ROOT-GRADED LIE ALGEBRAS
WITH COMPATIBLE GRADINGS

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Abstract. Lie algebras graded by a finite irreducible reduced root system \( \Delta \) will be generalized as predivision \( \Delta G \)-graded Lie algebras for an abelian group \( G \). In this paper such algebras are classified, up to central extensions, when \( \Delta = A_l \) for \( l \geq 3 \), \( D \) or \( E \), and \( G = \mathbb{Z}^n \).

Introduction

The concept of a Lie algebra over a field \( F \) of characteristic 0 graded by a finite irreducible reduced root system \( \Delta \) or a \( \Delta \)-graded Lie algebra was introduced by Berman and Moody [3]). It is a Lie algebra \( L \) together with a finite dimensional split simple Lie algebra \( \mathfrak{g} \), a split Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) and the root system \( \Delta \), so that \( \mathfrak{g} \) has the root space decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta} \mathfrak{g}_\mu \) with \( \mathfrak{h} = \mathfrak{g}_0 \), satisfying the following three conditions:

(i) \( L \) contains \( \mathfrak{g} \) as a subalgebra
(ii) \( L = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu \), where \( L_\mu = \{ x \in L \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h} \} \); and
(iii) \( L_0 = \sum_{\mu \in \Delta} [L_\mu, L_{-\mu}] \).

The subalgebra \( \mathfrak{g} = (\mathfrak{g}, \mathfrak{h}) \) is called the grading subalgebra of \( L \).

Berman and Moody classified \( \Delta \)-graded Lie algebras, up to central extensions, when \( \Delta \) has type \( A_l \), \( l \geq 2 \), \( D \) or \( E \) in [3], and then Benkart and Zelmanov completed the classification for the other types in [5] (see also [7] for the classification of \( \Delta \)-graded Lie algebras over rings where \( \Delta \) is not necessarily finite there, using Jordan methods).

Let us explain the case \( \Delta = A_l \), \( l \geq 3 \), in order to describe our motivation of this paper. By [3], an \( A_l \)-graded Lie algebra covers \( \text{psl}_{l+1}(A) \) for a unital associative algebra \( A \) (see Definition 2.9). Then Berman, Gao and Krylyuk showed in [4] that the core of an extended affine Lie algebra of type \( A_l \) for \( l \geq 3 \) is an \( A_l \)-graded Lie algebra and covers \( \text{sl}_{l+1}(\mathbb{C}_q) \) where \( \mathbb{C}_q = \mathbb{C}_q[t_1^\pm, \ldots, t_n^\pm] \) is a certain \( \mathbb{Z}^n \)-graded associative algebra called, a quantum torus over \( \mathbb{C} \) (see §2 below). The Lie algebra \( L = \text{sl}_{l+1}(\mathbb{C}_q) \) is not only graded by \( A_l \) but also graded by \( \mathbb{Z}^n \), and the \( \mathbb{Z}^n \)-grading \( L = \bigoplus_{\alpha \in \mathbb{Z}^n} L^\alpha \)
is compatible with the $A_l$-grading $L = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu$ in the sense that

$$L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{\alpha \in \mathbb{Z}^n} L_\mu^{\alpha} \quad \text{where} \quad L_\mu^{\alpha} = L_\mu \cap L^{\alpha}.$$  

We will call such a double grading a \textit{compatible $A_l\mathbb{Z}^n$-grading} (see Definition 2.6 for the general definition). Moreover, let $\{h_\mu \in \mathfrak{h} \mid \mu \in \Delta\}$ be the set of coroots where $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices in the grading subalgebra $\mathfrak{g} = sl_{l+1}(\mathbb{C})$. Then $L$ has the following two properties:

1. for any $\mu \in \Delta$ and any $0 \neq x \in L_\mu^{\alpha}$, there exists $y \in L_{-\mu}^{\alpha}$ such that $[x, y] = h_\mu$;
2. $\dim_{\mathbb{C}} L_\mu^{\alpha} = 1$ for all $\mu \in \Delta$ and $\alpha \in \mathbb{Z}^n$.

The property 1 will be called \textit{division} (see Definition 2.6 for the general definition). Our interest is to describe such Lie algebras without the property (2), namely, division $A_l\mathbb{Z}^n$-graded Lie algebras. One of the main results of the paper which is contained in Proposition 2.13 is the following:

\textbf{Result 1.} Let $l \geq 3$. Then any division $A_l\mathbb{Z}^n$-graded Lie algebra covers $\text{psl}_{l+1}(P)$ where $P$ is a division $\mathbb{Z}^n$-graded associative algebra (i.e., all nonzero homogeneous elements are invertible).

A division $\mathbb{Z}^n$-graded associative algebra over a field $F$ can be considered as a crossed product algebra $D \ast \mathbb{Z}^n$ for an associative division algebra $D$ over $F$ (see §1). Our next interest is to describe $D \ast \mathbb{Z}^n$ as a natural generalization of the algebra $F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ of Laurent polynomials or a quantum torus $F_q$.

A triple $(D, \varphi, q)$ is called a \textit{division $\mathbb{Z}^n$-grading triple} if

1. $D$ is an associative division algebra;
2. $\varphi = (\varphi_1, \ldots, \varphi_n)$ is an $n$-tuple of automorphisms $\varphi_i$ of $D$; and
3. $q = (q_{ij})$ is an $n \times n$ matrix over $D$ satisfying, for all $1 \leq i < j < k \leq n$,

$$q_{ii} = 1 \quad \text{and} \quad q_{ji}^{-1} = q_{ij},$$

$$\varphi_j \varphi_i = I(q_{ij}) \varphi_i \varphi_j,$$

$$\varphi_k(q_{ij}) = q_{jk} \varphi_j(q_{ik}) q_{ij} \varphi_i(q_{kj}) q_{ki};$$

where $I(q_{ij})$ is the inner automorphism of $D$ determined by $q_{ij}$, i.e.,

$$I(q_{ij})(d) = q_{ij} d q_{ij}^{-1} \quad \text{for} \quad d \in D.$$

We will show that $D \ast \mathbb{Z}^n$ can be considered as a generalization of the ring $D[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ of Laurent polynomials over $D$ in $n$-variables in the following sense:

$D[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] = \bigoplus_{\alpha \in \mathbb{Z}^n} D_{\alpha}$ is a $\mathbb{Z}^n$-graded algebra, where $t_\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, and the multiplication rule is determined by

$$t_it_i^{-1} = t_i^{-1}t_i = 1, \quad t_id = dt_i \quad \text{and} \quad t_jt_i = t_it_j \quad \text{for all} \quad d \in D \quad \text{and} \quad i, j.$$
Result 2. For any division $\mathbb{Z}^n$-grading triple $(D, \varphi, q)$, there exists a division $\mathbb{Z}^n$-graded associative algebra $D_{\varphi, q} = D_{\varphi, q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ such that $D_{\varphi, q} = \oplus_{\alpha \in \mathbb{Z}^n} D t_\alpha$ has the same $\mathbb{Z}^n$-grading as $D[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ above, and the multiplication rule is determined by

$$t_i t_{i}^{-1} t_i = 1, \quad t_i d = \varphi_i(d) t_i \quad \text{and} \quad t_j t_i = q_{ij} t_i t_j \quad \text{for all } d \in D \text{ and } i, j.$$

Conversely, any division $\mathbb{Z}^n$-graded associative algebra is isomorphic to $D_{\varphi, q}$ for some division $\mathbb{Z}^n$-grading triple $(D, \varphi, q)$ (see Theorem 3.3 for more precise statements).

Consequently, one gets that any division $A_l$-$\mathbb{Z}^n$-graded Lie algebra for $l \geq 3$ covers $psl_{l+1}(D_{\varphi, q})$. We will also classify division $\Delta \mathbb{Z}^n$-graded Lie algebras when $\Delta = D$ or $E$, which is simpler than the case $A$. Moreover, our concept “division” can be generalized as “predivision” (see Definition 2.6). Result 1 and 2 above will be proved in this more general set-up.

The organization of the paper is as follows. In §1 we review basic concepts of graded algebras and crossed product algebras. In §2 we observe some properties of $\Delta G$-graded Lie algebras. Then predivision or division $\Delta G$-graded Lie algebras are defined. After describing some examples of them, we classify predivision $\Delta G$-graded Lie algebras for $\Delta = A_l (l \geq 3)$, $D$ and $E$ types. In §3 we classify crossed product algebras $R \ast \mathbb{Z}^n$. Finally in §4 we give a summary of our results.

Result 2 above is part of my Ph.D thesis, written at the University of Ottawa. I would like to thank my supervisor, Professor Erhard Neher, for his encouragement and suggestions.

§ 1 Basic Concepts

For any group $G$ and any $G$-graded algebra $L = \oplus_{g \in G} L_g$, we denote

$$\text{supp } L := \{g \in G \mid L_g \neq (0)\}.$$

Then we have $L = \oplus_{g \in G'} L_g$ where $G' = \langle \text{supp } L \rangle$ is the subgroup of $G$ generated by $\text{supp } L$. Because of this, we will in the following always assume

$$G = \langle \text{supp } L \rangle. \quad (1.1)$$

Whenever a class of algebras has a notion of invertibility, one can make the following definition:

Definition 1.2. Let $G$ be a group. A $G$-graded algebra $P = \oplus_{g \in G} P_g$ is called a predivision $G$-graded algebra if $P_g$ contains an invertible element for all $g \in \text{supp } P$. Also, $P$ is called a division $G$-graded algebra if all nonzero homogeneous elements are invertible.

One can easily check that if $P$ is associative, then $\text{supp } P = G$ and $P$ is strongly graded, i.e., $P_g P_h = P_{gh}$ for all $g, h \in G$. This is not true if $P$ is a Jordan algebra (see [9]). Predivision $G$-graded associative algebras are realized as crossed product algebras, which we recall here:
**Definition 1.3.** Let \( R \) be a unital associative algebra over a field \( F \) and \( G \) a group. Let \( R \ast G \) be the free left \( R \)-module with basis \( G = \{ g \mid g \in G \} \), a copy of \( G \). Define a multiplication on \( R \ast G \) by linear extension of
\[
(r \bar{g})(s \bar{h}) = r \sigma_g(s) \tau(g,h) \bar{gh},
\]
for \( r, s \in R \) and \( g, h \in G \), where

- (action) \( \sigma : G \to \text{Aut}_F(R) \), the group of \( F \)-automorphisms of \( R \),
- (twisting) \( \tau : G \times G \to U(R) \), the group of units of \( R \),

are arbitrary maps and \( \sigma_g := \sigma(g) \). This \( R \ast G = (R,G,\sigma,\tau) \) is called a *crossed product algebra over \( F \)* if this multiplication is associative. It is easily seen that this is in fact an algebra over \( F \). If there is no action or twisting, that is, if \( \sigma_g = \text{id} \) and \( \tau(g,h) = 1 \) for all \( g, h \in G \), then \( R \ast G = R[G] \) is the ordinary group algebra. If the action is trivial, then \( R \ast G =: R^t[G] \) is called a *twisted group algebra*. Finally, if the twisting is trivial, then \( R \ast G =: RG \) is called a *skew group algebra*.

**Remark 1.4.** If a crossed product algebra \( R \ast G \) is commutative, then the action is clearly trivial, and so \( R \ast G = R^t[G] \).

The following lemma characterizes \( \sigma \) and \( \tau \) (see [8], Lemma 1.1 p.2). We denote by \( I(d) \) the inner automorphism determined by \( d \in U(R) \), i.e., \( I(d)(r) = drd^{-1} \) for \( r \in R \).

**1.5.** The associativity of \( R \ast G \) is equivalent to the following two conditions: for all \( g, h, k \in G \),

(i) \( \sigma_g \sigma_h = I(\tau(g,h)) \sigma_{gh} \),
(ii) \( \sigma_g(\tau(h,k)) \tau(g,hk) = \tau(g,h) \tau(gh,k) \).

**Remark 1.6.** If \( R \) is commutative, then the action \( \sigma : G \to \text{Aut}_F(R) \) becomes a group homomorphism by condition (i) in 1.5. So the action is really a “group action” in usual sense. Also, for a skew group algebra \( RG \), the action becomes a group homomorphism for the same reason. Conversely, any group action \( G \to \text{Aut}_F(R) \) defines a skew group algebra \( RG \).

