of Λ. Therefore, we get $(F_{\mathbf{h}_{1,n}}, \tau_2) \not\cong (F_{\boldsymbol{\varepsilon}}, *)$.

For (iii) and (iv), let $F_{\mathbf{h}_{l,n}} = F_{\mathbf{h}_{l,n}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. Let U be the subalgebra of $(F_{\mathbf{h}_{l,n}}, \tau_1)$ generated by $t_{2l-1}^{\pm 1}$, $t_{2l}^{\pm 1}$ and $t_{2l+1}^{\pm 1}$, and let V be the subalgebra of $(F_{\mathbf{h}_{l,n}}, \tau_2)$ generated by $t_{2l-3}^{\pm 1}$, $t_{2l-2}^{\pm 1}$, $t_{2l-1}^{\pm 1}$ and $t_{2l}^{\pm 1}$. Then we have $(U, \tau_1|_U) \cong (F_{\mathbf{h}_{1,3}}, \tau_1) \cong (F_{\mathbf{m}_3}, *)$ and $(U, \tau_2|_V) \cong (F_{\mathbf{h}_{2,4}}, \tau_2) \cong (F_{\mathbf{m}_4}, *)$ (see Example 2.4). Therefore, by Lemma 2.3 we obtain (iii) and (iv). ■

3 Semilattices

We review semilattices (see [1]). Let $\mathbb{E}$ be a Euclidean space. A subset S of $\mathbb{E}$ is called a *semilattice in* $\mathbb{E}$ if

(S1) $0 \in S$,
(S2) $S - 2S \subset S$,
(S3) S spans $\mathbb{E}$,
(S4) S is discrete in $\mathbb{E}$.

Also, a subset S of a free abelian group of finite rank is called a *semilattice in* Λ if (S1), (S2) and

(S3)′ S spans Λ.

If S is a semilattice in $\mathbb{E}$, then the group $\langle S\rangle$ generated by S is a lattice in $\mathbb{E}$ and S is a semilattice in $\langle S\rangle$. Also, if S is a semilattice in Λ, then S can be considered as a semilattice in some $\mathbb{E}$. Note that $2S$ is not a semilattice in $\langle S\rangle$, but a semilattice in $\mathbb{E}$. We define the *rank of a semilattice S in* $\mathbb{E}$ (resp. *in* Λ) as the dimension of $\mathbb{E}$ (resp. the rank of Λ). Two semilattices S and S' in $\mathbb{E}$ (resp. in Λ) are said to be *isomorphic* if there exists $\varphi \in \mathrm{GL}(\mathbb{E})$ (resp. $\varphi \in \mathrm{Aut}\,\Lambda$, the group of automorphisms of Λ) so that $\varphi(S) = S'$, and denoted $S \cong S'$. Semilattices S and S' in $\mathbb{E}$ are said to be *similar* if there exists $\varphi \in \mathrm{GL}(\mathbb{E})$ (resp. $\varphi \in \mathrm{Aut}\,\Lambda$) so that $\varphi(S+\sigma) = S'$ for some $\sigma \in S$, and we then write $S \sim S'$. The relations $\cong$ and $\sim$ are equivalence relations.

Example 3.1 Let $F_{\boldsymbol{\varepsilon}} = \bigoplus_{\alpha\in\Lambda} Ft_{\boldsymbol{\alpha}}$ be an elementary quantum torus. We fix a toral $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\rangle$-grading of $F_{\boldsymbol{\varepsilon}}$. Let τ be a graded involution of $F_{\boldsymbol{\varepsilon}}$, and let

$$S(\boldsymbol{\varepsilon}, \tau) := \{\boldsymbol{\alpha} \in \Lambda \mid \tau(t_{\boldsymbol{\alpha}}) = t_{\boldsymbol{\alpha}}\}.$$

Then $S(\boldsymbol{\varepsilon}, \tau)$ satisfies (S1) and (S2), and so $S(\boldsymbol{\varepsilon}, \tau)$ is a semilattice in some $\mathbb{E}$. In [1, p. 83], there is a description of $S(\boldsymbol{\varepsilon}, \tau)$ in terms of the coordinates of Λ relative to the basis $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\rangle$, namely, for $\boldsymbol{\alpha} = \alpha_1\boldsymbol{\sigma}_1 + \cdots + \alpha_n\boldsymbol{\sigma}_n \in \Lambda$, $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ and $\tau = (a_1, \dots, a_n)$,

$$S(\boldsymbol{\varepsilon}, \tau) = \Big\{ \boldsymbol{\alpha} \in \Lambda \;\Big|\; \sum_{i\in I_\tau} \alpha_i + \sum_{(i,j)\in J_{\boldsymbol{\varepsilon}}} \alpha_i\alpha_j \equiv 0 \mod 2 \Big\}$$

where $I_\tau = \{i \mid a_i = -1\}$ and $J_{\boldsymbol{\varepsilon}} = \{(i, j) \mid \varepsilon_{ij} = -1\}$.

