ROOT-GRADED LIE ALGEBRAS WITH COMPATIBLE GRADINGS

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Abstract. Lie algebras graded by a finite irreducible reduced root system Δ will be generalized as *predivision ΔG-graded Lie algebras* for an abelian group G. In this paper such algebras are classified, up to central extensions, when $\Delta = A_l$ for $l \geq 3$, D or E, and $G = \mathbb{Z}^n$.

Introduction

The concept of a Lie algebra over a field F of characteristic 0 graded by a finite irreducible reduced root system Δ or a Δ-graded Lie algebra was introduced by Berman and Moody [3]). It is a Lie algebra L together with a finite dimensional split simple Lie algebra $\mathfrak{g}$, a split Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the root system Δ, so that $\mathfrak{g}$ has the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \big(\oplus_{\mu\in\Delta}\ \mathfrak{g}_\mu \big)$ with $\mathfrak{h} = \mathfrak{g}_0$, satisfying the following three conditions:

(i) L contains $\mathfrak{g}$ as a subalgebra
(ii) $L = \oplus_{\mu\in\Delta\cup\{0\}}\ L_\mu$, where $L_\mu = \{x \in L \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$; and
(iii) $L_0 = \sum_{\mu\in\Delta}\ [L_\mu, L_{-\mu}]$.

The subalgebra $\mathfrak{g} = (\mathfrak{g}, \mathfrak{h})$ is called the *grading subalgebra of L*.

Berman and Moody classified Δ-graded Lie algebras, up to central extensions, when Δ has type A_l, $l \geq 2$, D or E in [3], and then Benkart and Zelmanov completed the classification for the other types in [5] (see also [7] for the classification of Δ-graded Lie algebras over rings where Δ is not necessarily finite there, using Jordan methods).

Let us explain the case $\Delta = A_l$, $l \geq 3$, in order to describe our motivation of this paper. By [3], an A_l-graded Lie algebra covers $psl_{l+1}(A)$ for a unital associative algebra A (see Definition 2.9). Then Berman, Gao and Krylyuk showed in [4] that the core of an extended affine Lie algebra of type A_l for $l \geq 3$ is an A_l-graded Lie algebra and covers $sl_{l+1}(\mathbb{C}_{\boldsymbol{q}})$ where $\mathbb{C}_{\boldsymbol{q}} = \mathbb{C}_{\boldsymbol{q}}[t_1^{\pm}, \dots, t_n^{\pm}]$ is a certain $\mathbb{Z}^n$-graded associative algebra called, a *quantum torus* over $\mathbb{C}$ (see §2 below). The Lie algebra $L = sl_{l+1}(\mathbb{C}_{\boldsymbol{q}})$ is not only graded by A_l but also graded by $\mathbb{Z}^n$, and the $\mathbb{Z}^n$-grading $L = \oplus_{\alpha\in\mathbb{Z}^n}\ L^\alpha$

is compatible with the A_l-grading $L = \oplus_{\mu\in\Delta\cup\{0\}} L_\mu$ in the sense that

$$L = \bigoplus_{\mu\in\Delta\cup\{0\}} \bigoplus_{\alpha\in\mathbb{Z}^n} L_\mu^\alpha \quad \text{where} \quad L_\mu^\alpha = L_\mu \cap L^\alpha.$$

We will call such a double grading a *compatible* $A_l\mathbb{Z}^n$*-grading* (see Definition 2.6 for the general definition). Moreover, let $\{h_\mu \in \mathfrak{h} \mid \mu \in \Delta\}$ be the set of coroots where $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices in the grading subalgebra $\mathfrak{g} = sl_{l+1}(\mathbb{C})$. Then L has the following two properties:

(1) for any $\mu \in \Delta$ and any $0 \neq x \in L_\mu^\alpha$, there exists $y \in L_{-\mu}^{-\alpha}$ such that $[x, y] = h_\mu$;
(2) $\dim_{\mathbb{C}} L_\mu^\alpha = 1$ for all $\mu \in \Delta$ and $\alpha \in \mathbb{Z}^n$.

The property (1) will be called *division* (see Definition 2.6 for the general definition). Our interest is to describe such Lie algebras without the property (2), namely, division $A_l\mathbb{Z}^n$-graded Lie algebras. One of the main results of the paper which is contained in Proposition 2.13 is the following:

Result 1. *Let $l \geq 3$. Then any division $A_l\mathbb{Z}^n$-graded Lie algebra covers $psl_{l+1}(P)$ where P is a division $\mathbb{Z}^n$-graded associative algebra (i.e., all nonzero homogeneous elements are invertible).*

A division $\mathbb{Z}^n$-graded associative algebra over a field F can be considered as a crossed product algebra $D * \mathbb{Z}^n$ for an associative division algebra D over F (see §1). Our next interest is to describe $D * \mathbb{Z}^n$ as a natural generalization of the algebra $F[t_1^{\pm1}, \dots, t_n^{\pm1}]$ of Laurent polynomials or a quantum torus $F_{\boldsymbol{q}}$.

A triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$ is called a *division $\mathbb{Z}^n$-grading triple* if

(1) D is an associative division algebra;
(2) $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$ is an n-tuple of automorphisms φ_i of D; and
(3) $\boldsymbol{q} = (q_{ij})$ is an $n \times n$ matrix over D satisfying, for all $1 \leq i < j < k \leq n$,

$$q_{ii} = 1 \quad \text{and} \quad q_{ji}^{-1} = q_{ij},$$
$$\varphi_j\varphi_i = \mathrm{I}(q_{ij})\varphi_i\varphi_j,$$
$$\varphi_k(q_{ij}) = q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{kj})q_{ki},$$

where $\mathrm{I}(q_{ij})$ is the inner automorphism of D determined by q_{ij}, i.e.,

$$\mathrm{I}(q_{ij})(d) = q_{ij} d q_{ij}^{-1} \quad \text{for } d \in D.$$

We will show that $D*\mathbb{Z}^n$ can be considered as a generalization of the ring $D[t_1^{\pm1}, \dots, t_n^{\pm1}]$ of Laurent polynomials over D in n-variables in the following sense:

$D[t_1^{\pm1}, \dots, t_n^{\pm1}] = \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} Dt_{\boldsymbol{\alpha}}$ is a $\mathbb{Z}^n$-graded algebra, where $t_{\boldsymbol{\alpha}} = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, and the multiplication rule is determined by

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i d = d t_i \quad \text{and} \quad t_j t_i = t_i t_j \quad \text{for all } d \in D \text{ and } i, j.$$

Result 2. *For any division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$, there exists a division $\mathbb{Z}^n$-graded associative algebra $D_{\boldsymbol{\varphi},\boldsymbol{q}} = D_{\boldsymbol{\varphi},\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ such that $D_{\boldsymbol{\varphi},\boldsymbol{q}} = \oplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} D t_{\boldsymbol{\alpha}}$ has the same $\mathbb{Z}^n$-grading as $D[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ above, and the multiplication rule is determined by*

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i d = \varphi_i(d) t_i \quad \text{and} \quad t_j t_i = q_{ij} t_i t_j \quad \text{for all } d \in D \text{ and } i, j.$$

Conversely, any division $\mathbb{Z}^n$-graded associative algebra is isomorphic to $D_{\boldsymbol{\varphi},\boldsymbol{q}}$ for some division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$ (see Theorem 3.3 for more precise statements).

Consequently, one gets that any division A_l-$\mathbb{Z}^n$-graded Lie algebra for $l \geq 3$ covers $psl_{l+1}(D_{\boldsymbol{\varphi},\boldsymbol{q}})$. We will also classify division $\Delta\mathbb{Z}^n$-graded Lie algebras when $\Delta = D$ or E, which is simpler than the case A. Moreover, our concept "division" can be generalized as "predivision" (see Definition 2.6). Result 1 and 2 above will be proved in this more general set-up.

The organization of the paper is as follows. In §1 we review basic concepts of graded algebras and crossed product algebras. In §2 we observe some properties of ΔG-graded Lie algebras. Then predivision or division ΔG-graded Lie algebras are defined. After describing some examples of them, we classify predivision ΔG-graded Lie algebras for $\Delta = A_l$ ($l \geq 3$), D and E types. In §3 we classify crossed product algebras $R * \mathbb{Z}^n$. Finally in §4 we give a summary of our results.

Result 2 above is part of my Ph.D thesis, written at the University of Ottawa. I would like to thank my supervisor, Professor Erhard Neher, for his encouragement and suggestions.

§ 1 Basic Concepts

For any group G and any G-graded algebra $L = \oplus_{g \in G} L_g$, we denote

$$\operatorname{supp} L := \{g \in G \mid L_g \neq (0)\}.$$

Then we have $L = \oplus_{g \in G'} L_g$ where $G' = \langle \operatorname{supp} L \rangle$ is the subgroup of G generated by $\operatorname{supp} L$. Because of this, we will in the following always assume

$$G = \langle \operatorname{supp} L \rangle. \tag{1.1}$$

Whenever a class of algebras has a notion of invertibility, one can make the following definition:

Definition 1.2. Let G be a group. A G-graded algebra $P = \oplus_{g \in G} P_g$ is called a *predivision G-graded algebra* if P_g contains an invertible element for all $g \in \operatorname{supp} P$. Also, P is called a *division G-graded algebra* if all nonzero homogeneous elements are invertible.