If \( d : G \to U(R) \) assigns to each element \( g \in G \) a unit \( d_g \), then \( \hat{G} = \{ d_g \bar{g} \mid g \in G \} \) yields another \( R \)-basis for \( R \ast G \) so that \( R \ast G \) is a crossed product algebra for the new basis. One calls this a *diagonal change of basis* ([8], p.3). Any crossed product algebra has an identity element. It is of the form \( 1 = u \bar{e} \) for some unit \( u \) in \( R \) where \( e \) is the identity element of \( G \) ([8], Exercise 2 p.9). We can and will assume that \( 1 = \bar{e} \), via a diagonal change of basis, and so \( \tau(g,e) = \tau(e,g) = 1 \) for all \( g \in G \). The embedding of \( R \) into \( R \ast G \) is then given by \( r \mapsto r \bar{e} \). Also, we have ([8], p.3)

\[
(1.7) \quad r \bar{g} \quad \text{is invertible if and only if} \quad r \in U(R).
\]
Now, it is clear that a crossed product algebra \( R \ast G = \oplus_{g \in G} Rg \) is a predivision \( G \)-graded associative algebra. Conversely, suppose that \( A = \oplus_{g \in G} A_g \) is a predivision \( G \)-graded associative algebra over \( F \). Then we have \( A = \oplus_{g \in G} Rg \) where \( R = A_k \) and an invertible element \( x_g \in A_g \), which exists since \( A \) is predivision graded and supp \( A = G \). Moreover, for \( h \in G \), we have \( x_g x_h = x_g x_h (x_g h)^{-1} x_g h \). So we can put \( \tau(g, h) := x_g x_h (x_g h)^{-1} \in U(R) \). Then we have \( x_g x_h = \tau(g, h) x_g h \). Also, let \( I(x_g) \) be the inner automorphism determined by \( x_g \) and let \( \sigma_g := I(x_g) \mid_R \). Then, \( \sigma_g \) is clearly an \( F \)-automorphism of \( R \) and for \( r, r' \in R \),
\[
(rx_g)(r' x_h) = r(x_g r' x_g^{-1}) x_g x_h = r \sigma_g(r') x_g x_h = r \sigma_g(r' \tau(g, h) x_g h).
\]
Hence \( A \) is a crossed product algebra \( R \ast G \) determined by these \( \sigma \) and \( \tau \). So the two concepts, a crossed product algebra \( R \ast G \) and a predivision \( G \)-graded associative algebra, coincide (see [8], Exercise 2 p.18). In particular, a division \( G \)-graded associative algebra is a crossed product algebra \( R \ast G \) where \( R \) is a division algebra.

By Remark 1.4, a predivision \( G \)-graded commutative associative algebra \( Z = \oplus_{g \in G} Z_g \) \((G \text{ is necessarily abelian})\) is a twisted group algebra \( K^G[G] \) where \( K := Z_e \). Moreover (see [8], Exercise 6 p.10):

1.8. If the abelian group \( G \) is free, then \( Z \) is a group algebra \( K^G[G] \). In particular, when \( G = \mathbb{Z}^n \), \( Z \) is the algebra \( K[z_{1,1}^\pm, \ldots , z_{n,1}^\pm] \) of Laurent polynomials for invertible elements \( z_i \in Z_{\varepsilon_i} \), \( i = 1, \ldots , n \), where \( \{\varepsilon_1, \ldots , \varepsilon_n\} \) is a basis of \( \mathbb{Z}^n \).

§ 2 Predivision \( \Delta \)-graded Lie algebras

In this section \( F \) is a field of characteristic 0 and \( \Delta \) is a finite irreducible reduced root system. We have defined a \( \Delta \)-graded Lie algebra \( L = \oplus_{\mu \in \Delta \cup \{0\}} L_{\mu} \) over \( F \) in Introduction. We note that the centre \( Z(L) \) of \( L \) is contained in \( L_0 \).

A homomorphism (resp. an isomorphism) \( \varphi : L \longrightarrow L' \) of \( \Delta \)-graded Lie algebras \( L = (L, \mathfrak{g}, \mathfrak{h}) \) and \( L' = (L', \mathfrak{g}', \mathfrak{h}') \), which have the same type \( \Delta \), is called a \( \Delta \)-homomorphism (resp. an \( \Delta \)-isomorphism) if \( \varphi(\mathfrak{g}) = \mathfrak{g}' \) and \( \varphi(\mathfrak{h}) = \mathfrak{h}' \) (cf. Definition 1.20 in [3]). Then one can check that \( \varphi(L_\alpha) \subset L'_\alpha \) for all \( \alpha \in \Delta \), and so \( \varphi(L_0) \subset L'_0 \). In other words, a \( \Delta \)-homomorphism is graded.

Recall that a cover \( \hat{L} = (\hat{L}, \pi) \) of a Lie algebra \( L \) is an epimorphism \( \pi : \hat{L} \longrightarrow L \) of Lie algebras so that \( \hat{L} \) is perfect, i.e., \( \hat{L} = [\hat{L}, \hat{L}] \), and ker \( \pi \) is contained in the centre of \( \hat{L} \). If \( \pi : \hat{L} \longrightarrow L \) is a cover of a \( \Delta \)-graded Lie algebra \( L \), then there exists a \( \Delta \)-grading of \( \hat{L} \) such that \( \pi \) is a \( \Delta \)-homomorphism (see Proposition 1.24 in [3]). However, it is not known whether or not, for \( \Delta \)-graded Lie algebras \( \hat{L} \) and \( L \), any cover \( \hat{L} \longrightarrow L \) is a \( \Delta \)-homomorphism. Thus we define the following:

**Definition 2.1.** For \( \Delta \)-graded Lie algebras \( \hat{L} \) and \( L \), if \( \pi : \hat{L} \longrightarrow L \) is a cover and a \( \Delta \)-homomorphism, \( \hat{L} = (\hat{L}, \pi) \) is called a \( \Delta \)-cover of \( L \). Also, for \( \Delta \)-graded Lie algebras \( L \) and \( L' \), if there exist a \( \Delta \)-graded Lie algebra \( \hat{L} \) and maps \( \pi : \hat{L} \longrightarrow L \) and \( \pi' : \hat{L} \longrightarrow L' \) such that \((\hat{L}, \pi)\) and \((\hat{L}, \pi')\) are both \( \Delta \)-covers, we say that \( L \) and \( L' \) are \( \Delta \)-isogeneous.
Example 2.2. Let $L = (L, \mathfrak{g}, \mathfrak{h})$ be a $\Delta$-graded Lie algebra with its centre $Z(L)$. Then, for any subspace $V$ of $Z(L)$, $L/V = (L/V, \mathfrak{g} + V, \mathfrak{h} + V)$ is a $\Delta$-graded Lie algebra, and the canonical epimorphism $L \twoheadrightarrow L/V$ is a $\Delta$-cover. In particular, $L$ and $L/V$ are $\Delta$-isogeneous.

We will show that if $L$ and $L'$ are $\Delta$-isogeneous, then $L/Z(L)$ and $L'/Z(L')$ are $\Delta$-isomorphic, i.e., there exists a $\Delta$-isomorphism between them.

Lemma 2.3. Let $\pi : \tilde{L} \twoheadrightarrow L$ be a $\Delta$-cover and $c : L \twoheadrightarrow L/Z(L)$ the canonical epimorphism. Then we have $Z(\tilde{L}) = \pi^{-1}(Z(L))$, and hence $\ker c \circ \pi = Z(\tilde{L})$.

Proof. It is clear that $Z(\tilde{L}) \subset \pi^{-1}(Z(L))$. For the other inclusion, let $x \in \pi^{-1}(Z(L))$. Then $x \in \tilde{L}_0$, and so for any $\alpha \in \Delta$, one has $[x, \tilde{L}_\alpha] \subset \tilde{L}_\alpha$. On the other hand, we have $[x, \tilde{L}_\alpha] \subset \ker \pi \subset Z(L) \subset \tilde{L}_0$. Hence $[x, \tilde{L}_\alpha] = (0)$ and we get $x \in Z(\tilde{L})$. \hfill \qed

Corollary 2.4. Suppose that $L$ and $L'$ are $\Delta$-isogeneous. Then $L/Z(L)$ and $L'/Z(L')$ are $\Delta$-isomorphic.

Proof. By our assumption, there exists a $\Delta$-graded Lie algebra $\tilde{L} = (\tilde{L}, \mathfrak{g}, \mathfrak{h})$ such that $\pi : \tilde{L} = (L, \mathfrak{g}, \mathfrak{h}) \twoheadrightarrow L$ and $\pi' : \tilde{L} \twoheadrightarrow L' = (L', \mathfrak{g}', \mathfrak{h}')$ are both $\Delta$-covers. Let $c : L \twoheadrightarrow L/Z(L)$ and $c' : L' \twoheadrightarrow L'/Z(L')$ be the canonical epimorphisms. Then, by Lemma 2.3, we have $\ker c \circ \pi = Z(\tilde{L}) = \ker c' \circ \pi'$. Hence there exists the induced isomorphism

$$\varphi : L/Z(L) = (L/Z(L), \mathfrak{g} + Z(L), \mathfrak{h} + Z(L)) \twoheadrightarrow L'/Z(L') = (L'/Z(L'), \mathfrak{g}' + Z(L'), \mathfrak{h}' + Z(L'))$$

such that $\varphi \circ c \circ \pi = c' \circ \pi'$. In particular, $\varphi(\mathfrak{g} + Z(L)) = \varphi \circ c \circ \pi(\mathfrak{g}) = c' \circ \pi'(\mathfrak{g}) = \mathfrak{g}' + Z(L')$ and similarly $\varphi(\mathfrak{h} + Z(L)) = \mathfrak{h}' + Z(L')$. Therefore, $\varphi$ is a $\Delta$-isomorphism. \hfill \qed

Remark 2.5. Any $\Delta$-graded Lie algebra is perfect. Also, any perfect Lie algebra $L$, we have $Z(L/Z(L)) = (0)$. Indeed, it is enough to show that if $x \in L$ satisfies $[x, L] \subset Z(L)$, then $x \in Z(L)$. Since $[x, L] = [x, [L, L]] \subset [[x, L], L] + [L, [x, L]] = (0)$, we get $x \in Z(L)$.

Now we define new concepts.

Definition 2.6. Let $L = (L, \mathfrak{g}, \mathfrak{h}) = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu$ be a $\Delta$-graded Lie algebra over $F$. Let $G$ be an abelian group. We say that $L$ admits a compatible $G$-grading or simply $L$ is a $\Delta G$-graded Lie algebra if

$$L = \bigoplus_{g \in G} L^g$$

is a $G$-graded Lie algebra such that $\mathfrak{g} \subset L^0$.

As a consequence, $L^g$ is a $\mathfrak{h}$-module for all $g \in G$ via the adjoint action. Hence we have $L^g = \bigoplus_{\mu \in \Delta \cup \{0\}} L^g_\mu$ where $L^g_\mu = L_\mu \cap L^g$ (see [6] Proposition 1, p.92). Therefore, $L_\mu = \bigoplus_{g \in G} L^g_\mu$ and

$$L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{g \in G} L^g_\mu.$$
Remark 2.7. (i) The compatible $G$-grading is completely determined by $L^g_\mu$ for all $\mu \in \Delta$ and $g \in G$ since $L^g_\mu = \sum_{\mu \in \Delta} \sum_{g=h+k} [L^h_\mu, L^k_{-\mu}]$.

(ii) Let $\text{supp} L_\mu := \{g \in G \mid L^g_\mu \neq (0)\}$. Then we have

$$\text{supp} L \subset \bigcup_{\mu \in \Delta} (\text{supp} L_\mu + \text{supp} L_{-\mu}),$$

where $\text{supp} L = \{g \in G \mid L^g \neq (0)\}$ as defined in the beginning of §1.

Let $Z(L)$ be the centre of $L$ and let

$$\{h_\mu \in \mathfrak{h} \mid \mu \in \Delta\}$$

be the set of coroots. Then a $\Delta G$-graded Lie algebra $L$ is called predivision if

(pd) for any $\mu \in \Delta$ and any $L^g_\mu \neq (0)$, there exist $x \in L^g_\mu$ and $y \in L^{-g}_{-\mu}$ such that $[x, y] \equiv h_\mu$ modulo $Z(L)$;

and division if

(d) for any $\mu \in \Delta$ and any $0 \neq x \in L^g_\mu$, there exists $y \in L^{-g}_{-\mu}$ such that $[x, y] \equiv h_\mu$ modulo $Z(L)$.