Now, if $S(\boldsymbol{\varepsilon}, \tau)$ satisfies (S3)′, it is a semilattice in Λ. For example, $S(\boldsymbol{\varepsilon}, *)$ is a semilattice in Λ since $\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \in S(\boldsymbol{\varepsilon}, *)$. Let

$$\Lambda^{(t)} = 2\mathbb{Z}\boldsymbol{\sigma}_1 + \cdots + 2\mathbb{Z}\boldsymbol{\sigma}_t + \mathbb{Z}\boldsymbol{\sigma}_{t+1} + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n.$$

Then one can see that

$$S(\mathbf{1}, \tau_1) = \Lambda^{(1)} \quad \text{and} \quad S(\mathbf{h}_{1,n}, \tau_2) = \Lambda^{(2)},$$

which are lattices, and so semilattices in some Euclidean space but not semilattices in Λ.

If $(F_\varepsilon, \tau) \cong (F_{\varepsilon'}, \tau')$, then by Lemma 1.2, there exists the induced automorphism p of Λ, and clearly we have $p\big(S(\varepsilon, \tau)\big) = S(\varepsilon', \tau')$. Therefore, by Theorem 2.7:

Corollary 3.2

$$S(\varepsilon, \tau) \cong \begin{cases} \Lambda^{(1)}, & \textit{or} \\ \Lambda^{(2)}, & \textit{or} \\ S(\boldsymbol{\eta}, *) & \textit{as semilattices in } \Lambda \end{cases}$$

for some elementary quantum matrix $\boldsymbol{\eta}$.

We will need the following fundamental property of semilattices, which is shown in [1, II.1.4].

Lemma 3.3 *Suppose that S is a semilattice in a lattice Λ. Then*

$$2\Lambda \subset S \subset \Lambda \quad \textit{and} \quad 2\Lambda + S \subset S. \tag{3.4}$$

Conversely, any generating subset S of Λ satisfying (3.4) is a semilattice in Λ. ■

Suppose that S is a semilattice in a lattice Λ. Then, by (3.4) above, one can write

$$S = \bigsqcup_{i=0}^{m} (\sigma_i + 2\Lambda) \quad \text{(disjoint union)} \quad \text{for some } \sigma_i \in S.$$

We call the integer $m+1$ the *index of S* and write it as $I(S)$, though Azam first defined the index as m (see [4, Definition 1.5, p. 3]). We have found our definition more convenient. Let $n := \operatorname{rank} \Lambda$. Then one can check that $n + 1 \leq I(S) \leq 2^n$. Azam showed that the index is a similarity invariant (see [4, Lemma 1.7, p. 3]).

4 Classification of $S(\varepsilon, *)$

Recall the notation $S(\varepsilon, \tau) = \{\boldsymbol{\alpha} \in \Lambda \mid \tau(t_{\boldsymbol{\alpha}}) = t_{\boldsymbol{\alpha}}\}$ for a quantum torus (F_ε, τ) with graded involution, where ε is any elementary quantum matrix and τ is any graded involution (Example 3.1). Also, we defined the main involution $*$ of F_ε for any elementary quantum matrix ε, and two special graded involutions τ_1 and τ_2 of $F_{\mathbf{h}_{l,n}}$ for the special elementary quantum matrix $\mathbf{h}_{l,n}$ in Section 2. Note that $n \geq 2l$ and $l \geq 0$. Also, τ_1 is defined when $n > 2l$ and τ_2 is defined when $l \geq 1$.