One can easily check that if P is associative, then $\operatorname{supp} P = G$ and P is strongly graded, i.e., $P_g P_h = P_{gh}$ for all $g, h \in G$. This is not true if P is a Jordan algebra (see [9]). Predivision G-graded associative algebras are realized as crossed product algebras, which we recall here:

Definition 1.3. Let R be a unital associative algebra over a field F and G a group. Let $R*G$ be the free left R-module with basis $\overline{G} = \{\overline{g} \mid g \in G\}$, a copy of G. Define a multiplication on $R*G$ by linear extension of

$$(r\overline{g})(s\overline{h}) = r\sigma_g(s)\tau(g,h)\overline{gh},$$

for $r, s \in R$ and $g, h \in G$, where

$$\begin{aligned} &\text{(action)} \quad \sigma : G \longrightarrow \mathrm{Aut}_F(R), \quad \text{the group of } F\text{-automorphisms of } R,\\ &\text{(twisting)} \quad \tau : G \times G \longrightarrow U(R), \quad \text{the group of units of } R, \end{aligned}$$

are arbitrary maps and $\sigma_g := \sigma(g)$. This $R*G = (R, G, \sigma, \tau)$ is called a *crossed product algebra over* F if this multiplication is associative. It is easily seen that this is in fact an algebra over F. If there is no action or twisting, that is, if $\sigma_g = \mathrm{id}$ and $\tau(g,h) = 1$ for all $g, h \in G$, then $R*G = R[G]$ is the ordinary *group algebra*. If the action is trivial, then $R*G =: R^t[G]$ is called a *twisted group algebra*. Finally, if the twisting is trivial, then $R*G =: RG$ is called a *skew group algebra*.

Remark 1.4. If a crossed product algebra $R*G$ is commutative, then the action is clearly trivial, and so $R*G = R^t[G]$.

The following lemma characterizes σ and τ (see [8], Lemma 1.1 p.2). We denote by $\mathrm{I}(d)$ the inner automorphism determined by $d \in U(R)$, i.e., $\mathrm{I}(d)(r) = drd^{-1}$ for $r \in R$.

1.5. *The associativity of $R*G$ is equivalent to the following two conditions: for all $g, h, k \in G$,*
(i) $\sigma_g\sigma_h = \mathrm{I}\big(\tau(g,h)\big)\sigma_{gh}$,
(ii) $\sigma_g\big(\tau(h,k)\big)\tau(g,hk) = \tau(g,h)\tau(gh,k)$.

Remark 1.6. If R is commutative, then the action $\sigma : G \longrightarrow \mathrm{Aut}_F(R)$ becomes a group homomorphism by condition (i) in 1.5. So the action is really a "group action" in usual sense. Also, for a skew group algebra RG, the action becomes a group homomorphism for the same reason. Conversely, any group action $G \longrightarrow \mathrm{Aut}_F(R)$ defines a skew group algebra RG.

If $d : G \longrightarrow U(R)$ assigns to each element $g \in G$ a unit d_g, then $\tilde{G} = \{d_g\overline{g} \mid g \in G\}$ yields another R-basis for $R*G$ so that $R*G$ is a crossed product algebra for the new basis. One calls this a *diagonal change of basis* ([8], p.3). Any crossed product algebra has an identity element. It is of the form $1 = u\overline{e}$ for some unit u in R where e is the identity element of G ([8], Exercise 2 p.9). We can and will assume that $1 = \overline{e}$, via a diagonal change of basis, and so $\tau(g,e) = \tau(e,g) = 1$ for all $g \in G$. The embedding of R into $R*G$ is then given by $r \mapsto r\overline{e}$. Also, we have ([8], p.3)

(1.7) $$r\overline{g} \quad \text{is invertible if and only if} \quad r \in U(R).$$

Now, it is clear that a crossed product algebra $R * G = \oplus_{g\in G} R\overline{g}$ is a predivision G-graded associative algebra. Conversely, suppose that $A = \oplus_{g\in G} A_g$ is a predivision G-graded associative algebra over F. Then we have $A = \oplus_{g\in G} Rx_g$ where $R = A_e$ and an invertible element $x_g \in A_g$, which exists since A is predivision graded and $\operatorname{supp} A = G$. Moreover, for $h \in G$, we have $x_g x_h = x_g x_h (x_{gh})^{-1} x_{gh}$. So we can put $\tau(g,h) := x_g x_h (x_{gh})^{-1} \in U(R)$. Then we have $x_g x_h = \tau(g,h) x_{gh}$. Also, let $I(x_g)$ be the inner automorphism determined by x_g and let $\sigma_g := I(x_g)\mid_R$. Then, σ_g is clearly an F-automorphism of R and for $r, r' \in R$,

$$(rx_g)(r'x_h) = r(x_g r' x_g^{-1}) x_g x_h = r\sigma_g(r') x_g x_h = r\sigma_g(r')\tau(g,h) x_{gh}.$$

Hence A is a crossed product algebra $R * G$ determined by these σ and τ. So the two concepts, a crossed product algebra $R * G$ and a predivision G-graded associative algebra, coincide (see [8], Exercise 2 p.18). In particular, a division G-graded associative algebra is a crossed product algebra $R * G$ where R is a division algebra.

By Remark 1.4, a predivision G-graded commutative associative algebra $Z = \oplus_{g\in G} Z_g$ (G is necessarily abelian) is a twisted group algebra $K^t[G]$ where $K := Z_e$. Moreover (see [8], Exercise 6 p.10):

1.8. *If the abelian group G is free, then Z is a group algebra $K[G]$. In particular, when $G = \mathbb{Z}^n$, Z is the algebra $K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ of Laurent polynomials for invertible elements $z_i \in Z_{\boldsymbol{\varepsilon}_i}$, $i = 1, \dots, n$, where $\{\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n\}$ is a basis of $\mathbb{Z}^n$.*

§ 2 Predivision ΔG-graded Lie algebras

In this section F is a field of characteristic 0 and Δ is a finite irreducible reduced root system. We have defined a Δ-graded Lie algebra $L = \oplus_{\mu\in\Delta\cup\{0\}} L_\mu$ over F in Introduction. We note that the centre $Z(L)$ of L is contained in L_0.

A homomorphism (resp. an isomorphism) $\varphi : L \longrightarrow L'$ of Δ-graded Lie algebras $L = (L, \mathfrak{g}, \mathfrak{h})$ and $L' = (L', \mathfrak{g}', \mathfrak{h}')$, which have the same type Δ, is called a Δ-*homomorphism* (resp. an Δ-*isomorphism*) if $\varphi(\mathfrak{g}) = \mathfrak{g}'$ and $\varphi(\mathfrak{h}) = \mathfrak{h}'$ (cf. Definition 1.20 in [3]). Then one can check that $\varphi(L_\alpha) \subset L'_\alpha$ for all $\alpha \in \Delta$, and so $\varphi(L_0) \subset L'_0$. In other words, a Δ-homomorphism is graded.

Recall that a *cover* $\tilde{L} = (\tilde{L}, \pi)$ of a Lie algebra L is an epimorphism $\pi : \tilde{L} \longrightarrow L$ of Lie algebras so that $\tilde{L}$ is perfect, i.e., $\tilde{L} = [\tilde{L}, \tilde{L}]$, and $\ker \pi$ is contained in the centre of $\tilde{L}$. If $\pi : \tilde{L} \longrightarrow L$ is a cover of a Δ-graded Lie algebra L, then there exists a Δ-grading of $\tilde{L}$ such that π is a Δ-homomorphism (see Proposition 1.24 in [3]). However, it is not known whether or not, for Δ-graded Lie algebras $\tilde{L}$ and L, any cover $\tilde{L} \longrightarrow L$ is a Δ-homomorphism. Thus we define the following:

Definition 2.1. For Δ-graded Lie algebras $\tilde{L}$ and L, if $\pi : \tilde{L} \longrightarrow L$ is a cover and a Δ-homomorphism, $\tilde{L} = (\tilde{L}, \pi)$ is called a Δ-*cover of* L. Also, for Δ-graded Lie algebras L and L', if there exist a Δ-graded Lie algebra $\tilde{L}$ and maps $\pi : \tilde{L} \longrightarrow L$ and $\pi' : \tilde{L} \longrightarrow L'$ such that $(\tilde{L}, \pi)$ and $(\tilde{L}, \pi')$ are both Δ-covers, we say that L and L' are Δ-*isogeneous*.

Example 2.2. Let $L = (L, \mathfrak{g}, \mathfrak{h})$ be a Δ-graded Lie algebra with its centre $Z(L)$. Then, for any subspace V of $Z(L)$, $L/V = (L/V, \mathfrak{g} + V, \mathfrak{h} + V)$ is a Δ-graded Lie algebra, and the canonical epimorphism $L \longrightarrow L/V$ is a Δ-cover. In particular, L and L/V are Δ-isogeneous.

We will show that if L and L' are Δ-isogeneous, then $L/Z(L)$ and $L'/Z(L')$ are Δ-isomorphic, i.e., there exists a Δ-isomorphism between them.

Lemma 2.3. *Let $\pi : \tilde{L} \longrightarrow L$ be a Δ-cover and $c : L \longrightarrow L/Z(L)$ the canonical epimorphism. Then we have $Z(\tilde{L}) = \pi^{-1}\big(Z(L)\big)$, and hence* $\ker c \circ \pi = Z(\tilde{L})$.