Note that (d) implies (pd), i.e., ‘division’ $\implies$ ‘predivision’. If $\dim_F L^g_\mu \leq 1$ for all $\mu \in \Delta$ and $g \in G$, then the two concepts, ‘predivision’ and ‘division’, coincide.

Example 2.8. (a) A $\Delta$-graded Lie algebra is a predivision $\Delta G_0$-graded for the trivial group $G_0 = \{0\}$.

(b) The core of an extended affine Lie algebra of reduced type $\Delta$ with nullity $n$ is a division $\Delta \Lambda$-graded Lie algebra over $\mathbb{C}$, where $\Lambda$ is a free abelian group of rank $n$. Indeed, it is known that such a core $L$ is a $\Delta$-graded Lie algebra over $\mathbb{C}$ and has a $\Lambda$-grading, say

$$L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{\delta \in \Lambda} L_{\mu+\delta},$$

where $\Lambda$ is defined as the group generated by isotropic roots $\delta$ in a vector space, which turns out to be a lattice of rank $n$, and so $\text{supp} L$ of the $\Lambda$-grading of $L$ is equal to $\Lambda$ (see for the details in [2]). Also, the grading subalgebra $\mathfrak{g}$ is contained in $\bigoplus_{\mu \in \Delta \cup \{0\}} L_{\mu}$ ($L_{\mu} = L_{\mu+0}$) so that the $\Lambda$-grading $L = \bigoplus_{\delta \in \Lambda} L^\delta$, where $L^\delta := \bigoplus_{\mu \in \Delta \cup \{0\}} L_{\mu+\delta}$, is compatible. Thus $L$ is a $\Delta \Lambda$-graded Lie algebra.

We recall one of the basic properties of extended affine Lie algebras (see [1]): For any $\mu \in \Delta$, $\delta \in \Lambda$ and any $0 \neq e_{\mu+\delta} \in L_{\mu+\delta}$, there exist some $f_{\mu+\delta} \in L_{-\mu-\delta}$ and $h_{\mu+\delta} \in L_0$ ($= L_{0+0}$) such that $\langle e_{\mu+\delta}, f_{\mu+\delta}, h_{\mu+\delta} \rangle$ is an $sl_2$-triplet, and in particular $[e_{\mu+\delta}, f_{\mu+\delta}] = h_{\mu+\delta}$.

One can check that $h_\mu - h_{\mu+\delta} \in Z(L)$ for all coroots $h_\mu = h_{\mu+0}$ of $\mathfrak{g}$. Therefore $L$ is a division $\Delta \Lambda$-graded Lie algebra. We note that $\dim C L_{\mu+\delta} \leq 1$ for all $\mu \in \Delta$ and $\delta \in \Lambda$, which is also one of the basic properties of extended affine Lie algebras.
(c) Let $Z = \bigoplus_{g \in G} Z_g$ be a $G$-graded commutative associative algebra over $F$ and let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta} \mathfrak{g}_{\mu}$ be a finite dimensional split simple Lie algebra over $F$ of type $\Delta$ with the set $\{h_\mu \in \mathfrak{h} \mid \mu \in \Delta\}$ of coroots. Then $L := \mathfrak{g} \otimes_F Z$ is a $\Delta G$-graded Lie algebra. In fact, $L = \bigoplus_{\mu \in \Delta \cup \{0\}} (\mathfrak{g}_\mu \otimes_F Z)$ for $\mathfrak{g}_0 = \mathfrak{h}$ is a $\Delta$-graded Lie algebra with grading subalgebra $\mathfrak{g} = \mathfrak{g} \otimes 1$. We put $L^g := \mathfrak{g} \otimes_F Z_g$ for all $g \in G$. Then $\text{supp} L = \text{supp} Z$ and $L = \bigoplus_{g \in G} L^g$ is a $G$-graded Lie algebra with $\mathfrak{g} \subset L^0$, i.e., compatible. Hence $L$ is a $\Delta G$-graded Lie algebra. We call the compatible $G$-grading of $L = \mathfrak{g} \otimes_F Z$ the natural compatible $G$-grading from the $G$-grading of $Z$.

Suppose that $Z = \bigoplus_{g \in G} K\overline{g}$ is a crossed product commutative algebra over $F$. Let $e \in \mathfrak{g}_\mu$ and $f \in \mathfrak{g}_{-\mu}$ such that $[e, f] = h_\mu$. Then $e \otimes \overline{g} \in L^{\mu}$, $f \otimes \overline{g}^{-1} \in L^{-\mu}$ and

$$[e \otimes \overline{g}, f \otimes \overline{g}^{-1}] = [e, f] \otimes \overline{g} \overline{g}^{-1} = h_\mu \otimes 1 = h_\mu$$

for all $g \in G$, and so $L$ is a predivision $\Delta G$-graded Lie algebra over $F$. Note that $Z(L) = (0)$. Also, if $K$ is a field, then $L$ is a division $\Delta G$-graded Lie algebra.

Suppose that $\overline{L} = (\overline{L}, \overline{\mathfrak{g}}, \overline{\mathfrak{h}}) = \bigoplus_{g \in G} \overline{L}^g$ is a $\Delta G$-graded Lie algebra and that $\pi : \overline{L} \longrightarrow L$ is a cover of a Lie algebra $L$. Then $L = \bigl( L, \pi(\mathfrak{g}), \pi(\mathfrak{h}) \bigr)$ becomes a $\Delta$-graded Lie algebra so that $(\overline{L}, \pi)$ is a $\Delta$-cover of $L$. Moreover, if $\text{ker} \pi$ is $G$-graded, then $L$ admits the induced compatible $G$-grading $L = \bigoplus_{g \in G} \pi(\overline{L}^g)$. In particular, the centre $Z(\overline{L})$ is always $G$-graded, $\overline{L}/Z(\overline{L})$ is a $\Delta G$-graded Lie algebra.

**Definition 2.9.** Let $P$ be a unital associative algebra over $F$ and let $\mathfrak{gl}_{l+1}(P)$ be the Lie algebra consisting of all $(l + 1) \times (l + 1)$ matrices over $P$ under the commutator product $(l \geq 1)$. Let $e_{ij}(a) \in \mathfrak{gl}_{l+1}(P)$ whose $(i, j)$-entry is $a$ and the other entries are all 0. We define $sl_{l+1}(P)$ as the subalgebra of $\mathfrak{gl}_{l+1}(P)$ generated by $e_{ij}(a)$ for all $a \in P$ and $1 \leq i \not= j \leq l + 1$. The centre $Z(sl_{l+1}(P))$ of $sl_{l+1}(P)$ consists of $\sum_{i=1}^{l+1} e_{ii}(a)$ for $a \in [P, P] \cap Z(P)$ where $[P, P]$ is the span of all commutators in $P$ and $Z(P)$ is the centre of $P$. We define $psl_{l+1}(P)$ as $sl_{l+1}(P)/Z(sl_{l+1}(P))$.

It is well-known that $sl_{l+1}(P)$ is an $A_l$-graded Lie algebra (see [3]): Denote $\{e_{ij}(b) \mid b \in B\}$ by $e_{ij}(B)$ for any subset $B \subset P$. Let

$$sl_{l+1}(F) = \mathfrak{h} \oplus \bigoplus_{1 \leq i \not= j \leq l+1} e_{ij}(F1) \subset sl_{l+1}(P),$$

be the split simple Lie algebra over $F$ of type $A_l$ where $\mathfrak{h}$ is the Cartan subalgebra consisting of diagonal matrices of $sl_{l+1}(F)$. Let $\varepsilon_i : \mathfrak{h} \longrightarrow F$ be the projection onto the $(i, j)$-entry for $i = 1, \ldots, l + 1$, and $\Delta := \{\varepsilon_i - \varepsilon_j \mid i \not= j\}$, which is a root system of type $A_l$. Then

$$sl_{l+1}(P) = L_0 \oplus \bigg( \bigoplus_{\varepsilon_i - \varepsilon_j \in \Delta} e_{ij}(P) \bigg).$$
where \( L_0 = \sum_{\varepsilon_i - \varepsilon_j \in \Delta} [e_{ij}(P), e_{ji}(P)] \), is an \( A_l \)-graded Lie algebra with grading subalgebra \( sl_{l+1}(F) \). Let \( Z := Z(sl_{l+1}(P)) \). We can and will identify \( sl_{l+1}(F) + Z \) with \( sl_{l+1}(F) \) and \( e_{ij}(P) + Z \) with \( e_{ij}(P) \), and so

\[
psl_{l+1}(P) = (L_0/Z) \oplus \left( \bigoplus_{\varepsilon_i - \varepsilon_j \in \Delta} e_{ij}(P) \right)
\]

is also an \( A_l \)-graded Lie algebra with the same grading subalgebra \( sl_{l+1}(F) \).

**Example 2.10.** Let \( sl_{l+1}(P) \) be the \( A_l \)-graded Lie algebra over \( F \) with grading subalgebra \( sl_{l+1}(F) \) described above. If \( P = \bigoplus_{g \in G} P_g \) is a \( G \)-graded algebra, then \( sl_{l+1}(P) \) admits a compatible \( G \)-grading. Indeed, let

\[
sl_{l+1}(P)^g := \left\{ \sum_{i,j} e_{ij}(P_g) \mid \sum_{i,j} e_{ij}(P_g) \subset sl_{l+1}(P) \right\}.
\]

Then \( sl_{l+1}(P) = \bigoplus_{g \in G} sl_{l+1}(P)^g \), and it is a \( G \)-graded Lie algebra with \( sl_{l+1}(F) \subset L^0 \), i.e., compatible. Note that \( \text{supp} \left( sl_{l+1}(P) \right) \supset \text{supp} \, P \), and so \( \langle \text{supp} \left( sl_{l+1}(P) \right) \rangle = G \). Also, \( psl_{l+1}(P) \) admits the induced compatible \( G \)-grading. We call the compatible \( G \)-grading of \( sl_{l+1}(P) \) or \( psl_{l+1}(P) \), i.e.,

\[
sl_{l+1}(P)^g_{\varepsilon_i - \varepsilon_j} = e_{ij}(P_g) = psl_{l+1}(P)^g_{\varepsilon_i - \varepsilon_j} \quad \text{for all } \varepsilon_i - \varepsilon_j \in \Delta \text{ and } g \in G,
\]

the natural compatible \( G \)-grading from the \( G \)-grading of \( P \).

If \( P = \bigoplus_{g \in G} Rg \) is a crossed product algebra, then

\[
[e_{ij}(\overline{g}), e_{ji}(\overline{g}^{-1})] = e_{ii}(1) - e_{jj}(1) = [e_{ij}(1), e_{ji}(1)] = h_{\varepsilon_i - \varepsilon_j}
\]

for all \( g \in G \). Thus \( sl_{l+1}(P) \) and \( psl_{l+1}(P) \) with the natural compatible \( G \)-gradings from the \( G \)-grading of \( P \) are predvision \( A_l \)-graded Lie algebras over \( F \). Also, if \( R \) is a division algebra, then the \( A_l \)-graded Lie algebras \( sl_{l+1}(P) \) and \( psl_{l+1}(P) \) are division.

For any associative algebra \( P \), one can define a new product, \( p \cdot q = \frac{1}{2}(pq + qp) \) for all \( p, q \in P \). Then \( P^+ : = (P, \cdot) \) is a Jordan algebra.

**Lemma 2.11.** (i) Suppose that the \( A_l \)-graded Lie algebra \( psl_{l+1}(P) \) described above admits a predvision (resp. division) compatible \( G \)-grading. Then if \( l \geq 2 \), \( P \) is a predvision (resp. division) \( G \)-graded algebra, and the \( G \)-grading of \( psl_{l+1}(P) \) is natural from the \( G \)-grading of \( P \).

If \( l = 1 \), then \( P^+ \) is a predvision (resp. division) \( G \)-graded Jordan algebra.

(ii) Suppose that the \( \Delta \)-graded Lie algebra \( g \otimes_F Z \) described in Example 2.8(c) admits a predvision (resp. division) compatible \( G \)-grading. Then \( Z \) is a predvision
(resp. division) $G$-graded algebra, and the $G$-grading of $g \otimes_F Z$ is natural from the $G$-grading of $Z$.