We will classify $S(\varepsilon, *)$ in this section. By Theorem 2.7, we already know that

$$S(\varepsilon, *) \cong \begin{cases} S(\mathbf{h}_{l,n}, *) \\ S(\mathbf{h}_{l,n}, \tau_1) & (l \geq 1) \\ S(\mathbf{h}_{l,n}, \tau_2) & (l \geq 2). \end{cases}$$

For simplicity, we put

$$S(n, l, \tau) := S(\mathbf{h}_{l,n}, \tau).$$

Let $F_{\mathbf{h}_{l,n}} = F_{\mathbf{h}_{l,n}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with a $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$-grading. Let

$$\begin{aligned} I\big(S(n,l,\tau)\big) &:= \big\{ (\kappa_1, \dots, \kappa_n) \in \{0,1\}^n \;\big|\; \kappa_1 \boldsymbol{\sigma}_1 + \cdots + \kappa_n \boldsymbol{\sigma}_n \in S(n,l,\tau) \big\} \\ &= \big\{ (\kappa_1, \dots, \kappa_n) \in \{0,1\}^n \;\big|\; \tau(t_1^{\kappa_1} \cdots t_n^{\kappa_n}) = t_1^{\kappa_1} \cdots t_n^{\kappa_n} \big\} \quad \text{and} \\ I\big(S(n,l,\tau)\big)^- &:= \{0,1\}^n \setminus I\big(S(n,l,\tau)\big) \\ &= \big\{ (\kappa_1, \dots, \kappa_n) \in \{0,1\}^n \;\big|\; \tau(t_1^{\kappa_1} \cdots t_n^{\kappa_n}) = -t_1^{\kappa_1} \cdots t_n^{\kappa_n} \big\}. \end{aligned}$$

So

$$2^n = |\{0,1\}^n| = \big| I\big(S(n,l,\tau)\big) \big| + \big| I\big(S(n,l,\tau)\big)^- \big|. \tag{0}$$

We note that $\big| I\big(S(n,l,\tau)\big) \big|$ is the index of the semilattice $S(n,l,\tau)$ in Λ if $S(n,l,\tau) = S(n,l,*)$, $S(n,l,\tau_1)$ for $l \geq 1$ or $S(n,l,\tau_2)$ for $l \geq 2$. Thus, if $\big| I\big(S(n,l_0,*)\big) \big|$, $\big| I\big(S(n,l_1,\tau_1)\big) \big|$ and $\big| I\big(S(n,l_2,\tau_2)\big) \big|$ are all distinct for any l_0, l_1, l_2, then the $S(n,l,*)$, $S(n,l,\tau_1)$ and $S(n,l,\tau_2)$ are pairwise non-similar. In fact, we can prove the following:

Lemma 4.1 *In the notation above, we have the index formulas*

$$\begin{aligned} \big| I\big(S(n,l,*)\big) \big| &= 2^{n-1} + 2^{n-l-1} \quad (l \geq 0), \\ \big| I\big(S(n,l,\tau_1)\big) \big| &= 2^{n-1} \quad (l \geq 0 \textit{ and } n > 2l) \\ \big| I\big(S(n,l,\tau_2)\big) \big| &= 2^{n-1} - 2^{n-l-1} \quad (l \geq 1). \end{aligned}$$

In particular, for arbitrary $l_0, l_1 \geq 0$ and $l_2 \geq 1$ such that $n \geq 2l_0, 2l_2$ and $n > 2l_1$,

$$\big| I\big(S(n,l_0,*)\big) \big| > \big| I\big(S(n,l_1,\tau_1)\big) \big| > \big| I\big(S(n,l_2,\tau_2)\big) \big|.$$

Proof For $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_n) \in \{0,1\}^n$ and $t^{\boldsymbol{\kappa}} := t_1^{\kappa_1} \cdots t_{2l}^{\kappa_{2l}} t_{2l+1}^{\kappa_{2l+1}} \cdots t_n^{\kappa_n}$, we have

$$(t^{\boldsymbol{\kappa}})^* = (t_2^{\kappa_2} t_1^{\kappa_1})(t_4^{\kappa_4} t_3^{\kappa_3}) \cdots (t_{2l}^{\kappa_{2l}} t_{2l-1}^{\kappa_{2l-1}}) t_{2l+1}^{\kappa_{2l+1}} \cdots t_n^{\kappa_n} = (-1)^{\sum_{i=1}^{l} \kappa_{2i-1}\kappa_{2i}} t^{\boldsymbol{\kappa}}.$$

Note that

$$t_{2i}^{\kappa_{2i}} t_{2i-1}^{\kappa_{2i-1}} = \begin{cases} t_{2i-1}^{\kappa_{2i-1}} t_{2i}^{\kappa_{2i}} & \text{if } (\kappa_{2i-1}, \kappa_{2i}) = (0,0), (0,1) \text{ or } (1,0) \\ -t_{2i-1}^{\kappa_{2i-1}} t_{2i}^{\kappa_{2i}} & \text{if } (\kappa_{2i-1}, \kappa_{2i}) = (1,1). \end{cases}$$