Proof. It is clear that $Z(\tilde{L}) \subset \pi^{-1}\big(Z(L)\big)$. For the other inclusion, let $x \in \pi^{-1}\big(Z(L)\big)$. Then $x \in \tilde{L}_0$, and so for any $\alpha \in \Delta$, one has $[x, \tilde{L}_\alpha] \subset \tilde{L}_\alpha$. On the other hand, we have $[x, \tilde{L}_\alpha] \subset \ker \pi \subset Z(\tilde{L}) \subset \tilde{L}_0$. Hence $[x, \tilde{L}_\alpha] = (0)$ and we get $x \in Z(\tilde{L})$. $\square$

Corollary 2.4. *Suppose that L and L' are Δ-isogeneous. Then $L/Z(L)$ and $L'/Z(L')$ are Δ-isomorphic.*

Proof. By our assumption, there exists a Δ-graded Lie algebra $\tilde{L} = (\tilde{L}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ such that $\pi : \tilde{L} = (L, \mathfrak{g}, \mathfrak{h}) \longrightarrow L$ and $\pi' : \tilde{L} \longrightarrow L' = (L', \mathfrak{g}', \mathfrak{h}')$ are both Δ-covers. Let $c : L \longrightarrow L/Z(L)$ and $c' : L' \longrightarrow L'/Z(L')$ be the canonical epimorphisms. Then, by Lemma 2.3, we have $\ker c \circ \pi = Z(\tilde{L}) = \ker c' \circ \pi'$. Hence there exists the induced isomorphism

$$\begin{aligned} \varphi : L/Z(L) =& \big(L/Z(L), \mathfrak{g} + Z(L), \mathfrak{h} + Z(L)\big) \\ &\longrightarrow L'/Z(L') = \big(L'/Z(L'), \mathfrak{g}' + Z(L'), \mathfrak{h}' + Z(L')\big) \end{aligned}$$

such that $\varphi \circ c \circ \pi = c' \circ \pi'$. In particular, $\varphi\big(\mathfrak{g} + Z(L)\big) = \varphi \circ c \circ \pi(\tilde{\mathfrak{g}}) = c' \circ \pi'(\tilde{\mathfrak{g}}) = \mathfrak{g}' + Z(L')$ and similarly $\varphi\big(\mathfrak{h} + Z(L)\big) = \mathfrak{h}' + Z(L')$. Therefore, φ is a Δ-isomorphism. $\square$

Remark 2.5. Any Δ-graded Lie algebra is perfect. Also, any perfect Lie algebra L, we have $Z\big(L/Z(L)\big) = (0)$. Indeed, it is enough to show that if $x \in L$ satisfies $[x, L] \subset Z(L)$, then $x \in Z(L)$. Since $[x, L] = \big[x, [L, L]\big] \subset \big[[x, L], L\big] + \big[L, [x, L] = (0)$, we get $x \in Z(L)$.

Now we define new concepts.

Definition 2.6. Let $L = (L, \mathfrak{g}, \mathfrak{h}) = \oplus_{\mu \in \Delta \cup \{0\}} L_\mu$ be a Δ-graded Lie algebra over F. Let G be an abelian group. We say that L admits a *compatible G-grading* or simply L is a *ΔG-graded Lie algebra* if

$L = \oplus_{g \in G} L^g$ is a G-graded Lie algebra such that $\mathfrak{g} \subset L^0$.

As a consequence, L^g is a $\mathfrak{h}$-module for all $g \in G$ via the adjoint action. Hence we have $L^g = \oplus_{\mu \in \Delta \cup \{0\}} L^g_\mu$ where $L^g_\mu = L_\mu \cap L^g$ (see [6] Proposition 1, p.92). Therefore, $L_\mu = \oplus_{g \in G} L^g_\mu$ and

$$L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{g \in G} L^g_\mu.$$

Remark 2.7. (i) The compatible G-grading is completely determined by L_μ^g for all $\mu \in \Delta$ and $g \in G$ since $L_0^g = \sum_{\mu\in\Delta} \sum_{g=h+k} [L_\mu^h, L_{-\mu}^k]$.

(ii) Let $\operatorname{supp} L_\mu := \{g \in G \mid L_\mu^g \neq (0)\}$. Then we have

$$\operatorname{supp} L \subset \bigcup_{\mu\in\Delta} (\operatorname{supp} L_\mu + \operatorname{supp} L_\mu),$$

where $\operatorname{supp} L = \{g \in G \mid L^g \neq (0)\}$ as defined in the beginning of §1.

Let $Z(L)$ be the centre of L and let

$$\{h_\mu \in \mathfrak{h} \mid \mu \in \Delta\}$$

be the set of coroots. Then a ΔG-graded Lie algebra L is called *predivision* if

(pd) for any $\mu \in \Delta$ and any $L_\mu^g \neq (0)$, there exist $x \in L_\mu^g$ and $y \in L_{-\mu}^{-g}$ such that $[x, y] \equiv h_\mu$ modulo $Z(L)$;

and *division* if

(d) for any $\mu \in \Delta$ and any $0 \neq x \in L_\mu^g$, there exists $y \in L_{-\mu}^{-g}$ such that $[x, y] \equiv h_\mu$ modulo $Z(L)$.

Note that (d) implies (pd), i.e., 'division' $\Longrightarrow$ 'predivision'. If $\dim_F L_\mu^g \leq 1$ for all $\mu \in \Delta$ and $g \in G$, then two concepts, 'predivision' and 'division', coincide.

Example 2.8. (a) A Δ-graded Lie algebra is a predivision ΔG_0-graded for the trivial group $G_0 = \{0\}$.

(b) The core of an extended affine Lie algebra of reduced type Δ with nullity n is a division $\Delta\Lambda$-graded Lie algebra over $\mathbb{C}$, where Λ is a free abelian group of rank n. Indeed, it is known that such a core L is a Δ-graded Lie algebra over $\mathbb{C}$ and has a Λ-grading, say

$$L = \bigoplus_{\mu\in\Delta\cup\{0\}} \bigoplus_{\delta\in\Lambda} L_{\mu+\delta},$$

where Λ is defined as the group generated by isotropic roots δ in a vector space, which turns out to be a lattice of rank n, and so $\operatorname{supp} L$ of the Λ-grading of L is equal to Λ (see for the details in [2]). Also, the grading subalgebra $\mathfrak{g}$ is contained in $\oplus_{\mu\in\Delta\cup\{0\}} L_\mu$ ($L_\mu = L_{\mu+0}$) so that the Λ-grading $L = \oplus_{\delta\in\Lambda} L^\delta$, where $L^\delta := \oplus_{\mu\in\Delta\cup\{0\}} L_{\mu+\delta}$, is compatible. Thus L is a $\Delta\Lambda$-graded Lie algebra.

We recall one of the basic properties of extended affine Lie algebras (see [1]): For any $\mu \in \Delta$, $\delta \in \Lambda$ and any $0 \neq e_{\mu+\delta} \in L_{\mu+\delta}$, there exist some $f_{\mu+\delta} \in L_{-\mu-\delta}$ and $h_{\mu+\delta} \in L_0$ $(= L_{0+0})$ such that $\langle e_{\mu+\delta}, f_{\mu+\delta}, h_{\mu+\delta}\rangle$ is an sl_2-triplet, and in particular $[e_{\mu+\delta}, f_{\mu+\delta}] = h_{\mu+\delta}$.

One can check that $h_\mu - h_{\mu+\delta} \in Z(L)$ for all coroots $h_\mu = h_{\mu+0}$ of $\mathfrak{g}$. Therefore L is a division $\Delta\Lambda$-graded Lie algebra. We note that $\dim_{\mathbb{C}} L_{\mu+\delta} \leq 1$ for all $\mu \in \Delta$ and $\delta \in \Lambda$, which is also one of the basic properties of extended affine Lie algebras.

(c) Let $Z = \oplus_{g\in G} Z_g$ be a G-graded commutative associative algebra over F and let $\mathfrak{g} = \mathfrak{h} \oplus \big(\oplus_{\mu\in\Delta} \ \mathfrak{g}_\mu \big)$ be a finite dimensional split simple Lie algebra over F of type Δ with the set $\{h_\mu \in \mathfrak{h} \mid \mu \in \Delta\}$ of coroots. Then $L := \mathfrak{g} \otimes_F Z$ is a ΔG-graded Lie algebra. In fact, $L = \oplus_{\mu\in\Delta\cup\{0\}} (\mathfrak{g}_\mu \otimes_F Z)$ for $\mathfrak{g}_0 = \mathfrak{h}$ is a Δ-graded Lie algebra with grading subalgebra $\mathfrak{g} = \mathfrak{g} \otimes 1$. We put $L^g := \mathfrak{g} \otimes_F Z_g$ for all $g \in G$. Then $\operatorname{supp} L = \operatorname{supp} Z$ and $L = \oplus_{g\in G} L^g$ is a G-graded Lie algebra with $\mathfrak{g} \subset L^0$, i.e, compatible. Hence L is a ΔG-graded Lie algebra. We call the compatible G-grading of $L = \mathfrak{g} \otimes_F Z$ the *natural compatible G-grading from the G-grading of Z*.

Suppose that $Z = \oplus_{g\in G} K\overline{g}$ is a crossed product commutative algebra over F. Let $e \in \mathfrak{g}_\mu$ and $f \in \mathfrak{g}_{-\mu}$ such that $[e, f] = h_\mu$. Then $e \otimes \overline{g} \in L^g_\mu$, $f \otimes \overline{g}^{-1} \in L^{-g}_{-\mu}$ and

$$[e \otimes \overline{g}, f \otimes \overline{g}^{-1}] = [e, f] \otimes \overline{g}\ \overline{g}^{-1} = h_\mu \otimes 1 = h_\mu$$

for all $g \in G$, and so L is a predivision ΔG-graded Lie algebra over F. Note that $Z(L) = (0)$. Also, if K is a field, then L is a division ΔG-graded Lie algebra.