**Proof.** (i): By our assumption, $psl_{l+1}(P) = psl_{l+1}(P)_0 \oplus ( \oplus_{\varepsilon_i-\varepsilon_j \in \Delta} e_{ij}(P))$ admits a predivision (resp. division) compatible $G$-grading, say

$$psl_{l+1}(P) = psl_{l+1}(P)_0 \oplus ( \oplus_{\varepsilon_i-\varepsilon_j \in \Delta} \oplus g \in G e_{ij}(P)^g).$$

Let

$$P'_{ij} := \{ p \in P \mid e_{ij}(p) \in e_{ij}(P)^g \}.$$

We claim that $P'_{ij} = P_{g}^r$ for all $\varepsilon_r - \varepsilon_s \in \Delta$. If $l = 1$, then $\Delta = \{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_1 \}$. For $p \in P^1_{g}$, we have

$$[[e_{12}(p), e_{21}(1)], e_{21}(1)] = -2e_{21}(p) \in e_{21}(P)^g$$

since $e_{21}(1) \in e_{21}(P)^0$. Thus $p \in P^1_{g}$ and we get $P^1_{g} \subset P^1_{g}$. The other inclusion can be obtained by the similar way. Hence the claim holds for $l = 1$.

In general, it is well-known that for any distinct $\alpha, \beta \in \Delta = A_l$, $l \geq 2$, $D$ or $E$, there exists a sequence $\alpha_1, \ldots, \alpha_t \in \Delta$ so that $\alpha_1 = \alpha$, $\alpha_t = \beta$ and $\alpha_{i+1} - \alpha_i \in \Delta$ for $i = 1, \ldots, t - 1$.

Now, for $l \geq 2$, it is enough to show that $P'_{ij} \subset P_{g}^r$. Let $p \in P'_{ij}$. We apply the above for $\alpha = \varepsilon_i - \varepsilon_j$ and $\beta = \varepsilon_r - \varepsilon_s$. For $p \in P'_{ij}$,

$$\cdots[[e_{ij}(p), e_{\alpha_1}(1)], e_{\alpha_2}(1)], \ldots, e_{\alpha_t}(1)] = \pm e_{\alpha_t}(p) = \pm e_{r_s}(p) \in e_{r_s}(P)^g$$

since $e_{\alpha_i}(1) \in L_{\alpha_i}$. Hence $p \in P_{g}^r$ and our claim is settled.

Thus one can write $P_g = P_{g}^r$ and $P = \oplus g \in G P_g$. Since, for $p \in P_g$ and $q \in P_h$ ($g, h \in G$),

$$[e_{ij}(p), e_{jk}(q)] = e_{ik}(pq) \in e_{ik}(P)^{g+h} \quad \text{if } l \geq 2 \text{ and } i \neq k,$$

$$[e_{12}(p), e_{21}(1), e_{12}(q)] = e_{12}(pq + qp) \in e_{12}(P)^{g+h} \quad \text{if } l = 1,$$

we have $pq \in P_{g+h}$ if $l \geq 2$ and $pq + qp \in P_{g+h}$ if $l = 1$. Also, one can see that $\text{supp} L \subset \text{supp} P + \text{supp} P$ (see Remark 2.7), and so $(\text{supp} P) \supset (\text{supp} L) = G$, whence $(\text{supp} P) = G$. Therefore, $P$ is a $G$-graded algebra if $l \geq 2$ and $P^+$ is a $G$-graded Jordan algebra if $l = 1$. Note that $e_{ij}(P)^g = e_{ij}(P_g)$ for all $\varepsilon_i - \varepsilon_j \in \Delta$ and $g \in G$, and hence the $G$-grading for $l \geq 2$ is natural (see Remark 2.7).

By (pd), for any $\varepsilon_i - \varepsilon_j \in \Delta$ and any $g \in \text{supp} P$, there exist $e_{ij}(p) \in e_{ij}(P_g)$ and $e_{ji}(q) \in e_{jk}(P_{-g})$ such that

$$[e_{ij}(p), e_{ji}(q)] = [e_{ij}(1), e_{ji}(1)] + z \quad \text{for some } z \in Z(sl_{l+1}(P)).$$

Hence $e_{ii}(pq) - e_{jj}(qp) = e_{ii}(1) - e_{jj}(1) + \sum_{k=1}^{l+1} e_{kk}(a)$ for some $a \in P$, and so $a = 0$ and $pq = qp = 1$, i.e., $p$ is invertible. Also, $p$ is invertible in $P \iff p$ is invertible in $P^+$. Therefore, $P = \oplus g \in G P_g$ is a predivision $G$-graded associatative algebra if $l \geq 2$, and $P^+ = \oplus g \in G P_g$ is a predivision $G$-graded Jordan algebra if $l = 1$. The statement for ‘division’ can be shown in the same manner.

(ii): Let $Z_g := \{ z \in Z \mid g \otimes z \subset (g \otimes_P Z)^g \}$. Then $Z = \oplus g \in G Z_g$ becomes a $G$-graded algebra. The rest can be shown in the same manner. □
Definition 2.12. For $\Delta$-graded Lie algebras $\tilde{L} = \oplus_{g \in G} \tilde{L}^g$ and $L = \oplus_{g \in G} L^g$, if $\Delta$-cover $\pi : \tilde{L} \rightarrow L$ satisfies $L^g = \pi(\tilde{L}^g)$ for all $g \in G$, then $\tilde{L} = (\tilde{L}, \pi)$ is called a $\Delta G$-cover of $L$. Also, for $\Delta$-graded Lie algebras $L$ and $L'$, if there exist a $\Delta$-graded Lie algebra $\tilde{L}$ and maps $\pi : \tilde{L} \rightarrow L$ and $\pi' : \tilde{L} \rightarrow L'$ such that $(\tilde{L}, \pi)$ and $(\tilde{L}, \pi')$ are both $\Delta$-covers, we say that $L$ and $L'$ are $\Delta$-isogeneous.

It is clear that if $\tilde{L}$ is a $\Delta$-cover of $L$, then

$$\tilde{L} \text{ is is predivision (resp. division)} \iff L \text{ is predivision (resp. division).}$$

Also, by Corollary 2.4, if $L$ and $L'$ are $\Delta$-isogeneous, then $L/Z(L)$ and $L'/Z(L')$ are $\Delta$-isomorphic, i.e., there exists a $\Delta$-isomorphism which is also $G$-graded between them. In particular, $\tilde{L}/Z(\tilde{L})$ and $L/Z(L)$ above are $\Delta$-isomorphic.

Proposition 2.13. (i) Let $l \geq 3$. Then a predivision (resp. division) $A_l$-$G$-graded Lie algebra $L$ over $F$ is an $A_l$-$G$-cover of $\text{psl}_{l+1}(P)$ admitting the natural compatible $G$-grading from the $G$-grading of a predivision (resp. division) $G$-graded associative algebra $P$ over $F$. Hence $L/Z(L)$ and $\text{psl}_{l+1}(P)$ are $\Delta$-isomorphic.

(ii) Let $\Delta = D$ or $E$ and let $\mathfrak{g}$ be a finite dimensional split simple Lie algebra $L$ over $F$ of type $\Delta$. Then a predivision (resp. division) $\Delta$-$G$-graded Lie algebra over $F$ is a $\Delta$-cover of $\mathfrak{g} \otimes_F Z$ admitting the natural compatible $G$-grading from the $G$-grading of a predivision (resp. division) $G$-graded commutative associative algebra $Z$ over $F$. Hence $L/Z(L)$ and $\mathfrak{g} \otimes_F Z$ are $\Delta$-isomorphic.

Proof. For (i), let $L$ be a predivision $A_l$-$G$-graded Lie algebra over $F$. Berman and Moody showed in [3] that $L$ is $A_l$-isogeneous to $(\text{sl}_{l+1}(P), \text{sl}_{l+1}(F))$ (the Steinberg Lie algebra $\text{st}_{l+1}(P)$ serves as an $A_l$-cover of $L$ and $\text{sl}_{l+1}(F)$). Hence, by Corollary 2.4, $L/Z(L)$ is $A_l$-isomorphic to $\text{psl}_{l+1}(P)$. Thus $(\text{psl}_{l+1}(P), \text{sl}_{l+1}(F))$ admits a compatible $G$-grading via the $A_l$-isomorphism from the compatible $G$-grading of $L/Z(L)$ induced by the compatible $G$-grading of $L$. Therefore, the statement follows from Lemma 2.11.

(ii): Let $L$ be a predivision $\Delta$-$G$-graded Lie algebra over $F$. Berman and Moody showed in [3] that $L$ is a $\Delta$-cover of $\mathfrak{g} \otimes_F Z$. Thus the statement follows from Lemma 2.11.  

In this paper we classify predivision $\Delta Z^n$-graded Lie algebras for $\Delta = A_l$, $l \geq 3$, $D$ or $E$, up to central extensions. By Proposition 2.13, our work is to classify crossed product algebras $R \ast Z^n$. We determine such algebras as a generalization of quantum tori. Namely, let $\mathfrak{q} = (q_{ij})$ be an $n \times n$ matrix over $F$ such that

$$q_{ii} = 1 \text{ and } q_{ij}^{-1} = q_{ij}.$$ 

The quantum torus $F_\mathfrak{q} = F_\mathfrak{q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ determined by $\mathfrak{q}$ is defined as the associative algebra over $F$ with $2n$ generators $t_1^{\pm 1}, \ldots, t_n^{\pm 1}$, and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \text{ and } t_j t_i = q_{ij} t_i t_j$$
for all \(1 \leq i, j \leq n\). Quantum tori are characterized as predivision \(\mathbb{Z}^n\)-graded associative algebras whose homogeneous spaces are all 1-dimensional (see [4]). Note that \(F_q\) is commutative \(\iff q = 1\) whose entries are all 1, i.e., \(F_1 = F[\{t_1, \ldots, t_n\}]\) is the algebra of Laurent polynomials. Also, a quantum torus is a twisted group algebra \(F^\tau[\mathbb{Z}^n]\).

\[ \text{§ 3 Classification of } R \ast \mathbb{Z}^n \]

Throughout this section \(F\) is an arbitrary field and \(G\) is an arbitrary group. For a \(G\)-graded algebra \(S = \bigoplus_{g \in G} S_g \) over \(F\) in general, we denote by \(\text{GrAut}_F(S)\) the group of graded automorphisms of \(S\), i.e.,

\[ \text{GrAut}_F(S) := \{ \sigma \in \text{Aut}_F(S) \mid \sigma(S_g) = S_g \text{ for all } g \in G \}. \]

**Lemma 3.1.** Let \((R \ast G) \ast M = (R \ast G, M, \eta, \xi)\) be a crossed product algebra over \(F\) and \((R \ast G) \ast M = (R \ast G, M, \eta, \xi)\) a crossed product algebra over \(F\) for a group \(M\), an action \(\eta\) and a twisting \(\xi\). Suppose that \(\eta(M) \subset \text{GrAut}_F(R \ast G)\) and that \(\xi(m, l) \in U(R)\) for all \(m, l \in M\). Then, \((R \ast G) \ast M\) is a crossed product algebra \(R \ast (G \times M) = (R, (G \times M), \sigma', \tau')\) over \(F\) for some action \(\sigma'\) and twisting \(\tau'\).