Hence, for

$$\bar{l} = \begin{cases} l-1 & \text{if } l \text{ is even} \\ l & \text{if } l \text{ is odd,} \end{cases}$$

we obtain, by counting the pairs $(\kappa_{2i-1}, \kappa_{2i}) = (1,1)$,

$$\begin{aligned} \big| I\big(S(n,l,*)\big) \big| &= 2^n - 2^{n-2l}\left(\binom{l}{1} 3^{l-1} + \binom{l}{3} 3^{l-3} + \cdots + \binom{l}{\bar{l}} 3^{l-\bar{l}} \right) \\ &= 2^n - 2^{n-2l}(2^{2l-1} - 2^{l-1}) \\ &= 2^{n-1} + 2^{n-l-1}, \end{aligned} \tag{1}$$

by comparing the binomial expansions of $(3+1)^l$ and $(3-1)^l$.

Next we show $\big|I\big(S(n,l,\tau_1)\big)\big| = 2^{n-1}$ for any $l \geq 0$. Let $A_0 := \big\{\boldsymbol{\kappa} \in \{0,1\}^n \bigm| \kappa_{2l+1} = 0\big\}$ and $A_1 := \big\{\boldsymbol{\kappa} \in \{0,1\}^n \bigm| \kappa_{2l+1} = 1\big\}$ so that

$$I\big(S(n,l,\tau_1)\big) = \Big(I\big(S(n,l,\tau_1)\big) \cap A_0\Big) \sqcup \Big(I\big(S(n,l,\tau_1)\big) \cap A_1\Big).$$

Note that $\tau_1(t_{2l+1}) = -t_{2l+1}$ and t_{2l+1} commutes with all t_i, and so $\Big|\Big(I\big(S(n,l,\tau_1)\big) \cap A_0\Big)\Big| = \big|I\big(S(n-1,l,*)\big)\big|$ and $\Big|\Big(I\big(S(n,l,\tau_1)\big) \cap A_1\Big)\Big| = \big|I\big(S(n-1,l,*)\big)^-\big|$. Thus, by (0), we get

$$\big|I\big(S(n,l,\tau_1)\big)\big| = \big|I\big(S(n-1,l,*)\big)\big| + \big|I\big(S(n-1,l,*)\big)^-\big| = 2^{n-1}.$$

Recall that τ_2 is defined only for $l \geq 1$, and so we can consider a partition of $\{0,1\}^n$ by the following four subsets B_k, $k = 1,2,3,4$, namely,

$$B_1 := \big\{\boldsymbol{\kappa} \in \{0,1\}^n \bigm| \kappa_{2l-1} = \kappa_{2l} = 0\big\},$$
$$B_2 := \big\{\boldsymbol{\kappa} \in \{0,1\}^n \bigm| \kappa_{2l-1} = 1, \kappa_{2l} = 0\big\},$$
$$B_3 := \big\{\boldsymbol{\kappa} \in \{0,1\}^n \bigm| \kappa_{2l-1} = 0, \kappa_{2l} = 1\big\},$$
$$B_4 := \big\{\boldsymbol{\kappa} \in \{0,1\}^n \bigm| \kappa_{2l-1} = \kappa_{2l} = 1\big\},$$

so that

$$I\big(S(n,l,\tau_2)\big) = \bigsqcup_{k=1}^{4}\Big(I\big(S(n,l,\tau_2)\big) \cap B_k\Big).$$

Since $\tau_2(t_{2l-1}) = -t_{2l-1}$, $\tau_2(t_{2l}) = -t_{2l}$ and $\tau_2(t_{2l-1}t_{2l}) = -t_{2l-1}t_{2l}$, and since t_{2l-1}, t_{2l} and $t_{2l-1}t_{2l}$ commute with all t_i for $i \neq 2l-1, 2l$, we have $\Big|\Big(I\big(S(n,l,\tau_2)\big) \cap B_1\Big)\Big| = \big|I\big(S(n-2,l-1,*)\big)\big|$ and $\Big|\Big(I\big(S(n,l,\tau_2)\big) \cap B_k\Big)\Big| = \big|I\big(S(n-2,l-1,*)\big)^-\big|$ for $k = 2,3,4$. Thus we get