Suppose that $\tilde{L} = (\tilde{L}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) = \oplus_{g\in G} \tilde{L}^g$ is a ΔG-graded Lie algebra and that $\pi : \tilde{L} \longrightarrow L$ is a cover of a Lie algebra L. Then $L = \big(L, \pi(\tilde{\mathfrak{g}}), \pi(\tilde{\mathfrak{h}})\big)$ becomes a Δ-graded Lie algebra so that $(\tilde{L}, \pi)$ is a Δ-cover of L. Moreover, if $\ker \pi$ is G-graded, then L admits the induced compatible G-grading $L = \oplus_{g\in G} \pi(\tilde{L}^g)$. In particular, the centre $Z(\tilde{L})$ is always G-graded, $\tilde{L}/Z(\tilde{L})$ is a ΔG-graded Lie algebra.

Definition 2.9. Let P be a unital associative algebra over F and let $\mathfrak{gl}_{l+1}(P)$ be the Lie algebra consisting of all $(l + 1) \times (l + 1)$ matrices over P under the commutator product $(l \geq 1)$. Let $e_{ij}(a) \in \mathfrak{gl}_{l+1}(P)$ whose (i, j)-entry is a and the other entries are all 0. We define $sl_{l+1}(P)$ as the subalgebra of $\mathfrak{gl}_{l+1}(P)$ generated by $e_{ij}(a)$ for all $a \in P$ and $1 \leq i \neq j \leq l + 1$. The centre $Z\big(sl_{l+1}(P)\big)$ of $sl_{l+1}(P)$ consists of $\sum_{i=1}^{l+1} e_{ii}(a)$ for $a \in [P, P] \cap Z(P)$ where $[P, P]$ is the span of all commutators in P and $Z(P)$ is the centre of P. We define $psl_{l+1}(P)$ as $sl_{l+1}(P)/Z\big(sl_{l+1}(P)\big)$.

It is well-known that $sl_{l+1}(P)$ is an A_l-graded Lie algebra (see [3]): Denote $\{e_{ij}(b) \mid b \in B\}$ by $e_{ij}(B)$ for any subset $B \subset P$. Let

$$sl_{l+1}(F) = \mathfrak{h} \oplus \bigoplus_{1\leq i\neq j\leq l+1} e_{ij}(F1) \subset sl_{l+1}(P),$$

be the split simple Lie algebra over F of type A_l where $\mathfrak{h}$ is the Cartan subalgebra consisting of diagonal matrices of $sl_{l+1}(F)$. Let $\varepsilon_i : \mathfrak{h} \longrightarrow F$ be the projection onto the (i, j)-entry for $i = 1, \ldots , l + 1$, and $\Delta := \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, which is a root system of type A_l. Then

$$sl_{l+1}(P) = L_0 \oplus \Bigg(\bigoplus_{\varepsilon_i-\varepsilon_j\in\Delta} e_{ij}(P) \Bigg),$$

where $L_0 = \sum_{\varepsilon_i-\varepsilon_j\in\Delta} [e_{ij}(P), e_{ji}(P)]$, is an A_l-graded Lie algebra with grading subalgebra $sl_{l+1}(F)$. Let $Z := Z\big(sl_{l+1}(P)\big)$. We can and will identify $sl_{l+1}(F)+Z$ with $sl_{l+1}(F)$ and $e_{ij}(P)+Z$ with $e_{ij}(P)$, and so

$$psl_{l+1}(P) = (L_0/Z) \oplus \Big(\bigoplus_{\varepsilon_i-\varepsilon_j\in\Delta} e_{ij}(P) \Big)$$

is also an A_l-graded Lie algebra with the same grading subalgebra $sl_{l+1}(F)$.

Example 2.10. Let $sl_{l+1}(P)$ be the A_l-graded Lie algebra over F with grading subalgebra $sl_{l+1}(F)$ described above. If $P = \oplus_{g\in G} P_g$ is a G-graded algebra, then $sl_{l+1}(P)$ admits a compatible G-grading. Indeed, let

$$sl_{l+1}(P)^g := \big\{ \sum_{i,j} e_{ij}(P_g) \mid \sum_{i,j} e_{ij}(P_g) \subset sl_{l+1}(P) \big\}.$$

Then $sl_{l+1}(P) = \oplus_{g\in G} sl_{l+1}(P)^g$, and it is a G-graded Lie algebra with $sl_{l+1}(F) \subset L^0$, i.e., compatible. Note that $\operatorname{supp}\big(sl_{l+1}(P)\big) \supset \operatorname{supp} P$, and so $\big\langle \operatorname{supp}\big(sl_{l+1}(P)\big)\big\rangle = G$. Also, $psl_{l+1}(P)$ admits the induced compatible G-grading. We call the compatible G-grading of $sl_{l+1}(P)$ or $psl_{l+1}(P)$, i.e.,

$$sl_{l+1}(P)^g_{\varepsilon_i-\varepsilon_j} = e_{ij}(P_g) = psl_{l+1}(P)^g_{\varepsilon_i-\varepsilon_j} \quad \text{for all } \varepsilon_i-\varepsilon_j \in \Delta \text{ and } g \in G,$$

the *natural compatible G-grading from the G-grading of P*.

If $P = \oplus_{g\in G} R\overline{g}$ is a crossed product algebra, then

$$[e_{ij}(\overline{g}), e_{ji}(\overline{g}^{-1})] = e_{ii}(1) - e_{jj}(1) = [e_{ij}(1), e_{ji}(1)] = h_{\varepsilon_i-\varepsilon_j}$$

for all $g \in G$. Thus $sl_{l+1}(P)$ and $psl_{l+1}(P)$ with the natural compatible G-gradings from the G-grading of P are predivision A_lG-graded Lie algebras over F. Also, if R is a division algebra, then the A_lG-graded Lie algebras $sl_{l+1}(P)$ and $psl_{l+1}(P)$ are division.

For any associative algebra P, one can define a new product, $p \cdot q = \frac{1}{2}(pq+qp)$ for all $p, q \in P$. Then $P^+ := (P, \cdot)$ is a Jordan algebra.

Lemma 2.11. (i) *Suppose that the A_l-graded Lie algebra $psl_{l+1}(P)$ described above admits a predivision (resp. division) compatible G-grading. Then if $l \geq 2$, P is a predivision (resp. division) G-graded algebra, and the G-grading of $psl_{l+1}(P)$ is natural from the G-grading of P.*

If $l = 1$, then P^+ is a predivision (resp. division) G-graded Jordan algebra.

(ii) *Suppose that the Δ-graded Lie algebra $\mathfrak{g} \otimes_F Z$ described in Example 2.8(c) admits a predivision (resp. division) compatible G-grading. Then Z is a predivision* *(resp. division) G-graded algebra, and the G-grading of $\mathfrak{g} \otimes_F Z$ is natural from the G-grading of Z.*

Proof. (i): By our assumption, $psl_{l+1}(P) = psl_{l+1}(P)_0 \oplus \big(\oplus_{\varepsilon_i - \varepsilon_j \in \Delta} \ e_{ij}(P)\big)$ admits a predivision (resp. division) compatible G-grading, say

$$psl_{l+1}(P) = psl_{l+1}(P)_0 \oplus \big(\oplus_{\varepsilon_i - \varepsilon_j \in \Delta} \ \oplus_{g \in G} \ e_{ij}(P)^g\big).$$

Let

$$P_g^{ij} := \{p \in P \mid e_{ij}(p) \in e_{ij}(P)^g\}.$$

We claim that $P_g^{ij} = P_g^{rs}$ for all $\varepsilon_r - \varepsilon_s \in \Delta$. If $l = 1$, then $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_1\}$. For $p \in P_g^{12}$, we have

$$[[e_{12}(p), e_{21}(1)], e_{21}(1)] = -2e_{21}(p) \in e_{21}(P)^g$$

since $e_{21}(1) \in e_{21}(P)^0$. Thus $p \in P_g^{21}$ and we get $P_g^{12} \subset P_g^{21}$. The other inclusion can be obtained by the similar way. Hence the claim holds for $l = 1$.

In general, it is well-known that for any distinct $\alpha, \beta \in \Delta = A_l$, $l \geq 2$, D or E, there exists a sequence $\alpha_1, \dots, \alpha_t \in \Delta$ so that $\alpha_1 = \alpha$, $\alpha_t = \beta$ and $\alpha_{i+1} - \alpha_i \in \Delta$ for $i = 1, \dots, t-1$.

Now, for $l \geq 2$, it is enough to show that $P_g^{ij} \subset P_g^{rs}$. Let $p \in P_g^{ij}$. We apply the above for $\alpha = \varepsilon_i - \varepsilon_j$ and $\beta = \varepsilon_r - \varepsilon_s$. For $p \in P_g^{ij}$,

$$[\cdot\cdot[[e_{ij}(p), e_{\alpha_2}(1)], e_{\alpha_3}(1)], \dots, e_{\alpha_t}(1)] = \pm e_{\alpha_t}(p) = \pm e_{rs}(p) \in e_{rs}(P)^g$$

since $e_{\alpha_i}(1) \in L^0_{\alpha_i}$. Hence $p \in P_g^{rs}$ and our claim is settled.