**Proof.** We have

\[ (R \ast G) \ast M = \bigoplus_{m \in M} (R \ast G) \overline{m} = \bigoplus_{m \in M} \left( \bigoplus_{g \in G} R \overline{g} \right) \overline{m} = \bigoplus_{(g, m) \in G \times M} R \overline{g m} \]
as free \(R\)-modules. We define \(\eta_m = \eta(m) \mid R_1\) an \(F\)-automorphism of \(R\) for every \(m \in M\). Also for \(h \in G, \overline{h}\) is a unit in \((R \ast G)\) (see 1.6). Since \(\eta_m\) is a graded automorphism of \(R \ast G\) by our first assumption, \(\eta_m(\overline{h}) = d_{m,h} \overline{h}\) for some \(d_{m,h} \in U(R)\). Therefore, for \(r \overline{g m} \in R \overline{g m}\) and \(s \overline{h l} \in R \overline{h l}\), we have

\[ (r \overline{g m})(s \overline{h l}) = r \overline{g \eta_m(s) \eta_m(\overline{h}) \xi(m,l) \overline{m l}} \]

\[ = r \overline{g \eta_m(s) \eta_m(\overline{h}) \xi(m,l) \overline{m l}} \]

\[ = r \overline{g \eta_m(s) \eta_m(\overline{h}) \xi(m,l) \overline{m l}} \]

\[ = r \overline{g \eta_m(s) \eta_m(\overline{h}) \xi(m,l) \overline{m l}} \]

\[ = r \overline{g \eta_m(s) \eta_m(\overline{h}) \xi(m,l) \overline{m l}} \]

Thus we have the action

\[ \sigma' : G \times M \longrightarrow \text{Aut}_F R \text{ by } \sigma'_{(g, m)} = \sigma_g \eta_m, \]

and the twisting \(\tau' : (G \times M) \times (G \times M) \longrightarrow U(R)\) by

\[ \tau'(\langle g, m \rangle, \langle h, l \rangle) = \sigma_g(d_{m,h}) \sigma_g(\xi(m,l)) \tau(g, h). \]
Since the crossed product algebra \((R \ast G) \ast M\) is associative, we get
\[
(R \ast G) \ast M = R \ast (G \times M) = (R, G \times M, \sigma', \tau').
\]

A triple \((R, \varphi, q)\) where \(R\) is a unital associative algebra over \(F\),
\[
\varphi = (\varphi_1, \ldots, \varphi_n)
\]
is an \(n\)-tuple of \(F\)-automorphisms \(\varphi_i\) of \(R\), and \(q = (q_{ij})\) is an \(n \times n\) matrix over \(R\) satisfying, for all \(1 \leq i < j < k \leq n\),
\begin{align*}
(G1) & \quad q_{ii} = 1 \quad \text{and} \quad q_{ji}^{-1} = q_{ij}, \\
(G2) & \quad \varphi_j \varphi_i = I(q_{ij}) \varphi_i \varphi_j, \\
(G3) & \quad \varphi_k(q_{ij}) = q_{jk} \varphi_j(q_{kk}) \varphi_i(q_{kj}) q_{ki},
\end{align*}
is called a \(\mathbb{Z}^n\)-grading triple, and a division \(\mathbb{Z}^n\)-grading triple if \(R\) is a division algebra.

For a \(\mathbb{Z}^n\)-grading triple, we introduce several notations and prove some identities.

**Notations.**
\begin{align*}
(N1) & \quad \varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n, \\
& \quad \text{i.e., the } i\text{-th coordinate is 1 and the others are 0.} \\
(N2) & \quad q_{ij}^{(m)} := \begin{cases} q_{ij} \varphi_i(q_{ij}) \varphi_i^2(q_{ij}) \cdots \varphi_i^{m-1}(q_{ij}) = \prod_{l=0}^{m-1} \varphi_i^l(q_{ij}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \varphi_i^{-1}(q_{ji}) \varphi_i^{-2}(q_{ji}) \cdots \varphi_i^{m}(q_{ji}) = \prod_{l=1}^{m} \varphi_i^l(q_{ji}), & \text{if } m < 0, \end{cases} \\
& \quad \text{and } q_{ij}^{-(m)} := (q_{ij}^{(m)})^{-1}. \\
(N3) & \quad \varphi^{(\alpha)}_k := \begin{cases} \text{id}, & \text{if } k = 0, 1 \\
\varphi_1^{\alpha_1} \cdots \varphi_{k-1}^{\alpha_{k-1}}, & \text{if } k > 1, \end{cases} \\
& \quad \text{and } \varphi^{\alpha} := \varphi_1^{\alpha_1} \cdots \varphi_n^{\alpha_n}. \\
(N4) & \quad q_{\varepsilon_j, \alpha} := \prod_{i=1}^{j-1} \varphi^{(\alpha)^i}(q_{ij}^{(\alpha_i)^i}) \quad \text{with } \alpha_0 = q_{0j} = 1. \\
(N5) & \quad q_{\varepsilon_j, \alpha}^{(m)} := \begin{cases} \prod_{l=m-1}^{0} \varphi_j^{l}(q_{\varepsilon_j, \alpha}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \prod_{l=m}^{-1} \varphi_j^{l}(q_{\varepsilon_j, \alpha}), & \text{if } m < 0. \end{cases} \\
(N6) & \quad q_{\alpha, \beta} := \prod_{j=n}^{1} \varphi_j^{(\alpha_j)}(q_{\varepsilon_j, \beta}).
\end{align*}
Lemma 3.2. For \( m \in \mathbb{Z} \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \), we have

\[
\begin{align*}
(1) & \quad \varphi_i^{-m}(q_{ij}^{(-m)}) = q_{ij}^{(-m)}, \\
(2) & \quad \varphi_j \varphi_i^m = I(q_{ij}^{(m)}) \varphi_i^m \varphi_j, \\
(3) & \quad \varphi_j \varphi(\alpha)_i = \begin{cases} \\
I(\prod_{l=1}^{i-1} \varphi(\alpha)_i(q_{ij}^{(\alpha_l)})) \varphi(\alpha)_i \varphi_j & \text{for } j \geq i, \\
I(\prod_{l=1}^{j-1} \varphi(\alpha)_i(q_{ij}^{(\alpha_l)})) \varphi(\alpha + \epsilon_i) & \text{for } j < i,
\end{cases} \\
(4) & \quad q_{ij}^{(m+1)} = q_{ij} \varphi_i(q_{ij}^{(m)}) \text{ and } q_{ij}^{(-m+1)} = \varphi_i(q_{ij}^{(-m)}) q_{ji}, \\
(5) & \quad \varphi_k(q_{ij}^{(m)}) = q_{jk} \varphi_j(q_{ik}^{(m)}) \varphi_i(q_{kj}) q_{ik}^{(-m)}. 
\end{align*}
\]

Proof. For (1), we have from (N2),

\[
q_{ij}^{(-m)} = \begin{cases} \\
\varphi_i^{-1}(q_{ji}) \cdots \varphi_i(q_{ji}) q_{ji} = \prod_{l=m-1}^{1} \varphi_i(q_{ji}), & \text{if } m > 0 \\
1, & \text{if } m = 0 \\
\varphi_i^m(q_{ij}) \cdots \varphi_i^{-2}(q_{ij}) \varphi_i^{-1}(q_{ij}) = \prod_{l=m}^{1} \varphi_i(q_{ij}), & \text{if } m < 0.
\end{cases}
\]

So we get

\[
\varphi_i^{-m}(q_{ij}^{(-m)}) = \begin{cases} \\
\varphi_i^{-1}(q_{ji}) \cdots \varphi_i^{-m}(q_{ji}) = \prod_{l=-1}^{-m} \varphi_i(q_{ji}), & \text{if } m > 0 \\
1, & \text{if } m = 0 \\
q_{ij} \varphi_i(q_{ij}) \cdots \varphi_i^{-m-1}(q_{ij}) = \prod_{l=1}^{-m-1} \varphi_i(q_{ij}), & \text{if } m < 0,
\end{cases}
\]

which is exactly \( q_{ij}^{(-m)} \).

For (2), the case \( m = 0 \) is clear. Assume that \( m > 0 \). Put \( q := q_{ij} \) for simplicity. Then we have

\[
\begin{align*}
\varphi_j \varphi_i^m & = \varphi_j \varphi_i^{m-1} \varphi_i \\
& = I(q^{(m-1)}) \varphi_i^{m-1} \varphi_j \varphi_i & \text{by induction on } m \\
& = I(q^{(m-1)}) \varphi_i^{m-1} I(q) \varphi_i \varphi_j & \text{by (G2)} \\
& = I(q^{(m-1)}) I(\varphi_i^{m-1}(q)) \varphi_i^m \varphi_j \\
& = I(q^{(m)}) \varphi_i^m \varphi_j.
\end{align*}
\]

Also, \((\varphi_j \varphi_i^m)^{-1} = (I(q_{ij}^{(m)}) \varphi_i^m \varphi_j)^{-1}\) for \( m > 0 \), and so

\[
\varphi_i^{-m} \varphi_j^{-1} = \varphi_j^{-1} \varphi_i^{-m} (I(q_{ij}^{(-m)}) \varphi_i^{-m} \varphi_j)^{-1} \varphi_i^{-m} \varphi_j = \varphi_j^{-1} I(q_{ij}^{(-m)}) \varphi_i^{-m},
\]

by (1). Hence we get \( \varphi_j \varphi_i^{-m} = I(q_{ij}^{(-m)}) \varphi_i^{-m} \varphi_j \), and (2) holds for all \( m \in \mathbb{Z} \).
For (3), when \( j \geq i \), using (2), we have
\[
\varphi_j \varphi_i^{(\alpha)} = \varphi_j \varphi_1^{\alpha_1} \cdots \varphi_i^{\alpha_{i-1}} \\
= I(q_{ij}^{(\alpha_1)}) \varphi_1^{\alpha_1} \varphi_j \varphi_2^{\alpha_2} \cdots \varphi_i^{\alpha_{i-1}} \\
= I(q_{ij}^{(\alpha_1)}) \varphi_1^{\alpha_1} I(q_{2j}^{(\alpha_2)}) \varphi_j \varphi_2^{\alpha_2} \varphi_j \varphi_3^{\alpha_3} \cdots \varphi_i^{\alpha_{i-1}} \\
\cdots \\
= I(q_{ij}^{(\alpha_1)}) \varphi_1^{\alpha_1} \varphi_j \varphi_2^{\alpha_2} I(q_{3j}^{(\alpha_3)}) \varphi_j \varphi_3^{\alpha_3} \cdots I(q_{i-1,j}^{(\alpha_{i-1})}) \varphi_j \varphi_i^{\alpha_{i-1}} \\
= I(\prod_{l=1}^{i-1} \varphi^{(\alpha_l)}(q_{lj}^{(\alpha_l)})) \varphi_j \varphi_i^{(\alpha_{i-1})}. \quad \text{(Note \( \varphi^{(\alpha)} = \text{id} \) when \( i = 1 \)}
\]

When \( j < i \), we have
\[
\varphi_j \varphi_i^{(\alpha)} = \varphi_j \varphi_1^{\alpha_1} \cdots \varphi_i^{\alpha_{i-1}} \\
= I(q_{ij}^{(\alpha_1)}) \varphi_1^{\alpha_1} \varphi_j \varphi_2^{\alpha_2} \cdots \varphi_j \varphi_i^{\alpha_{i-1}} \\
\cdots \\
= I(q_{ij}^{(\alpha_1)}) \varphi_1^{\alpha_1} \cdots I(q_{j-1,j}^{(\alpha_{j-1})}) \varphi_{j-1}^{\alpha_{j-1}} I(q_{jj}^{(\alpha_j)}) \varphi_j^{\alpha_j} \varphi_j \cdots \varphi_i^{\alpha_{i-1}} \\
= I(q_{ij}^{(\alpha_1)}) \varphi_1^{\alpha_1} \cdots I(q_{j-1,j}^{(\alpha_{j-1})}) \varphi_{j-1}^{\alpha_{j-1}} \varphi_j^{\alpha_j} \cdots \varphi_i^{\alpha_{i-1}} \\
= I(\prod_{l=1}^{j-1} \varphi^{(\alpha_l)}(q_{lj}^{(\alpha_l)})) \varphi_j \varphi_i^{(\alpha_{i-1})+\epsilon_j}). \quad \text{(Note \( \varphi^{(\alpha)} = \text{id} \) when \( j = 1 \)}
\]

For the first formula of (4), the case \( m = 0 \) is clear. We put \( q := q_{ij} \), \( p := q^{-1} \) and \( \varphi := \varphi_i \) for simplicity. For \( m > 0 \), we have
\[
q^{(m+1)} = q \varphi(q) \varphi^2(q) \cdots \varphi^m(q) \\
= q \varphi(q) \varphi(q) \cdots \varphi(q) = q \varphi(q^{(m)}).
\]

For \( m = -1 \), we have \( q^{(-1+1)} = 1 \), while \( q \varphi(q^{(-1)}) = q \varphi^{-1}(p) = 1 \). For \( m < -1 \), we have
\[
q^{(m+1)} = \varphi^{-1}(p) \varphi^{-2}(p) \cdots \varphi^{m+1}(p) \\
= q \varphi^{-1}(p) \varphi^{-2}(p) \cdots \varphi^{m+1}(p) \\
= q \varphi(q^{-1}(p) \varphi^{-2}(p) \cdots \varphi^{m}(p)) = q \varphi(q^{(m)}).
\]