$$\begin{aligned}
\big|I\big(S(n,l,\tau_2)\big)\big| &= \big|I\big(S(n-2,l-1,*)\big)\big| + 3\big|I\big(S(n-2,l-1,*)\big)^-\big| \\
&= 2^{n-2} + 2\big|I\big(S(n-2,l-1,*)\big)^-\big| \quad \text{by (0)} \\
&= 2^{n-2} + 2\big(2^{n-2} - (2^{(n-2)-1} + 2^{(n-2)-(l-1)-1})\big) \quad \text{by (0) and (1)} \\
&= 2^{n-1} - 2^{n-l-1}.
\end{aligned}$$

■

Thus, by the inequalities in Lemma 4.1, the three semilattices

$S(n,l,*)$, $S(n,l,\tau_1)$ $(l \geq 1)$ and $S(n,l,\tau_2)$ $(l \geq 2)$ are pairwise non-similar in Λ.

End of Proof of Theorem 2.7 If $(F_{\mathbf{h}_{l,n}}, *) \cong (F_{\mathbf{h}_{l,n}}, \tau_2)$, then $S(n,l,*) \cong S(n,l,\tau_2)$ as semilattices in Λ. Hence as a corollary of Lemma 4.1, we get $(F_{\mathbf{h}_{l,n}}, *) \ncong (F_{\mathbf{h}_{l,n}}, \tau_2)$

for $l \geq 2$. That is, we get one of the assertions in Theorem 2.7 whose proof was postponed there. ■

Moreover, by the index formulas in Lemma 4.1,

l is a similarity invariant for the semilattices $S(n, l, *)$ and $S(n, l, \tau_2)$ ($l \geq 2$) in Λ.

To show that l is a similarity invariant for $S(n, l, \tau_1)$, we would like to have a new similarity invariant since the index of $S(n, l, \tau_1)$ is constant for $l \geq 1$. Thus we define the following:

Definition 4.2 Let S be a semilattice in a lattice Λ. For $\gamma \in S$, if $\gamma + \sigma \in S$ for all $\sigma \in S$, then γ is called a *saturated element of* S. We denote the subset of saturated elements of S by $\Sigma(S)$. Then $\Sigma(S)$ is a subgroup of Λ containing 2Λ. We define the *saturation number* $\mathfrak{s} = \mathfrak{s}(S)$ *of* S as

$$|\Lambda/\Sigma(S)| = 2^{\mathfrak{s}}.$$

Lemma 4.3

(i) $\Sigma(S) = \Sigma(S + \sigma)$ *for any semilattice* S *in* Λ *and any* $\sigma \in S$.
(ii) *The saturation number is a similarity invariant.*

Proof (i) Let $\gamma \in \Sigma(S)$. Then $\gamma - \sigma \in S$ for any $\sigma \in S$, and so $\Sigma(S) \subset S + \sigma$. Moreover, for the semilattice $S + \sigma$ and any $\rho + \sigma \in S + \sigma$, we have $\gamma + \rho + \sigma \in S + \sigma$ since $\gamma + \rho \in S$. Hence $\Sigma(S) \subset \Sigma(S + \sigma)$ for any $\sigma \in S$. Since $-2\sigma \in S$, we have $-\sigma \in S + \sigma$. Hence $\Sigma(S + \sigma) \subset \Sigma(S)$, which shows (i).

(ii) By (i), we have $\mathfrak{s}(S) = \mathfrak{s}(S + \sigma)$ for any $\sigma \in S$. Hence we only need to show that the saturation number is an isomorphism invariant. Suppose $p(S) = S'$ for some $p \in \operatorname{Aut}\Lambda$. Then one can easily see that $p\big(\Sigma(S)\big) = \Sigma(S')$. Therefore, $|\Lambda/\Sigma(S)| = \big|\Lambda/p\big(\Sigma(S)\big)\big| = |\Lambda/\Sigma(S')|$, *i.e.*, $\mathfrak{s}$ is an isomorphism invariant. ■

Remark One can easily show that $\Sigma(S) = \bigcap_{\sigma \in S}(S + \sigma)$.

Corollary 4.4 *Let* $l \geq 1$. *Then* $\Sigma\big(S(n, l, \tau_1)\big) = \Lambda^{(2l+1)}$, *and hence* l *is a similarity invariant for the semilattices* $S(n, l, \tau_1)$ *in* Λ.