Thus one can write $P_g = P_g^{ij}$ and $P = \oplus_{g \in G} \ P_g$. Since, for $p \in P_g$ and $q \in P_h$ $(g, h \in G)$,

$$[e_{ij}(p), e_{jk}(q)] = e_{ik}(pq) \in e_{ik}(P)^{g+h} \quad \text{if } l \geq 2 \text{ and } i \neq k,$$

$$[e_{12}(p), e_{21}(1)], e_{12}(q)] = e_{12}(pq + qp) \in e_{12}(P)^{g+h} \quad \text{if } l = 1,$$

we have $pq \in P_{g+h}$ if $l \geq 2$ and $pq + qp \in P_{g+h}$ if $l = 1$. Also, one can see that $\operatorname{supp} L \subset \operatorname{supp} P + \operatorname{supp} P$ (see Remark 2.7), and so $\langle \operatorname{supp} P\rangle \supset \langle \operatorname{supp} L\rangle = G$, whence $\langle \operatorname{supp} P\rangle = G$. Therefore, P is a G-graded algebra if $l \geq 2$ and P^+ is a G-graded Jordan algebra if $l = 1$. Note that $e_{ij}(P)^g = e_{ij}(P_g)$ for all $\varepsilon_i - \varepsilon_j \in \Delta$ and $g \in G$, and hence the G-grading for $l \geq 2$ is natural (see Remark 2.7).

By (pd), for any $\varepsilon_i - \varepsilon_j \in \Delta$ and any $g \in \operatorname{supp} P$, there exist $e_{ij}(p) \in e_{ij}(P_g)$ and $e_{ji}(q) \in e_{jk}(P_{-g})$ such that

$$[e_{ij}(p), e_{ji}(q)] = [e_{ij}(1), e_{ji}(1)] + z \quad \text{for some } z \in Z\big(sl_{l+1}(P)\big).$$

Hence $e_{ii}(pq) - e_{jj}(qp) = e_{ii}(1) - e_{jj}(1) + \sum_{k=1}^{l+1} \ e_{kk}(a)$ for some $a \in P$, and so $a = 0$ and $pq = qp = 1$, i.e., p is invertible. Also, p is invertible in $P \Leftrightarrow p$ is invertible in P^+. Therefore, $P = \oplus_{g \in G} \ P_g$ is a predivision G-graded associative algebra if $l \geq 2$, and $P^+ = \oplus_{g \in G} \ P_g$ is a predivision G-graded Jordan algebra if $l = 1$. The statement for 'division' can be shown in the same manner.

(ii): Let $Z_g := \{z \in Z \mid \mathfrak{g} \otimes z \subset (\mathfrak{g} \otimes_F Z)^g\}$. Then $Z = \oplus_{g \in G} \ Z_g$ becomes a G-graded algebra. The rest can be shown in the same manner. □

Definition 2.12. For ΔG-graded Lie algebras $\tilde{L} = \oplus_{g\in G} \tilde{L}^g$ and $L = \oplus_{g\in G} L^g$, if Δ-cover $\pi : \tilde{L} \longrightarrow L$ satisfies $L^g = \pi(\tilde{L}^g)$ for all $g \in G$, then $\tilde{L} = (\tilde{L}, \pi)$ is called a *ΔG-cover of L*. Also, for ΔG-graded Lie algebras L and L', if there exist a ΔG-graded Lie algebra $\tilde{L}$ and maps $\pi : \tilde{L} \longrightarrow L$ and $\pi' : \tilde{L} \longrightarrow L'$ such that $(\tilde{L}, \pi)$ and $(\tilde{L}, \pi')$ are both ΔG-covers, we say that L and L' are *ΔG-isogeneous.*

It is clear that if $\tilde{L}$ is a ΔG-cover of L, then

$$\tilde{L} \text{ is is predivision (resp. division)} \quad \Longleftrightarrow \quad L \text{ is predivision (resp. division).}$$

Also, by Corollary 2.4, if L and L' are ΔG-isogeneous, then $L/Z(L)$ and $L'/Z(L')$ are *ΔG-isomorphic*, i.e., there exists a Δ-isomorphism which is also G-graded between them. In particular, $\tilde{L}/Z(\tilde{L})$ and $L/Z(L)$ above are ΔG-isomorphic.

Proposition 2.13. (i) *Let $l \geq 3$. Then a predivision (resp. division) $A_l G$-graded Lie algebra L over F is an $A_l G$-cover of $psl_{l+1}(P)$ admitting the natural compatible G-grading from the G-grading of a predivision (resp. division) G-graded associative algebra P over F. Hence $L/Z(L)$ and $psl_{l+1}(P)$ are ΔG-isomorphic.*

(ii) *Let $\Delta = D$ or E and let $\mathfrak{g}$ be a finite dimensional split simple Lie algebra L over F of type Δ. Then a predivision (resp. division) ΔG-graded Lie algebra over F is a ΔG-cover of $\mathfrak{g} \otimes_F Z$ admitting the natural compatible G-grading from the G-grading of a predivision (resp. division) G-graded commutative associative algebra Z over F. Hence $L/Z(L)$ and $\mathfrak{g} \otimes_F Z$ are ΔG-isomorphic.*

Proof. For (i), let L be a predivision $A_l G$-graded Lie algebra over F. Berman and Moody showed in [3] that L is A_l-isogeneous to $\big(sl_{l+1}(P), sl_{l+1}(F)\big)$ (the Steinberg Lie algebra $st_{l+1}(P)$ serves as an A_l-cover of L and $sl_{l+1}(P)$). Hence, by Corollary 2.4, $L/Z(L)$ is A_l-isomorphic to $psl_{l+1}(P)$. Thus $\big(psl_{l+1}(P), sl_{l+1}(F)\big)$ admits a compatible G-grading via the A_l-isomorphism from the compatible G-grading of $L/Z(L)$ induced by the compatible G-grading of L. Therefore, the statement follows from Lemma 2.11.

(ii): Let L be a predivision ΔG-graded Lie algebra over F. Berman and Moody showed in [3] that L is a Δ-cover of $\mathfrak{g} \otimes_F Z$. Thus the statement follows from Lemma 2.11. $\square$

In this paper we classify predivision $\Delta\mathbb{Z}^n$-graded Lie algebras for $\Delta = A_l$, $l \geq 3$, D or E, up to central extensions. By Proposition 2.13, our work is to classify crossed product algebras $R * \mathbb{Z}^n$. We determine such algebras as a generalization of *quantum tori.* Namely, let $\boldsymbol{q} = (q_{ij})$ be an $n \times n$ matrix over F such that

$$q_{ii} = 1 \quad \text{and} \quad q_{ji} = q_{ij}^{-1}.$$

The *quantum torus $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ determined by $\boldsymbol{q}$* is defined as the associative algebra over F with $2n$ generators $t_1^{\pm 1}, \dots, t_n^{\pm 1}$, and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad \text{and} \quad t_j t_i = q_{ij} t_i t_j$$

for all $1 \le i, j \le n$. Quantum tori are characterized as predivision $\mathbb{Z}^n$-graded associative algebras whose homogeneous spaces are all 1-dimensional (see [4]). Note that $F_{\boldsymbol{q}}$ is commutative $\Longleftrightarrow$ $\boldsymbol{q} = \boldsymbol{1}$ whose entries are all 1, i.e., $F_{\boldsymbol{1}} = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the algebra of Laurent polynomials. Also, a quantum torus is a twisted group algebra $F^t[\mathbb{Z}^n]$.

§ 3 Classification of $R * \mathbb{Z}^n$

Throughout this section F is an arbitrary field and G is an arbitrary group. For a G-graded algebra $S = \oplus_{g \in G}\, S_g$ over F in general, we denote by $\mathrm{GrAut}_F(S)$ the group of graded automorphisms of S, i.e.,

$$\mathrm{GrAut}_F(S) := \{\sigma \in \mathrm{Aut}_F(S) \mid \sigma(S_g) = S_g \text{ for all } g \in G\}.$$

Lemma 3.1. *Let $R * G = (R, G, \sigma, \tau)$ be a crossed product algebra over F and $(R * G) * M = (R * G, M, \eta, \xi)$ a crossed product algebra over F for a group M, an action η and a twisting ξ. Suppose that $\eta(M) \subset GrAut_F(R * G)$ and that $\xi(m, l) \in U(R)$ for all $m, l \in M$. Then, $(R * G) * M$ is a crossed product algebra $R * (G \times M) = \big(R, (G \times M), \sigma', \tau'\big)$ over F for some action σ' and twisting τ'.*

Proof. We have

$$(R * G) * M = \oplus_{m \in M}\ (R * G)\overline{m} = \oplus_{m \in M}\ (\oplus_{g \in G}\ R\overline{g})\overline{m} = \oplus_{(g,m) \in G \times M}\ R\overline{g}\overline{m}$$

as free R-modules. We define $\eta_m = \eta(m) \mid_{R1}$ an F-automorphism of R for every $m \in M$. Also for $h \in G$, $\overline{h}$ is a unit in $R * G$ (see 1.6). Since η_m is a graded automorphism of $R*G$ by our first assumption, $\eta_m(\overline{h}) = d_{m,h}\overline{h}$ for some $d_{m,h} \in U(R)$. Therefore, for $r\overline{g}\overline{m} \in R\overline{g}\overline{m}$ and $s\overline{h}\overline{l} \in R\overline{h}\overline{l}$, we have

$$\begin{aligned}
(r\overline{g}\overline{m})(s\overline{h}\overline{l}) &= r\overline{g}\eta_m(s\overline{h})\overline{m}\overline{l} \\
&= r\overline{g}\eta_m(s)\eta_m(\overline{h})\xi(m,l)\overline{ml} \\
&= r\overline{g}\eta_m(s)d_{m,h}\overline{h}\xi(m,l)\overline{ml} \\
&= r\overline{g}\eta_m(s)d_{m,h}\sigma_h\big(\xi(m,l)\big)\overline{h}\overline{ml} \quad \text{(by our second assumption)} \\
&= r\sigma_g\eta_m(s)\sigma_g(d_{m,h})\sigma_{gh}\big(\xi(m,l)\big)\overline{g}\overline{h}\overline{ml} \\
&= r\sigma_g\eta_m(s)\sigma_g(d_{m,h})\sigma_{gh}\big(\xi(m,l)\big)\tau(g,h)\overline{gh}\ \overline{ml}.
\end{aligned}$$