The second formula follows from the first since \( q_{ij}^{-(m+1)} = (q_{ij}^{(m+1)})^{-1} \).
For (5), the case \( m = 0 \) is clear. Assume that \( m > 0 \). Then we have

\[
\varphi_k(q_{ij}^{(m)}) = \varphi_k(q_{ij})\varphi_k(q_{ij}^{(m-1)}) \quad \text{by (3)}
\]

\[
= \varphi_k(q_{ij})q_{ik}\varphi_i(q_{ij}^{(m-1)})q_{ki} \quad \text{by (G2)}
\]

\[
= q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_j(q_{kj}^{(m-1)})q_{ki}q_{ik}\varphi_i(q_{jk}^{(m-1)})q_{ij}^{(m-1)}(q_{kj})(q_{ik}^{-(m-1)})q_{ki}
\]

by (G3) and induction on \( m \)

\[
= q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_j(q_{ik}^{(m-1)})\varphi_i(q_{ij}^{(m-1)})\varphi_i(q_{kj}^{-(m-1)})q_{ki}
\]

\[
= q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{ik}^{(m-1)})\varphi_i(q_{kj})\varphi_i(q_{ik}^{-(m)})
\]

by (G2) and (3)

\[
= q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{ik}^{(m-1)})\varphi_i(q_{kj})\varphi_i(q_{ik}^{-(m)}) \quad \text{by (3)}
\]

Also, one has \( (\varphi_k(q_{ij}^{(m)}))^{-1} = q_{jk}\varphi_j(q_{ik}^{(m)})q_{ij}^{(m)}\varphi_i(q_{kj})q_{ik}^{-(m)} \) for \( m > 0 \), and so

\[
\varphi_k(q_{ij}^{-(m)}) = q_{ik}^{(m)}\varphi_i(q_{kj})q_{ij}^{-(m)}\varphi_j(q_{ik}^{-(m)})q_{kj}
\]

Applying \( \varphi_i^{-(m)} \) in both hands, we get

\[
\varphi_i^{-(m)}\varphi_k(q_{ij}^{-(m)}) = \varphi_i^{-(m)}q_{ik}^{(m)}\varphi_i(q_{kj})\varphi_j(q_{ik}^{-(m)})q_{kj}
\]

\[
= \varphi_i^{-(m)}q_{ik}^{(m)}q_{kj}q_{ij}^{-(m)}\varphi_i^{-(m)}q_{ik}^{-(m)}q_{kj}^{-(m)}
\]

by (1).

Then, by (1) and (2), we have

\[
I(q_{ik}^{-(m)})\varphi_k(q_{ij}^{-(m)}) = q_{ik}^{-(m)}q_{jk}q_{ij}^{-(m)}I(q_{ij}^{-(m)})\varphi_j(q_{ik}^{-(m)})\varphi_i^{-(m)}(q_{kj})
\]

and we obtain

\[
\varphi_k(q_{ij}^{-(m)}) = q_{jk}\varphi_j(q_{ik}^{-(m)})q_{ij}^{-(m)}\varphi_i^{-(m)}(q_{kj})q_{ik}^{-(m)} \quad \text{for } m > 0.
\]

Hence, (5) holds for all \( m \in \mathbb{Z} \). □

Now we are ready to state our theorem.

**Theorem 3.3.** Let \((R, \varphi, q)\) be a \( \mathbb{Z}^n \)-grading triple and let \( R_{\varphi, q} := \oplus_{\alpha \in \mathbb{Z}^n} R_{t_{\alpha}} \) be a free left \( R \)-module with basis \( \{t_{\alpha} \mid \alpha \in \mathbb{Z}^n\} \). Then there exists a unique associative multiplication on \( R_{\varphi, q} \) such that, for \( t_i := t_{\alpha_i}, i = 1, \ldots, n, \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( r \in R \),

\[
t_{\alpha} = t_{\alpha_1}^{\alpha_1} \cdots t_{\alpha_n}^{\alpha_n}, \quad t_{i}t_{i}^{-1} = t_{i}^{-1}t_{i} = 1, \quad t_{i}r = \varphi_i(r)t_{i} \quad \text{and} \quad t_{j}t_{i} = q_{ij}t_{i}t_{j}.
\]

(3.4)
Moreover, for $r t \alpha, r' t \beta \in R \varphi, q$, we have
\[ r t \alpha r' t \beta = r \varphi^\alpha(r') q_{\alpha, \beta} t \alpha + \beta, \]
where $\varphi^\alpha$ and $q_{\alpha, \beta}$ are defined in (N3) and (N6). In particular, $R \varphi, q$ is a crossed product algebra $R \ast \mathbb{Z}^n$ with
\[
\begin{align*}
\text{(action)} & \quad \sigma : \mathbb{Z}^n \to \text{Aut}_F(R) \quad \text{by} \quad \sigma(\alpha) = \varphi^\alpha \\
\text{(twisting)} & \quad \tau : \mathbb{Z}^n \times \mathbb{Z}^n \to U(R) \quad \text{by} \quad \tau(\alpha, \beta) = q_{\alpha, \beta}.
\end{align*}
\]

Conversely, for any crossed product algebra $R \ast \mathbb{Z}^n$, there exists a $\mathbb{Z}^n$-grading triple $(R, \varphi, q)$ such that $R \ast \mathbb{Z}^n = R \varphi, q$.

**Proof.** We first consider a crossed product algebra $R \ast \mathbb{Z}$. Let $t := \bar{t} \in R \ast \mathbb{Z}$. Then, $t^m$ is a unit in $R \bar{m}$ for all $m \in \mathbb{Z}$. Using the diagonal basis change, one can take an $R$-basis $\{t^m \mid m \in \mathbb{Z}\}$. So we have $t^m t^l = t^{m+l}$ for all $m, l \in \mathbb{Z}$. Hence, $R \ast \mathbb{Z} = RZ$ is a skew group algebra. Let $\psi$ be the action of 1, i.e., $(r t 1) = \psi(r)t$ for $r \in R$. (Note that $1 = \bar{0}$.) Then the action of $m$ is $\psi^m$, i.e.,
\[ t^m(r 1) = \psi^m(r)t^m. \]
Conversely, it is clear that any $F$-automorphism $\psi$ of $R$ determines a skew group algebra $RZ$ by the action $m \mapsto \psi^m$ (see Remark 1.3). We denote this $RZ$ by $R[t; \psi]$.

Let $R^{(1)} := R[t_1; \psi_1]$ where $\psi_1 = \varphi_1$. Let $\psi_2$ be a graded $F$-automorphism $\psi_2$ of $R^{(1)}$ and $R^{(2)} := R^{(1)}[t_2; \psi_2]$. Then, by Lemma 3.1, we get $R^{(2)} = (RZ)Z = R \ast \mathbb{Z}^2$. Repeating this process $n$ times, one can construct $R \ast \mathbb{Z}^n$ inductively. Namely, for a crossed product algebra $R^{(k-1)} = R \ast \mathbb{Z}^{k-1}$, if we specify an $F$-graded automorphism $\psi_k$ of $R^{(k-1)}$, then
\[ R^{(k)} := R^{(k-1)}[t_k; \psi_k] = R \ast \mathbb{Z}^k, \]
and we obtain $R^{(n)} = R \ast \mathbb{Z}^n$. Thus, our task is to specify $\psi_k$ on $R^{(k-1)}$ and to show that $\psi_k$ is a graded $F$-automorphism. We note that
\[ \{t_1^{\alpha_1} \cdots t_{k-1}^{\alpha_{k-1}} \mid (\alpha_1, \ldots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}\} \]
is a basis of the free $R$-module $R^{(k-1)}$. For convenience, we put
\[ t^{(\alpha)_k} = t_1^{\alpha_1} \cdots t_{k-1}^{\alpha_{k-1}}, \]
and define an $F$-linear transformation $\psi_k$ on $R^{(k-1)}$ by
\[ \psi_k(r t^{(\alpha)_k}) = \varphi_k(r) \prod_{i=1}^{k-1} \varphi^{(\alpha)_i}(q_{ik}^{(\alpha)_i}) \cdot t^{(\alpha)_k} \quad \text{for} \quad r \in R, \]
which is clearly graded. If $\psi_k(r t^{(\alpha)_k}) = 0$, then $\varphi_k(r) = 0$, and hence $r = 0$, and so $\psi_k$ is injective. Since
\[ \psi_k \left( \varphi_k^{-1} \left( r \prod_{i=1}^{k-1} \varphi^{(\alpha)_i}(q_{ik}^{(\alpha)_i}) \right)^{-1} \right) t^{(\alpha)_k} = r t^{(\alpha)_k}, \]
$\psi_k$ is surjective. Therefore, $\psi_k$ is an $F$-linear graded isomorphism on $R^{(k-1)}$. So it remains to prove that $\psi_k$ is a homomorphism. For this purpose, we use a well-known fact.
3.5. Let $A$ and $B$ be unital associative algebras over $F$ and $f$ a $F$-linear map from $A$ into $B$. Let $\{t_i\}_{i \in I}$ be a generating set of the $F$-algebra $A$. Then, $f$ is a homomorphism if and only if $f(t_i y) = f(t_i) f(y)$ for all $i \in I$ and $y \in A$. Moreover, if $\{t_i^{\pm 1}\}_{i \in J}$ is a generating set of $A$, then $f$ is a homomorphism if and only if $f(t_i y) = f(t_i) f(y)$ and $f(t_i^{-1}) = f(t_i)^{-1}$ for all $i \in I$ and $y \in A$.

We have a generating set $R \cup \{t_1^{\pm 1}, \ldots, t_{k-1}^{\pm 1}\}$ of $R^{(k-1)}$ over $F$, and

$$
\psi_k(t_j^{-1}) = q_{jk}^{(-1)} t_j^{-1} = \varphi_j^{-1}(q_{jk}) t_j^{-1}
= (t_j \varphi_j^{-1}(q_{jk}))^{-1} = (q_{jk} t_j)^{-1} = \psi_k(t_j)^{-1}.
$$

So, by 3.5, we only need to show that, for all $r, r' \in R$ and $1 \leq j \leq k - 1$,

\begin{align*}
(A) & \quad \psi_k(rr' t^{(\alpha)_k}) = \psi_k(r) \psi_k(r' t^{(\alpha)_k}), \\
(B) & \quad \psi_k(t_j rt^{(\alpha)_k}) = \psi_k(t_j) \psi_k(rt^{(\alpha)_k}).
\end{align*}

For (A), we have

\begin{align*}
\psi_k(rr' t^{(\alpha)_k}) &= \varphi_k(r r') \prod_{i=1}^{k-1} \varphi^{(\alpha)}_i(q^{(\alpha_i)}_{ik}) t^{(\alpha)_k} \\
&= \varphi_k(r) \varphi_k(r') \prod_{i=1}^{k-1} \varphi^{(\alpha)}_i(q^{(\alpha_i)}_{ik}) t^{(\alpha)_k} \\
&= \psi_k(r) \psi_k(r' t^{(\alpha)_k}).
\end{align*}

For (B), we first note that there is the embedding of $R^{(j)}$ into $R^{(k-1)}$ for $1 \leq j \leq k - 1$, and so

$$
t_j t^{(\alpha)_j} = \psi_j(t^{(\alpha)_j}) t_j = \varphi_j(r) \prod_{i=1}^{j-1} \varphi^{(\alpha)}_i(q^{(\alpha_i)}_{ij}) t^{(\alpha)_j} t_j.
$$

Thus we have

\begin{align*}
\psi_k(t_j rt^{(\alpha)_k}) &= \psi_k(\varphi_j(r) t_j t^{(\alpha)_k}) \\
&= \psi_k(\varphi_j(r) t_j^{\alpha_j+1} \cdots t_{k-1}^{\alpha_{k-1}}) \\
&= \psi_k(\varphi_j(r) \prod_{i=1}^{j-1} \varphi^{(\alpha)}_i(q^{(\alpha_i)}_{ij}) t^{(\alpha+\varepsilon)_k}) \\
&= \varphi_k \varphi_j(r) \prod_{i=1}^{j-1} \varphi_k \varphi^{(\alpha)}_i(q^{(\alpha_i)}_{ij}) \prod_{i=1}^{k-1} \varphi^{(\alpha+\varepsilon)_i}(q^{(\alpha_i+\delta_{ij})}_{ik}) t^{(\alpha+\varepsilon)_k} \\
&= ABC t^{(\alpha+\varepsilon)_k},
\end{align*}

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Thus, after cancellations, we get

\[ A = \varphi_k \varphi_j (r) \]

First of all, we have

\[ A = \varphi_k \varphi_j (r) = q_{jk} \varphi_j \varphi_k (r) q_{kj} \quad \text{by (G2).} \]

Secondly, by Lemma 3.3(2) and (4), we have

\[ \varphi_k \varphi_i (q_{ij}^{(\alpha_i)}) \]

\[ = \left[ \prod_{l=1}^{i-1} \varphi_i (q_{ik}^{(\alpha_i)}) \right] \varphi_i (q_{ij}^{(\alpha_i)}) \left[ \prod_{l=1}^{i-1} \varphi_i (q_{lk}^{(\alpha_i)}) \right]^{-1} \]

\[ = \left[ \prod_{l=1}^{i-1} \varphi_i (q_{ik}^{(\alpha_i)}) \right] \varphi_i (q_{jk}^{(\alpha_i)}) q_{ij}^{(\alpha_i)} q_{ij}^{\alpha_i} q_{ik}^{\alpha_i} \left[ \prod_{l=1}^{i-1} \varphi_i (q_{lk}^{(\alpha_i)}) \right]^{-1} \].