Proof Recall our notation $S(n, l, \tau_1) = S(\mathbf{h}_{l,n}, \tau_1) = \{\boldsymbol{\alpha} \in \Lambda \mid \tau_1(t_{\boldsymbol{\alpha}}) = t_{\boldsymbol{\alpha}}\}$ for the quantum torus $F_{\mathbf{h}_{l,n}} = F_{\mathbf{h}_{l,n}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with a $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$-grading. By Lemma 2.5, the grading group of the centre $Z(F_{\mathbf{h}_{l,n}}, \tau_1)$ is equal to $\Lambda^{(2l+1)}$. Thus it is clear from this that $\Sigma\big(S(n, l, \tau_1)\big) \supset \Lambda^{(2l+1)}$. For the other inclusion, suppose $\Sigma\big(S(n, l, \tau_1)\big) \setminus \Lambda^{(2l+1)} \neq \varnothing$. Then there exists $\boldsymbol{\kappa} := \kappa_1\boldsymbol{\sigma}_1 + \cdots + \kappa_{2l+1}\boldsymbol{\sigma}_{2l+1} \in \Sigma\big(S(n, l, \tau_1)\big)$, where $\kappa_i = 0$ or 1 but not all $\kappa_1, \dots, \kappa_{2l}$ are 0. Then for $\kappa_j \neq 0$ with $j \leq 2l$, we have $\boldsymbol{\sigma}_k \in S(n, l, \tau_1)$ where

$$k = \begin{cases} j + 1 & \text{if } j \text{ is odd} \\ j - 1 & \text{if } j \text{ is even,} \end{cases}$$

and $\boldsymbol{\kappa} + \boldsymbol{\sigma}_k \notin S(n, l, \tau_1)$ since $\tau_1(t_1^{\kappa_1} \cdots t_{2l+1}^{\kappa_{2l+1}} t_k) = t_k t_1^{\kappa_1} \cdots t_{2l+1}^{\kappa_{2l+1}} = -t_1^{\kappa_1} \cdots t_{2l+1}^{\kappa_{2l+1}} t_k$. This is a contradiction. Hence $\Sigma\big(S(n, l, \tau_1)\big) = \Lambda^{(2l+1)}$. Thus $\mathfrak{s}\big(S(n, l, \tau_1)\big) = 2l + 1$, and hence l is a similarity invariant by Lemma 4.3. ■

Remarks 4.5

(i) One can also check that $\Sigma\big(S(n, l, *)\big) = \Sigma\big(S(n, l, \tau_2)\big) = \Lambda^{(2l)}$. So this is another reason why l is a similarity invariant for $S(n, l, *)$ or $S(n, l, \tau_2)$.
(ii) $S(n, l, \tau_1)$ for $l \geq 1$ give us $[\frac{n}{2}]$ semilattices in Λ which have the same index but are not similar, where $[\frac{n}{2}]$ is the greatest integer less than or equal to $\frac{n}{2}$.

We summarize the results about the semilattices above as a theorem.

Theorem 4.6 *Let $S(\boldsymbol{\varepsilon}, *)$ be the semilattice in Λ defined in Example 3.1. Then $S(\boldsymbol{\varepsilon}, *)$ is isomorphic to*

$$\begin{cases} S(\mathbf{h}_{l,n}, *) & (l \geq 0), \text{ or} \\ S(\mathbf{h}_{l,n}, \tau_1) & (l \geq 1), \text{ or} \\ S(\mathbf{h}_{l,n}, \tau_2) & (l \geq 2), \end{cases}$$

and any two of these three semilattices are not similar. Moreover, for each of these l is a similarity invariant.

In particular, the number of similarity classes of $S(\boldsymbol{\varepsilon}, *)$ is

$$\begin{cases} 3\left[\frac{n}{2}\right] & \text{if } n \geq 4 \\ 2 & \text{if } n = 2, 3 \\ 1 & \text{if } n = 1. \end{cases}$$

Proof We only need to show the last statement. Since $l \leq \left[\frac{n}{2}\right]$, there are $\left[\frac{n}{2}\right] + 1$ similarity classes from $S(\mathbf{h}_{l,n}, *)$ for $n \geq 1$, $\left[\frac{n}{2}\right]$ classes from $S(\mathbf{h}_{l,n}, \tau_1)$ for $n \geq 2$ and $\left[\frac{n}{2}\right] - 1$ classes from $S(\mathbf{h}_{l,n}, \tau_2)$ for $n \geq 4$. Summing them up, we get the results. ■

Remark 4.7 The number of similarity classes of semilattices in Λ is at least $2^n - n$, which is bigger than the number above if $n \geq 3$. Thus if n is not too small, one can say that the semilattices $S(\boldsymbol{\varepsilon}, *)$ are far from exhausting all semilattices in Λ.