Thus we have the action

$$\sigma' : G \times M \longrightarrow \mathrm{Aut}_F R \quad \text{by} \quad \sigma'_{(g,m)} = \sigma_g\eta_m,$$

and the twisting $\tau' : (G \times M) \times (G \times M) \longrightarrow U(R)$ by

$$\tau'\big((g,m),(h,l)\big) = \sigma_g(d_{m,h})\sigma_{gh}\big(\xi(m,l)\big)\tau(g,h).$$

Since the crossed product algebra $(R * G) * M$ is associative, we get

$$(R * G) * M = R * (G \times M) = (R, G \times M, \sigma', \tau'). \quad \square$$

A triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ where R is a unital associative algebra over F,

$$\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$$

is an n-tuple of F-automorphisms φ_i of R, and $\boldsymbol{q} = (q_{ij})$ is an $n \times n$ matrix over R satisfying, for all $1 \le i < j < k \le n$,

(G1) $$q_{ii} = 1 \quad \text{and} \quad q_{ji}^{-1} = q_{ij},$$

(G2) $$\varphi_j \varphi_i = \mathrm{I}(q_{ij}) \varphi_i \varphi_j,$$

(G3) $$\varphi_k(q_{ij}) = q_{jk} \varphi_j(q_{ik}) q_{ij} \varphi_i(q_{kj}) q_{ki},$$

is called a $\mathbb{Z}^n$*-grading triple*, and a *division* $\mathbb{Z}^n$*-grading triple* if R is a division algebra.

For a $\mathbb{Z}^n$-grading triple, we introduce several notations and prove some identities.

Notations.

(N1) $$\boldsymbol{\varepsilon}_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n,$$

i.e., the i-th coordinate is 1 and the others are 0.

(N2) $$q_{ij}^{(m)} := \begin{cases} q_{ij} \varphi_i(q_{ij}) \varphi_i^2(q_{ij}) \cdots \varphi_i^{m-1}(q_{ij}) = \prod_{l=0}^{m-1} \varphi_i^l(q_{ij}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \varphi_i^{-1}(q_{ji}) \varphi_i^{-2}(q_{ji}) \cdots \varphi_i^{m}(q_{ji}) = \prod_{l=-1}^{m} \varphi_i^l(q_{ji}), & \text{if } m < 0, \end{cases}$$

$$\text{and} \quad q_{ij}^{-(m)} := (q_{ij}^{(m)})^{-1}.$$

For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ and $k = 0, 1, 2, \dots, n$,

(N3) $$\varphi^{(\boldsymbol{\alpha})_k} := \begin{cases} \mathrm{id}, & \text{if } k = 0, 1 \\ \varphi_1^{\alpha_1} \cdots \varphi_{k-1}^{\alpha_{k-1}}, & \text{if } k > 1, \end{cases}$$

$$\text{and} \quad \varphi^{\boldsymbol{\alpha}} := \varphi_1^{\alpha_1} \cdots \varphi_n^{\alpha_n}.$$

(N4) $$q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\alpha}} := \prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)}) \quad \text{with} \quad \alpha_0 = q_{0j} = 1.$$

(N5) $$q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\alpha}}^{(m)} := \begin{cases} \prod_{l=m-1}^{0} \varphi_j^l(q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\alpha}}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \prod_{l=m}^{-1} \varphi_j^l(q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\alpha}}^{-1}), & \text{if } m < 0. \end{cases}$$

(N6) $$q_{\boldsymbol{\alpha}, \boldsymbol{\beta}} := \prod_{j=n}^{1} \varphi^{(\boldsymbol{\alpha})_j}(q_{\boldsymbol{\varepsilon}_j, \boldsymbol{\beta}}^{(\alpha_j)}).$$

Lemma 3.2. *For $m \in \mathbb{Z}$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we have*

$$\varphi_i^{-m}(q_{ij}^{-(m)}) = q_{ij}^{(-m)}, \tag{1}$$

$$\varphi_j \varphi_i^m = \mathrm{I}(q_{ij}^{(m)}) \varphi_i^m \varphi_j, \tag{2}$$

$$\varphi_j \varphi^{(\boldsymbol{\alpha})_i} = \begin{cases} \mathrm{I}\big(\prod_{l=1}^{i-1} \ \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\big) \varphi^{(\boldsymbol{\alpha})_i} \varphi_j & \textit{for } j \geq i, \\ \mathrm{I}\big(\prod_{l=1}^{j-1} \ \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\big) \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i} & \textit{for } j < i, \end{cases} \tag{3}$$

$$q_{ij}^{(m+1)} = q_{ij}\varphi_i(q_{ij}^{(m)}) \quad \textit{and} \quad q_{ij}^{-(m+1)} = \varphi_i(q_{ij}^{-(m)}) q_{ji}, \tag{4}$$

$$\varphi_k(q_{ij}^{(m)}) = q_{jk}\varphi_j(q_{ik}^{(m)}) q_{ij}^{(m)} \varphi_i^m(q_{kj}) q_{ik}^{-(m)}. \tag{5}$$

Proof. For (1), we have from (N2),

$$q_{ij}^{-(m)} = \begin{cases} \varphi_i^{m-1}(q_{ji}) \cdots \varphi_i(q_{ji}) q_{ji} = \prod_{l=m-1}^{1} \ \varphi_i^l(q_{ji}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \varphi_i^m(q_{ij}) \cdots \varphi_i^{-2}(q_{ij}) \varphi_i^{-1}(q_{ij}) = \prod_{l=m}^{-1} \ \varphi_i^l(q_{ij}), & \text{if } m < 0. \end{cases}$$

So we get

$$\varphi_i^{-m}(q_{ij}^{-(m)}) = \begin{cases} \varphi_i^{-1}(q_{ji}) \cdots \varphi_i^{-m}(q_{ji}) = \prod_{l=-1}^{-m} \ \varphi_i^l(q_{ji}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ q_{ij}\varphi_i(q_{ij}) \cdots \varphi_i^{-m-1}(q_{ij}) = \prod_{l=1}^{-m-1} \ \varphi_i^l(q_{ij}), & \text{if } m < 0, \end{cases}$$

which is exactly $q_{ij}^{(-m)}$.

For (2), the case $m = 0$ is clear. Assume that $m > 0$. Put $q := q_{ij}$ for simplicity. Then we have

$$\begin{aligned} \varphi_j \varphi_i^m &= \varphi_j \varphi_i^{m-1} \varphi_i \\ &= \mathrm{I}(q^{(m-1)}) \varphi_i^{m-1} \varphi_j \varphi_i \quad \text{by induction on } m \\ &= \mathrm{I}(q^{(m-1)}) \varphi_i^{m-1} \mathrm{I}(q) \varphi_i \varphi_j \quad \text{by (G2)} \\ &= \mathrm{I}(q^{(m-1)}) \mathrm{I}\big(\varphi_i^{m-1}(q)\big) \varphi_i^m \varphi_j \\ &= \mathrm{I}(q^{(m)}) \varphi_i^m \varphi_j. \end{aligned}$$

Also, $(\varphi_j \varphi_i^m)^{-1} = (\mathrm{I}(q_{ij}^{(m)}) \varphi_i^m \varphi_j)^{-1}$ for $m > 0$, and so

$$\varphi_i^{-m} \varphi_j^{-1} = \varphi_j^{-1} \varphi_i^{-m} (\mathrm{I}(q_{ij}^{-(m)}) = \varphi_j^{-1} \mathrm{I}\big(\varphi_i^{-m}(q_{ij}^{-(m)})\big) \varphi_i^{-m} = \varphi_j^{-1} \mathrm{I}(q_{ij}^{(-m)}) \varphi_i^{-m},$$

by (1). Hence we get $\varphi_j \varphi_i^{-m} = \mathrm{I}(q_{ij}^{(-m)}) \varphi_i^{-m} \varphi_j$, and (2) holds for all $m \in \mathbb{Z}$.