Note that

\[ \varphi_i (q_{ki}^{(\alpha_i)}) \left[ \prod_{l=1}^{i-1} \varphi_i (q_{lk}^{(\alpha_i)}) \right]^{-1} = \left[ \prod_{l=1}^{i} \varphi_i (q_{lk}^{(\alpha_i)}) \right]^{-1} \]

and

\[ \varphi_i (q_{kj}^{(\alpha_i)}) = \varphi_{i+1} (q_{kj}^{(\alpha_i)}). \]

So we have

\[ (\varphi_k \varphi_i (q_{ij}^{(\alpha_i)}))(\varphi_k \varphi_i (q_{ij}^{(\alpha_i+1)})) = \left[ \prod_{l=1}^{i-1} \varphi_i (q_{lk}^{(\alpha_i)}) \right] \varphi_i (q_{jk}^{(\alpha_i)}) q_{ij}^{(\alpha_i+1)} \left[ \prod_{l=1}^{i} \varphi_i (q_{lk}^{(\alpha_i)}) \right]^{-1} \times \varphi_{i+1} (q_{i+1,k}^{(\alpha_i)}) q_{i+1,j}^{(\alpha_i+1)} \left[ \prod_{l=1}^{i} \varphi_i (q_{lk}^{(\alpha_i)}) \right]^{-1}. \]

Thus, after cancellations, we get

\[ B = \prod_{i=1}^{j-1} \varphi_k \varphi_i (q_{ij}^{(\alpha_i)}) \]

\[ = q_{jk} \left[ \prod_{i=1}^{j-1} \varphi_i (q_{ik}^{(\alpha_i)}) q_{ij}^{(\alpha_i)} \right] \varphi_j (q_{kj}) \left[ \prod_{i=1}^{j-1} \varphi_i (q_{ik}^{(\alpha_i)}) \right]^{-1}. \]

Thirdly, we have

\[ C = \prod_{i=1}^{k-1} \varphi_i (q_{ik}^{(\alpha_i+\delta_i)}) \]

\[ = \left[ \prod_{i=1}^{j-1} \varphi_i (q_{ik}^{(\alpha_i)}) \right] \varphi_j (q_{jk}^{(\alpha_j+1)}) \left[ \prod_{i=j+1}^{k-1} \varphi_i (q_{ik}^{(\alpha_i)}) \right] \]

\[ = \left[ \prod_{i=1}^{j-1} \varphi_i (q_{ik}^{(\alpha_i)}) \right] \varphi_j (q_{jk} \varphi_j (q_{jk}^{(\alpha_j)})) \left[ \prod_{i=j+1}^{k-1} \varphi_i (q_{ik}^{(\alpha_i)}) \right]. \]
by Lemma 3.2(4). Consequently, after cancellations and notifying \( q_{ii} = 1 \), we obtain

\[
\psi_k(t_j r t^{(\alpha)_k}) = ABC t^{(\alpha + \varepsilon)_k}
\]

\((\star)\)

\[
= q_{jk} \varphi_j \varphi_k(r) \prod_{i=1}^{j} \varphi^{(\alpha)}(\varphi_j(q_{ik}^{(\alpha)})) q_{ij}^{(\alpha)} \prod_{i=j+1}^{k-1} \varphi^{(\alpha + \varepsilon)}(q_{ik}^{(\alpha)}) t^{(\alpha + \varepsilon)_k}.
\]

On the other hand, we have

\[
\psi_k(t_j) \psi_k(r t^{(\alpha)_k}) = q_{jk} t_j \varphi_k(r) \prod_{i=1}^{k-1} \varphi^{(\alpha)}(q_{ik}^{(\alpha)}) t^{(\alpha)_k}
\]

\[
= q_{jk} \varphi_j \left[ \varphi_k(r) \prod_{i=1}^{k-1} \varphi^{(\alpha)}(q_{ik}^{(\alpha)}) \right] t_j t^{(\alpha)_k}
\]

\[
= q_{jk} \varphi_j \varphi_k(r) \prod_{i=1}^{k-1} \varphi^{(\alpha)}(q_{ik}^{(\alpha)}) \prod_{l=1}^{j-1} \varphi^{(\alpha)}(q_{lj}^{(\alpha)}) t^{(\alpha + \varepsilon)_k}.
\]

We rewrite \( D := \prod_{i=1}^{k-1} \varphi_j \varphi^{(\alpha)}(q_{ik}^{(\alpha)}) \). To find an expression for \( D \), we use the following lemma:

**Lemma 3.6.** Let \( A \) be a unital associative algebra, \( a_0 = 1, a_1, \ldots, a_k \in A \) units and \( b_1, \ldots, b_k \in A \). Then we have

\[
(1) \quad \prod_{i=1}^{k} \left( I \left( \prod_{l=1}^{i-1} a_l \right) (b_i) \right) = \prod_{i=1}^{k} a_i b_i \left( \prod_{l=1}^{k} a_l \right)^{-1}.
\]

\[
(2) \quad \prod_{i=j+1}^{k} \left( I \left( \prod_{l=1}^{j-1} a_l \right) (b_i) \right) = I \left( \prod_{i=1}^{j-1} a_l \right) \left( \prod_{i=j+1}^{k} b_i \right).
\]

**Proof.** (1) is straightforward and (2) is obvious. □

By Lemma 3.2(3), we have, for \( i < j \),

\[
\varphi_j \varphi^{(\alpha)}(q_{ik}^{(\alpha)}) = I \left( \prod_{l=1}^{i-1} \varphi^{(\alpha)}(q_{il}^{(\alpha)}) \right) \left( \varphi^{(\alpha)}(q_{ij}^{(\alpha)}) \right).
\]

So, by Lemma 3.6(1), we get

\[
\prod_{i=1}^{j} \varphi_j \varphi^{(\alpha)}(q_{ik}^{(\alpha)}) = \prod_{i=1}^{j} \varphi^{(\alpha)}(\varphi_j(q_{ik}^{(\alpha)})) \left( \prod_{l=1}^{j-1} \varphi^{(\alpha)}(q_{lj}^{(\alpha)}) \right)^{-1}.
\]
By Lemma 3.2(3), we have, for $j < i$,
\[ \varphi_j \varphi^{(\alpha)}_i(q_{ik}^{(\alpha_i)}) = I \left( \prod_{l=1}^{j-1} \varphi(\alpha)_l(q_{lj}^{(\alpha_i)}) \right) \left( \varphi(\alpha+\varepsilon_j)_i(q_{ik}^{(\alpha_i)}) \right). \]

So, by Lemma 3.6(2), we get
\[ \prod_{i=j+1}^{k-1} \varphi_j \varphi^{(\alpha)}_i(q_{ik}^{(\alpha_i)}) = I \left( \prod_{l=1}^{j-1} \varphi(\alpha)_l(q_{lj}^{(\alpha_i)}) \right) \left( \prod_{i=j+1}^{k-1} \varphi(\alpha+\varepsilon_j)_i(q_{ik}^{(\alpha_i)}) \right). \]

Hence we get
\[ D = \prod_{i=1}^{k-1} \varphi_j \varphi^{(\alpha)}_i(q_{ik}^{(\alpha_i)}) \]
\[ = \prod_{i=1}^{j} \varphi(\alpha)_i(q_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}) \prod_{i=j+1}^{k-1} \varphi(\alpha+\varepsilon_j)_i(q_{ik}^{(\alpha_i)})t(\alpha+\varepsilon_j)_k \left[ \prod_{l=1}^{j-1} \varphi(\alpha)_l(q_{lj}^{(\alpha_i)}) \right]^{-1}. \]

Consequently, we obtain
\[ \psi_k(t_j) \psi_k(rt^{(\alpha)_k}) = q_{jk} \varphi_j \varphi_k(r) \prod_{i=1}^{j} \varphi(\alpha)_i(q_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}) \prod_{i=j+1}^{k-1} \varphi(\alpha+\varepsilon_j)_i(q_{ik}^{(\alpha_i)})t(\alpha+\varepsilon_j)_k, \]
which is exactly (*). Hence we have shown (B) and constructed a crossed product algebra $R \ast \mathbb{Z}^k = R^{(k)}$ for $k = 1, \ldots, n$ from $(R, \varphi, q)$.

Let us put $R_{\varphi,q} := R^{(n)} = \bigoplus_{\alpha \in \mathbb{Z}^n} R_{t^\alpha}$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ and $t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$. Since $\psi_k |_{R} = \varphi_k$ for $k = 1, \ldots, n$, we have $t_ir = \varphi_i(r) t_i$. Also, we have $t_j t_i = \psi_j(t_i) t_j = q_{ij} t_i t_j$ for $1 \leq i < j \leq n$, and so $t_j t_i = q_{ij} t_i t_j$ for all $1 \leq i, j \leq n$. Hence, our $R_{\varphi,q}$ satisfies (3.4). The uniqueness of the multiplication on $R_{\varphi,q}$ is clear since $R \cup \{ \sum_{i=1}^{k} t_{i}^{\pm 1} \}$ is a generating set of $R_{\varphi,q}$.

Now, one can easily check that $\psi^{(\alpha)}_j(t^{(\beta)_j}) = q_{\epsilon_j,\beta}^{(\alpha)} t^{(\beta)_j}$. So for $rt^{(\alpha)}_\alpha, r't^{(\beta)}_\beta \in R_{\varphi,q}$, we get
\[ rt^{(\alpha)}_\alpha r't^{(\beta)}_\beta = r \varphi^{(\alpha)}(r') t^{(\alpha)}_\alpha t^{(\beta)}_\beta \]
\[ = r \varphi^{(\alpha)}(r') t^{(\alpha)}_\alpha q_{\epsilon_n,\beta}^{(\alpha_n)} t^{(\beta)}_n t^{\beta_n} \]
\[ = r \varphi^{(\alpha)}(r') t^{(\alpha)}_\alpha q_{\epsilon_n,\beta}^{(\alpha_n)} t^{(\beta)}_n t^{\alpha_n+\beta_n} \]
\[ = r \varphi^{(\alpha)}(r') t^{(\alpha)}_\alpha q_{\epsilon_n,\beta}^{(\alpha_n)} t^{(\beta)}_n t^{\alpha_n+\beta_n} \]
\[ \ldots \]
\[ = r \varphi^{(\alpha)}(r') t^{(\alpha)}_\alpha q_{\epsilon_n,\beta}^{(\alpha_n)} \varphi^{(\alpha)}(q_{\epsilon_n,\beta}^{(\alpha_n)}) t^{\alpha_n+\beta_n} \]
\[ = r \varphi^{(\alpha)}(r') t^{\alpha_n+\beta_n}. \]
Conversely, for any crossed product algebra $R \ast \mathbb{Z}^n = (R, \mathbb{Z}^n, \tau, \sigma) = \oplus_{\alpha \in \mathbb{Z}^n} R\alpha$, we take a new $R$-basis $\{t_\alpha \mid \alpha \in \mathbb{Z}^n\}$ of $R \ast \mathbb{Z}^n$ where $t_\alpha = \varepsilon_1^{a_1} \cdots \varepsilon_n^{a_n}$. We set $q_{ij} := \tau(\varepsilon_j, \varepsilon_i)$ for $1 \leq i \leq j \leq n$, $q_{ij} := q_{ji}^{-1}$ and $\varphi_i := \sigma_{\varepsilon_i}$. Note that $\tau(\varepsilon_i, \varepsilon_j) = 1$. Then one can check that the triple $(R, \varphi, q)$ is a $\mathbb{Z}^n$-grading triple:

\[(G1)\] is clear. Let $t_i := \varepsilon_i$ for $i = 1, \ldots, n$. Then, for $i \leq j$ and $r \in R$, we have

\[t_j t_i r = \varphi_j \varphi_i(r) t_i t_j = \varphi_j \varphi_i(r) q_{ij} t_i t_j \text{ and } t_i t_j r = q_{ij} \varphi_i \varphi_j(r) t_i t_j.\]

Hence, $\varphi_j \varphi_i(r) q_{ij} = q_{ij} \varphi_i \varphi_j(r)$, i.e., $(G2)$ holds. For $i \leq j \leq k$, we have $t_k t_j t_i = t_k q_{ij} t_i t_j = \varphi_k(q_{ij})(q_{ik}) t_i t_j t_k = \varphi_k(q_{ij}) \varphi_i(q_{jk}) t_i t_j t_k \text{ and } t_k t_j t_i = q_{jk} \varphi_j(q_{ik}) t_i t_j t_k$. Hence, $\varphi_k(q_{ij}) \varphi_i(q_{jk}) = q_{jk} \varphi_j(q_{ik}) q_{ij}$, i.e., $(G3)$ holds.