5 Extended Affine Root Systems of Type C

We review the description of extended affine root systems of type C_r for $r \geq 3$ following [1, p. 34]. Let Λ be a lattice and S be a semilattice in a Euclidean space $\mathbb{E}$ so that

$$S + 2\Lambda \subset S \quad \text{and} \quad \Lambda + S \subset \Lambda. \tag{5.1}$$

Then an extended affine root system R of type C_r ($r \geq 3$) contains an irreducible root system $\Delta = \Delta_{sh} \sqcup \Delta_{lg}$ of type C_r, where Δ_{sh} (resp. Δ_{lg}) is the set of short (resp. long) roots, so that

$$R = R(\Lambda, S) = \Lambda \sqcup \Big(\bigsqcup_{\mu \in \Delta_{sh}} (\mu + \Lambda) \Big) \sqcup \Big(\bigsqcup_{\mu \in \Delta_{lg}} (\mu + S) \Big). \tag{5.2}$$

The rank of the lattice Λ is called the *nullity* of R.

If (Λ, S) and (Λ', S') are pairs of a lattice and a semilattice in $\mathbb{E}$ satisfying (5.1), we say that (Λ, S) and (Λ', S') are *isomorphic*, written $(\Lambda, S) \cong (\Lambda', S')$, if there exists $\varphi \in \mathrm{GL}(\mathbb{E})$ such that $\varphi(\Lambda) = \Lambda'$ and $\varphi(S) = S'$. Also, we say that (Λ, S) and (Λ', S') are *similar*, written $(\Lambda, S) \sim (\Lambda', S')$, if there exists $\lambda \in S$ such that $(\Lambda, S + \lambda) \cong (\Lambda', S')$. Note that $(\Lambda, S + \lambda)$ is a pair of a lattice and a semilattice satisfying (5.1) (see [1, Definition 4.8, p. 45]). The relations $\cong$ and $\sim$ are equivalence relations. It is shown in [1, Theorem 3.1, p. 39] that the root systems $R(\Lambda, S)$ and $R(\Lambda', S')$ are isomorphic if and only if $(\Lambda, S) \sim (\Lambda', S')$.

In general, (5.1) implies that $2\Lambda \subset S \subset \Lambda$, and so $2\Lambda \subset \langle S \rangle \subset \Lambda$. Thus we have

$$|\Lambda / \langle S \rangle| = 2^t, \quad \text{where } 0 \le t \le n.$$

The integer $t = t(\Lambda, S)$ is called the *twist number* of the pair (Λ, S). The twist number is a similarity invariant of the pair (see [1, Definition 4.11, p. 46]), and so the twist number is an isomorphism invariant of the root system $R(\Lambda, S)$.

Example 5.3 Let Λ be a lattice with basis $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\}$. Then the pair $(\Lambda, \Lambda^{(t)})$ satisfies (5.1) with twist number t, where $\Lambda^{(t)}$ is defined in Example 3.1. Moreover, for any semilattice S' in $\mathbb{Z}\boldsymbol{\sigma}_{t+1} + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n$, the pair $(\Lambda, 2\mathbb{Z}\boldsymbol{\sigma}_1 + \cdots + 2\mathbb{Z}\boldsymbol{\sigma}_t + S')$ satisfies (5.1) with twist number t [1, Proposition 4.17, p. 47].

The root systems of extended affine Lie algebras are extended affine root systems. However, it was conjectured in [1] that an extended affine root system is not necessarily the root system of an extended affine Lie algebra. Allison and Gao have shown in [2] that the twist numbers of root systems of extended affine Lie algebras of type C_r ($r \ge 3$) do not exceed 3. Precisely, they showed that such a root system R is given by

$$R\big(\Lambda, S(\boldsymbol{\varepsilon}, \tau)\big) \quad \text{if } r \ge 4,$$

where $S(\boldsymbol{\varepsilon}, \tau)$ is the semilattice of $(F_{\boldsymbol{\varepsilon}}, \tau)$ for any elementary quantum matrix $\boldsymbol{\varepsilon}$ and any graded involution τ defined in Example 3.1 and Λ is a toral grading of $F_{\boldsymbol{\varepsilon}}$. If $r = 3$, then