For (3), when $j \geq i$, using (2), we have

$$\begin{aligned}
\varphi_j \varphi_i^{(\boldsymbol{\alpha})_i} &= \varphi_j \varphi_1^{\alpha_1} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \varphi_j \varphi_2^{\alpha_2} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \mathrm{I}(q_{2j}^{(\alpha_2)}) \varphi_2^{\alpha_2} \varphi_j \varphi_3^{\alpha_3} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&\cdots\cdots \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \mathrm{I}(q_{2j}^{(\alpha_2)}) \varphi_2^{\alpha_2} \mathrm{I}(q_{3j}^{(\alpha_3)}) \varphi_3^{\alpha_3} \cdots \mathrm{I}(q_{i-1,j}^{(\alpha_{i-1})}) \varphi_{i-1}^{\alpha_{i-1}} \varphi_j \\
&= \mathrm{I}\Big(\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\Big) \varphi^{(\boldsymbol{\alpha})_i} \varphi_j. \quad (\text{Note } \varphi^{(\boldsymbol{\alpha})_0} = \mathrm{id} \text{ when } i = 1)
\end{aligned}$$

When $j < i$, we have

$$\begin{aligned}
\varphi_j \varphi_i^{(\boldsymbol{\alpha})_i} &= \varphi_j \varphi_1^{\alpha_1} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \varphi_j \varphi_2^{\alpha_2} \cdots \varphi_j^{\alpha_j} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&\cdots\cdots \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \cdots \mathrm{I}(q_{j-1,j}^{(\alpha_{j-1})}) \varphi_{j-1}^{\alpha_{j-1}} \mathrm{I}(q_{jj}^{(\alpha_j)}) \varphi_j^{\alpha_j} \varphi_j \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \cdots \mathrm{I}(q_{j-1,j}^{(\alpha_{j-1})}) \varphi_{j-1}^{\alpha_{j-1}} \varphi^{\alpha_j+1} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= \mathrm{I}\Big(\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\Big) \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}. \quad (\text{Note } \varphi^{(\boldsymbol{\alpha})_0} = \mathrm{id} \text{ when } j = 1)
\end{aligned}$$

For the first formula of (4), the case $m = 0$ is clear. We put $q := q_{ij}$, $p := q^{-1}$ and $\varphi := \varphi_i$ for simplicity. For $m > 0$, we have

$$\begin{aligned}
q^{(m+1)} &= q\varphi(q)\varphi^2(q) \cdots \varphi^m(q) \\
&= q\varphi\big(q\varphi(q) \cdots \varphi^{m-1}(q)\big) = q\varphi(q^{(m)}).
\end{aligned}$$

For $m = -1$, we have $q^{(-1+1)} = 1$, while $q\varphi(q^{(-1)}) = q\varphi\varphi^{-1}(p) = 1$. For $m < -1$, we have

$$\begin{aligned}
q^{(m+1)} &= \varphi^{-1}(p)\varphi^{-2}(p) \cdots \varphi^{m+1}(p) \\
&= qp\varphi^{-1}(p)\varphi^{-2}(p) \cdots \varphi^{m+1}(p) \\
&= q\varphi\big(\varphi^{-1}(p)\varphi^{-2}(p) \cdots \varphi^m(p)\big) = q\varphi(q^{(m)}).
\end{aligned}$$

The second formula follows from the first since $q_{ij}^{-(m+1)} = (q_{ij}^{(m+1)})^{-1}$.

For (5), the case $m=0$ is clear. Assume that $m>0$. Then we have

$$\begin{aligned}
&\varphi_k(q_{ij}^{(m)})\\
=&\varphi_k(q_{ij})\varphi_k\varphi_i(q_{ij}^{(m-1)}) \quad \text{by (3)}\\
=&\varphi_k(q_{ij})q_{ik}\varphi_i\varphi_k(q_{ij}^{(m-1)})q_{ki} \quad \text{by (G2)}\\
=&q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{kj})q_{ki}q_{ik}\varphi_i\big(q_{jk}\varphi_j(q_{ik}^{(m-1)})q_{ij}^{(m-1)}\varphi_i^{m-1}(q_{kj})(q_{ik}^{-(m-1)})\big)q_{ki}
\end{aligned}$$

by (G3) and induction on m

$$\begin{aligned}
=&q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i\varphi_j(q_{ik}^{(m-1)})\varphi_i(q_{ij}^{(m-1)})\varphi_i^m(q_{kj})\varphi_i(q_{ik}^{-(m-1)})q_{ki}\\
=&q_{jk}\varphi_j(q_{ik})q_{ij}q_{ji}\varphi_j\varphi_i(q_{ik}^{(m-1)})q_{ij}\varphi_i(q_{ij}^{(m-1)})\varphi_i^m(q_{kj})\varphi_i(q_{ik}^{-(m)})
\end{aligned}$$

by (G2) and (3)

$$\begin{aligned}
=&q_{jk}\varphi_j(q_{ik})\varphi_j\varphi_i(q_{ik}^{(m-1)})\varphi_i(q_{ij}^{(m)})\varphi_i^m(q_{kj})\varphi_i(q_{ik}^{-(m)}) \quad \text{by (3)}\\
=&q_{jk}\varphi_j(q_{ik}^{(m)})q_{ij}^{(m)}\varphi_i^m(q_{kj})q_{ik}^{-(m)} \quad \text{by (3).}
\end{aligned}$$

Also, one has $\big(\varphi_k(q_{ij}^{(m)})\big)^{-1}=\big(q_{jk}\varphi_j(q_{ik}^{(m)})q_{ij}^{(m)}\varphi_i^m(q_{kj})q_{ik}^{-(m)}\big)^{-1}$ for $m>0$, and so $\varphi_k(q_{ji}^{-(m)})=q_{ik}^{(m)}\varphi_i^m(q_{jk})q_{ij}^{-(m)}\varphi_j(q_{ik}^{-(m)})q_{kj}$. Applying φ_i^{-m} in both hands, we get

$$\begin{aligned}
\varphi_i^{-m}\varphi_k(q_{ij}^{-(m)}) &= \varphi_i^{-m}\big(q_{ik}^{(m)}\varphi_i^m(q_{jk})q_{ij}^{-(m)}\varphi_j(q_{ik}^{-(m)})q_{kj}\big)\\
&= \varphi_i^{-m}(q_{ik}^{(m)})q_{jk}q_{ij}^{(-m)}\varphi_i^{-m}\varphi_j(q_{ik}^{-(m)})\varphi_i^{-m}(q_{kj}) \quad \text{by (1).}
\end{aligned}$$

Then, by (1) and (2), we have

$$I(q_{ik}^{-(-m)})\varphi_k(q_{ij}^{(-m)}) = q_{ik}^{-(-m)}q_{jk}q_{ij}^{(-m)}I(q_{ij}^{-(-m)})\varphi_j(q_{ik}^{(-m)})\varphi_i^{-m}(q_{kj}),$$

and we obtain

$$\varphi_k(q_{ij}^{(-m)}) = q_{jk}\varphi_j(q_{ik}^{(-m)})q_{ij}^{(-m)}\varphi_i^{-m}(q_{kj})q_{ik}^{-(-m)} \quad \text{for } m>0.$$

Hence, (5) holds for all $m\in\mathbb{Z}$. □

Now we are ready to state our theorem.

Theorem 3.3. *Let $(R,\boldsymbol{\varphi},\boldsymbol{q})$ be a $\mathbb{Z}^n$-grading triple and let $R_{\boldsymbol{\varphi},\boldsymbol{q}}:=\oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} Rt_{\boldsymbol{\alpha}}$ be a free left R-module with basis $\{t_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha}\in\mathbb{Z}^n\}$. Then there exists a unique associative multiplication on $R_{\boldsymbol{\varphi},\boldsymbol{q}}$ such that, for $t_i:=t_{\boldsymbol{\varepsilon}_i}$, $i=1,\ldots,n$, $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)$ and $r\in R$,*

$$(3.4) \qquad t_{\boldsymbol{\alpha}}=t_1^{\alpha_1}\cdots t_n^{\alpha_n}, \quad t_it_i^{-1}=t_i^{-1}t_i=1, \quad t_ir=\varphi_i(r)t_i \quad \textit{and} \quad t_jt_i=q_{ij}t_it_j.$$

Moreover, for $rt_{\boldsymbol{\alpha}}, r't_{\boldsymbol{\beta}} \in R_{\boldsymbol{\varphi},\boldsymbol{q}}$, we have

$$rt_{\boldsymbol{\alpha}} r't_{\boldsymbol{\beta}} = r\varphi^{\boldsymbol{\alpha}}(r') q_{\boldsymbol{\alpha},\boldsymbol{\beta}} t_{\boldsymbol{\alpha}+\boldsymbol{\beta}},$$

where $\varphi^{\boldsymbol{\alpha}}$ and $q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are defined in (N3) and (N6). In particular, $R_{\boldsymbol{\varphi},\boldsymbol{q}}$ is a crossed product algebra $R * \mathbb{Z}^n$ with

$$\text{(action)} \quad \sigma : \mathbb{Z}^n \longrightarrow Aut_F(R) \quad \text{by} \quad \sigma(\boldsymbol{\alpha}) = \varphi^{\boldsymbol{\alpha}}$$

$$\text{(twisting)} \quad \tau : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow U(R) \quad \text{by} \quad \tau(\boldsymbol{\alpha}, \boldsymbol{\beta}) = q_{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

Conversely, for any crossed product algebra $R * \mathbb{Z}^n$, there exists a $\mathbb{Z}^n$-grading triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ such that $R * \mathbb{Z}^n = R_{\boldsymbol{\varphi},\boldsymbol{q}}$.

Proof. We first consider a crossed product algebra $R * \mathbb{Z}$. Let $t := \overline{1} \in R * \mathbb{Z}$. Then, t^m is a unit in $R\overline{m}$ for all $m \in \mathbb{Z}$. Using the diagonal basis change, one can take an R-basis $\{t^m \mid m \in \mathbb{Z}\}$. So we have $t^m t^l = t^{m+l}$ for all $m, l \in \mathbb{Z}$. Hence, $R * \mathbb{Z} = R\mathbb{Z}$ is a skew group algebra. Let ψ be the action of 1, i.e., $t(r1) = \psi(r)t$ for $r \in R$. (Note that $1 = \overline{0}$.) Then the action of m is ψ^m, i.e.,

$$t^m(r1) = \psi^m(r)t^m.$$

Conversely, it is clear that any F-automorphism ψ of R determines a skew group algebra $R\mathbb{Z}$ by the action $m \mapsto \psi^m$ (see Remark 1.3). We denote this $R\mathbb{Z}$ by $R[t; \psi]$.