Finally, it is clear that $R \ast \mathbb{Z}^n = \oplus_{\alpha \in \mathbb{Z}^n} R\alpha$ satisfies (3.4). Therefore, we obtain $R \ast \mathbb{Z}^n = R\varphi, q$. □

Thus the following is clear:

**Corollary 3.7.** Let $(D, \varphi, q)$ be a division $\mathbb{Z}^n$-grading triple. Then, $D\varphi, q$ is a division $\mathbb{Z}^n$-graded algebra. Conversely, for any division $\mathbb{Z}^n$-graded algebra $A$, there exists a division $\mathbb{Z}^n$-grading triple $(D, \varphi, q)$ such that $A = D\varphi, q$.

**Remark.** What we have shown in Theorem 3.3 can be written in the following way:

Let $B := \{\varepsilon_1, \ldots, \varepsilon_n\}$ and $C := \{\langle \varepsilon_j, \varepsilon_i \rangle \mid 1 \leq i < j \leq n\}$. Suppose that maps

\[\sigma : B \longrightarrow \text{Aut}_F(R) \text{ and } \tau : C \longrightarrow U(R)\]

satisfy

\[(a) \quad \sigma_{\varepsilon_j} \sigma_{\varepsilon_i} = I(\tau(\varepsilon_j, \varepsilon_i)) \sigma_{\varepsilon_i} \sigma_{\varepsilon_j} \quad \text{and} \]

\[(b) \quad \sigma_{\varepsilon_k} \big(\tau(\varepsilon_j, \varepsilon_i)\big) \tau(\varepsilon_k, \varepsilon_i) = \tau(\varepsilon_k, \varepsilon_j) \sigma_{\varepsilon_j} \big(\tau(\varepsilon_k, \varepsilon_i)\big) \tau(\varepsilon_k, \varepsilon_j)\]

for all $1 \leq i < j < k \leq n$. Then there exist unique action $\tilde{\sigma} : \mathbb{Z}^n \longrightarrow \text{Aut}_F(R)$ and twisting $\tilde{\tau} : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow U(R)$ such that $\tilde{\sigma} |_B = \sigma$, $\tilde{\tau} |_C = \tau$ and

\[(c) \quad \tilde{\tau}(\alpha_1 \varepsilon_1 + \cdots + \alpha_i \varepsilon_i, \alpha_j \varepsilon_j + \cdots + \alpha_n \varepsilon_n) = 1 \quad \text{for all } 1 \leq i < j \leq n.\]

Conversely, for any crossed product algebra $R \ast \mathbb{Z}^n$, we can use the diagonal basis change so that the action and twisting satisfy (a), (b) and (c).

In a certain situation, the condition (G3) of a $\mathbb{Z}^n$-grading triple is not needed. We use the notation $[a, b] = aba^{-1}b^{-1}$ for $a, b \in U(R)$.

**Lemma 3.8.** Let $R$ be a unital associative algebra over $F$, $\varphi = (I(d_1), \ldots, I(d_n))$ an $n$-tuple of inner automorphisms $\varphi_i$ of $R$ for some $d_1, \ldots, d_n \in U(R)$ and $q = (q_{ij})$ an $n \times n$ matrix over $F$. Suppose that a triple $(R, \varphi, q)$ satisfies (G1) and (G2). Then, $(R, \varphi, q)$ is a $\mathbb{Z}^n$-grading triple.

**Proof.** We only need to check (G3). By (G1) and (G2), we have, for all $1 \leq i, j \leq n$, $I(d_j)I(d_i) = I(q_{ij})I(d_i)I(d_j)$. So for all $r \in R$, $d_j d_i r d_i^{-1} d_j^{-1} = q_{ij} d_i d_j r d_j^{-1} d_i^{-1} q_{ji}$ and
hence \( rd_i^{-1}d_j^{-1}q_{ij}d_id_j = d_i^{-1}d_j^{-1}q_{ij}d_id_jr \), i.e., \( d_i^{-1}d_j^{-1}q_{ij}d_id_j =: c_{ij} \) is in the centre of \( R \). Note that \( c_{ji}^{-1} = c_{ij} \). Thus we have

\[ q_{ij} = c_{ij}[d_j,d_i]. \]

Using this identity, we get (G3): for all \( 1 \leq i < j < k \leq n \),

\[
q_{jk} \varphi_j(q_{ik})q_{ij} \varphi_i(q_{kj})q_{ki} =
\begin{align*}
&= c_{jk}[d_k,d_j]c_{ik}[d_k,d_i]d_j^{-1}c_{ij}[d_j,d_i]d_ic_{kj}[d_j,d_k]d_i^{-1}c_{ki}[d_i,d_k] \\
&= d_kc_{ij}[d_j,d_i]d_k^{-1} = \varphi_k(q_{ij}). 
\end{align*}
\]

By this lemma, if \( R \) is a finite dimensional central simple associative algebra, the defining identities of a \( \mathbb{Z}^n \)-grading triple are just (G1) and (G2).

**Remark 3.9.** (1) For a \( \mathbb{Z}^n \)-grading triple \((R, \varphi, q)\), if \( \varphi = 1 := (\text{id}, \ldots, \text{id}) \), then the crossed product algebra \( R_{1,q} \) has the trivial action by Theorem 3.3. So, \( R_{1,q} = R^e[\mathbb{Z}^n] \) is a twisted group algebra.

(2) For a \( \mathbb{Z}^n \)-grading triple \((R, \varphi, q)\), if \( q = 1_n = 1 := \left( \begin{array}{ccc} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{array} \right) \), then a crossed product algebra \( R_{\varphi,1} \) has the trivial twisting by Theorem 3.3. So, \( R_{\varphi,1} = R\mathbb{Z}^n \) is a skew group algebra.

(3) By (G2), \((R, \varphi, 1)\) is a \( \mathbb{Z}^n \)-grading triple if and only if

\[ (*) \quad \varphi_j \varphi_i = \varphi_i \varphi_j \quad \text{for all } i, j. \]

Finally, we give some examples.

**Example.** (1) Let \( F_q \) be an arbitrary quantum torus and \( R \) an arbitrary associative algebra. Then it is easy to see that \( R \otimes_F F_q \) is a predivision \( \mathbb{Z}^n \)-graded associative algebra (division \( \mathbb{Z}^n \)-graded if \( R \) is a division algebra) and is isomorphic to \( R_{1,q} \). Note also if \( R \) is a field, then this example becomes a quantum torus over \( R \). Conversely, for a division \( \mathbb{Z}^n \)-grading triple \((D, \varphi, q)\), if \( \varphi = 1 \), then \( I(q_{ij}) = \text{id} \) for all \( q_{ij} \), by (G2). Hence \( q_{ij} \) is in the centre of \( D \), say \( K \), and we can show that \( D_{1,q} \cong D \otimes_K K_q \). Therefore, \( D_{\varphi,q} \) is a tensor product with \( D \) and some quantum torus if and only if \( \varphi = 1 \).

(2) Let \( Q = \langle i, j \rangle \) be a quaternion algebra over a field, where \( i \) and \( j \) are the standard generators, \( \varphi = \varphi_3 = (I(i), I(j), I(ij)) \) and \( 1 = 1_3 \). Then one can easily check \((*)\) in Remark 3.9(3), and hence \( Q_{\varphi,1} \) is a predivision \( \mathbb{Z}^3 \)-graded associative algebra.
(3) Let \( K = \mathbb{Q}(\zeta_5) \) be a cyclotomic extension of \( \mathbb{Q} \) (the field of rational numbers) where \( \zeta := \zeta_5 \) is a primitive 5th root of unity, and \( \varphi \) the automorphism of \( K \) defined by \( \varphi(\zeta) = \zeta^2 \). Let \( \varphi = (\varphi, \varphi^2, \varphi^3) \) and

\[
q = \begin{pmatrix}
1 & \zeta & \zeta^2 \\
\zeta^{-1} & 1 & \zeta^{-1} \\
\zeta^3 & \zeta & 1
\end{pmatrix}.
\]

Then one can easily check that \((K, \varphi, q)\) is a division \( \mathbb{Z}^3 \)-grading triple, and hence \( K_{\varphi, q} \) is a division \( \mathbb{Z}^3 \)-graded associative algebra over \( \mathbb{Q} \).

(4) Let \( \mathbb{H} = \langle i, j \rangle \) be Hamilton’s quaternion over \( \mathbb{R} \) (the field of real numbers), i.e., the unique quaternion division algebra over \( \mathbb{R} \). Put \( k := ij \). Let \( \varphi = (I(d_1), I(d_2), I(d_3)) \) where \( d_1 = 1 + i, d_2 = 1 + j \) and \( d_3 = 1 + k \). We put \( q_{ij} = 2[d_j, d_i] \) for \( 1 \leq i < j \leq 3 \), \( q_{ji} = q_{ij}^{-1} \) and \( q_{ii} = 1 \). Then, \((\mathbb{H}, \varphi, q)\) satisfies (G1) and (G2), and

\[
q = \begin{pmatrix}
1 & 1 - i + j - k & 1 - i + j + k \\
(1 - i + j - k)^{-1} & 1 & 1 - i - j + k \\
(1 - i + j + k)^{-1} & (1 - i - j + k)^{-1} & 1
\end{pmatrix}.
\]

By Lemma 3.8, this is a division \( \mathbb{Z}^3 \)-grading triple and hence \( \mathbb{H}_{\varphi, q} \) is a division \( \mathbb{Z}^3 \)-graded associative algebra over \( \mathbb{R} \).

§ 4 Conclusion

By 1.8, Example 2.8(c), Example 2.10, Proposition 2.13, Theorem 3.3 and Corollary 3.7, one can summarize our results as follows:

**Corollary.** (i) Any predivision (resp. division) \( A_l \mathbb{Z}^n \)-graded Lie algebra over \( F \) for \( l \geq 3 \) is an \( A_l \mathbb{Z}^n \)-cover of \( \text{psl}_{l+1}(R_{\varphi, q}) \) for some (resp. division) \( \mathbb{Z}^n \)-grading triple \((R, \varphi, q)\). Conversely, any \( \text{psl}_{l+1}(R_{\varphi, q}) \) for \( l \geq 1 \) is a predivision (resp. division) \( A_l \mathbb{Z}^n \)-graded Lie algebra.

(ii) Any predivision (resp. division) \( \Delta \mathbb{Z}^n \)-graded Lie algebra over \( F \) for \( \Delta = D \) or \( E \) is a \( \Delta \mathbb{Z}^n \)-cover of \( g \otimes_F K[z_1^\pm, \ldots, z_n^\pm] \) where \( g \) is a finite dimensional split simple Lie algebra over \( F \) of type \( D \) or \( E \) and \( K \) is a unital commutative associative algebra over \( F \) (resp. \( K \) is a field extension of \( F \)). Conversely, for any finite dimensional split simple Lie algebra \( g \) over \( F \) of any type \( \Delta \), \( g \otimes_F K[z_1^\pm, \ldots, z_n^\pm] \) is a predivision (resp. division) \( \Delta \mathbb{Z}^n \)-graded Lie algebra.

**References**


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