$$R\big(\Lambda, S(\boldsymbol{\varepsilon}, \tau)\big) \text{ or } R(\Lambda, \Lambda^{(3)}),$$

where the second one comes from the octonion torus with standard involution (see [2, List 6.1, p. 46, and Proposition 4.25, p. 20]). Then they calculated the twist number of $\big(\Lambda, S(\boldsymbol{\varepsilon}, \tau)\big)$, and showed that such numbers do not exceed 2 (see [2, Theorem 6.2 (b), p. 46]). This fact also follows from our Corollary 3.2. Namely, we have

$$\big(\Lambda, S(\boldsymbol{\varepsilon}, \tau)\big) \cong \begin{cases} (\Lambda, \Lambda^{(1)}), & \text{or} \\ (\Lambda, \Lambda^{(2)}), & \text{or} \\ \big(\Lambda, S(\boldsymbol{\eta}, *)\big) \end{cases}$$

for some elementary quantum matrix $\boldsymbol{\eta}$, and so

$$t\big(\Lambda, S(\boldsymbol{\eta}, *)\big) = 0, \quad t(\Lambda, \Lambda^{(1)}) = 1 \quad \text{and} \quad t(\Lambda, \Lambda^{(2)}) = 2.$$

Note that in general, even if $t = t(\Lambda, S) = 1$, 2 or 3, there are many non-isomorphic semilattices S with the same twist number if n is not too small, as we suggested in Example 5.3. In fact, if $n \geq 5$, then there are at least two non-isomorphic semilattices S (exactly two if $t = 3$). However, in the pairs arising from root systems of extended affine Lie algebras, there is only one, up to isomorphism, in each case, *i.e.*, $\Lambda^{(1)}$ for $t = 1$, $\Lambda^{(2)}$ for $t = 2$ and $\Lambda^{(3)}$ for $t = 3$.

As a corollary of Theorem 4.6, we get:

Corollary 5.4 *Let $R = R(\Lambda, S)$ be the root system of an extended affine Lie algebra of type C_r $(r \geq 3)$. Then if $r \geq 4$, R is isomorphic to*

$$\begin{cases} R\big(\Lambda, S(\mathbf{h}_{l,n}, *)\big) & (l \geq 0), \text{ or} \\ R\big(\Lambda, S(\mathbf{h}_{l,n}, \tau_1)\big) & (l \geq 0), \text{ or} \\ R\big(\Lambda, S(\mathbf{h}_{l,n}, \tau_2)\big) & (l \geq 1), \end{cases}$$

and if $r = 3$, R is isomorphic to

$$\begin{cases} R\big(\Lambda, S(\mathbf{h}_{l,n}, *)\big) & (l \geq 0), \text{ or} \\ R\big(\Lambda, S(\mathbf{h}_{l,n}, \tau_1)\big) & (l \geq 0), \text{ or} \\ R\big(\Lambda, S(\mathbf{h}_{l,n}, \tau_2)\big) & (l \geq 1), \text{ or} \\ R(\Lambda, \Lambda^{(3)}). \end{cases}$$

Any two of these root systems are not isomorphic. Moreover, for each of these l is an isomorphic invariant.

In particular, the number of isomorphism classes of R for $r \geq 4$ (resp. $r = 3$) is

$$\begin{cases} 3\left[\frac{n}{2}\right] + 2\ (3\left[\frac{n}{2}\right] + 3) & \text{if } n \geq 4 \\ 4\ (5) & \text{if } n = 3 \\ 4\ (4) & \text{if } n = 2 \\ 2\ (2) & \text{if } n = 1. \end{cases}$$

Finally, by Remark 4.7, we have:

Corollary 5.5 *Let $r \geq 3$. Let $\mathcal{R}_t$ be the set of isomorphism classes of root systems of type C_r with nullity n and twist number t, and let $\mathcal{L}\mathcal{R}_t$ be the subset of $\mathcal{R}_t$ consisting of isomorphism classes of the root systems of extended affine Lie algebras of type C_r with nullity n and twist number t. Then $\mathcal{L}\mathcal{R}_t = \varnothing$ for all $t > 3$. Moreover, for $t = 0$, 1, 2 or 3, $\mathcal{L}\mathcal{R}_t$ is a proper subset of $\mathcal{R}_t$ if $n \geq 5$.*

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