Let $R^{(1)} := R[t_1; \psi_1]$ where $\psi_1 = \varphi_1$. Let ψ_2 be a graded F-automorphism ψ_2 of $R^{(1)}$ and $R^{(2)} := R^{(1)}[t_2; \psi_2]$. Then, by Lemma 3.1, we get $R^{(2)} = (R\mathbb{Z})\mathbb{Z} = R * \mathbb{Z}^2$. Repeating this process n times, one can construct $R * \mathbb{Z}^n$ inductively. Namely, for a crossed product algebra $R^{(k-1)} = R * \mathbb{Z}^{k-1}$, if we specify an F-graded automorphism ψ_k of $R^{(k-1)}$, then

$$R^{(k)} := R^{(k-1)}[t_k; \psi_k] = R * \mathbb{Z}^k,$$

and we obtain $R^{(n)} = R * \mathbb{Z}^n$. Thus, our task is to specify ψ_k on $R^{(k-1)}$ and to show that ψ_k is a graded F-automorphism where $k \geq 2$. We note that

$$\{t_1^{\alpha_1} \cdots t_{k-1}^{\alpha_{k-1}} \mid (\alpha_1, \ldots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}\}$$

is a basis of the free R-module $R^{(k-1)}$. For convenience, we put

$$t^{(\boldsymbol{\alpha})_k} = t_1^{\alpha_1} \cdots t_{k-1}^{\alpha_{k-1}},$$

and define an F-linear transformation ψ_k on $R^{(k-1)}$ by

$$\psi_k(rt^{(\boldsymbol{\alpha})_k}) = \varphi_k(r) \left[\prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) \right] t^{(\boldsymbol{\alpha})_k} \quad \text{for} \quad r \in R,$$

which is clearly graded. If $\psi_k(rt^{(\boldsymbol{\alpha})_k}) = 0$, then $\varphi_k(r) = 0$, and hence $r = 0$, and so ψ_k is injective. Since

$$\psi_k \left(\varphi_k^{-1} \left(r \left[\prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) \right]^{-1} \right) t^{(\boldsymbol{\alpha})_k} \right) = rt^{(\boldsymbol{\alpha})_k},$$

ψ_k is surjective. Therefore, ψ_k is an F-linear graded isomorphism on $R^{(k-1)}$. So it remains to prove that ψ_k is a homomorphism. For this purpose, we use a well-known fact.

3.5. *Let A and B be unital associative algebras over F and f a F-linear map from A into B. Let $\{t_i\}_{i\in I}$ be a generating set of the F-algebra A. Then, f is a homomorphism if and only if $f(t_i y) = f(t_i)f(y)$ for all $i \in I$ and $y \in A$. Moreover, if $\{t_i^{\pm 1}\}_{i\in I}$ is a generating set of A, then f is a homomorphism if and only if $f(t_i y) = f(t_i)f(y)$ and $f(t_i^{-1}) = f(t_i)^{-1}$ for all $i \in I$ and $y \in A$.*

We have a generating set $R \cup \{t_1^{\pm 1}, \dots, t_{k-1}^{\pm 1}\}$ of $R^{(k-1)}$ over F, and

$$\begin{aligned}\psi_k(t_j^{-1}) &= q_{jk}^{(-1)} t_j^{-1} = \varphi_j^{-1}(q_{kj}) t_j^{-1} \\ &= (t_j \varphi_j^{-1}(q_{jk}))^{-1} = (q_{jk} t_j)^{-1} = \psi_k(t_j)^{-1}.\end{aligned}$$

So, by 3.5, we only need to show that, for all $r, r' \in R$ and $1 \le j \le k-1$,

$$\psi_k(rr' t^{(\boldsymbol{\alpha})_k}) = \psi_k(r)\psi_k(r' t^{(\boldsymbol{\alpha})_k}), \tag{A}$$

$$\psi_k(t_j r t^{(\boldsymbol{\alpha})_k}) = \psi_k(t_j)\psi_k(r t^{(\boldsymbol{\alpha})_k}). \tag{B}$$

For (A), we have

$$\begin{aligned}\psi_k(rr' t^{(\boldsymbol{\alpha})_k}) &= \varphi_k(rr') \prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) t^{(\boldsymbol{\alpha})_k} \\ &= \varphi_k(r)\varphi_k(r') \prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(d_{ik}^{(\alpha_i)}) t^{(\boldsymbol{\alpha})_k} \\ &= \psi_k(r)\psi_k(r' t^{(\boldsymbol{\alpha})_k}).\end{aligned}$$

For (B), we first note that there is the embedding of $R^{(j)}$ into $R^{(k-1)}$ for $1 \le j \le k-1$, and so

$$t_j t^{(\boldsymbol{\alpha})_j} = \psi_j(t^{(\boldsymbol{\alpha})_j}) t_j = \varphi_j(r) \prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)}) t^{(\boldsymbol{\alpha})_j} t_j.$$

Thus we have

$$\begin{aligned}\psi_k(t_j r t^{(\boldsymbol{\alpha})_k}) &= \psi_k(\varphi_j(r) t_j t^{(\boldsymbol{\alpha})_k}) \\ &= \psi_k(\varphi_j(r)(\psi_j(t^{(\boldsymbol{\alpha})_j}) t_j^{\alpha_j+1} \cdots t_{k-1}^{\alpha_{k-1}}) \\ &= \psi_k(\varphi_j(r) \prod_{i=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)}) t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k}) \\ &= \varphi_k\varphi_j(r) \prod_{i=1}^{j-1} \varphi_k \varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)}) \prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i+\delta_{ij})}) t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k} \\ &:= ABC t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k},\end{aligned}$$

where $A = \varphi_k\varphi_j(r)$, $B = \prod_{i=1}^{j-1} \ \varphi_k\varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)})$ and $C = \prod_{i=1}^{k-1} \ \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i+\delta_{ij})})$. First of all, we have

$$A = \varphi_k\varphi_j(r) = q_{jk}\varphi_j\varphi_k(r)q_{kj} \quad \text{by (G2)}.$$

Secondly, by Lemma 3.3(2) and (4), we have

$$\begin{aligned}
&\varphi_k\varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)}) \\
&= \left[\prod_{l=1}^{i-1} \ \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]\varphi^{(\boldsymbol{\alpha})_i}\varphi_k(q_{ij}^{(\alpha_i)})\left[\prod_{l=1}^{i-1} \ \varphi^{(\boldsymbol{\alpha})_l}(d_{lk}^{(\alpha_l)})\right]^{-1} \\
&= \left[\prod_{l=1}^{i-1} \ \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]\varphi^{(\boldsymbol{\alpha})_i}\big(q_{jk}\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}\varphi_i^{\alpha_i}(q_{kj})q_{ik}^{-(\alpha_i)}\big)\left[\prod_{l=1}^{i-1} \ \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]^{-1}.
\end{aligned}$$

Note that

$$\varphi^{(\boldsymbol{\alpha})_i}(q_{ki}^{-(\alpha_i)})\left[\prod_{l=1}^{i-1} \ \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]^{-1} = \left[\prod_{l=1}^{i} \ \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]^{-1}$$
$$\text{and} \quad \varphi^{(\boldsymbol{\alpha})_i}\varphi_i^{\alpha_i}(q_{kj}) = \varphi^{(\boldsymbol{\alpha})_{i+1}}(q_{kj}).$$

So we have

$$\begin{aligned}
\big(\varphi_k\varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)})\big)\big(\varphi_k\varphi^{(\boldsymbol{\alpha})_{i+1}}(q_{i+1,j}^{(\alpha_{i+1})})\big) &= \left[\prod_{l=1}^{i-1} \ \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]\varphi^{(\boldsymbol{\alpha})_i}\big(q_{jk}\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}\big) \\
&\times \varphi^{(\boldsymbol{\alpha})_{i+1}}\big(\varphi_j(q_{i+1,k}^{(\alpha_{i+1})})q_{i+1,j}^{(\alpha_{i+1})}\big)\left[\prod_{l=1}^{i} \ \varphi^{(\boldsymbol{\alpha})_l}(q_{lk}^{(\alpha_l)})\right]^{-1}.
\end{aligned}$$

Thus, after cancellations, we get

$$\begin{aligned}
B &= \prod_{i=1}^{j-1} \ \varphi_k\varphi^{(\boldsymbol{\alpha})_i}(q_{ij}^{(\alpha_i)}) \\
&= q_{jk}\left[\prod_{i=1}^{j-1} \ \varphi^{(\boldsymbol{\alpha})_i}\big(\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}\big)\right]\varphi^{(\boldsymbol{\alpha})_j}(q_{kj})\left[\prod_{i=1}^{j-1} \ \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\right]^{-1}.
\end{aligned}$$

Thirdly, we have

$$\begin{aligned}
C &= \prod_{i=1}^{k-1} \ \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i+\delta_{ij})}) \\
&= \left[\prod_{i=1}^{j-1} \ \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\right]\varphi^{(\boldsymbol{\alpha})_j}(q_{jk}^{(\alpha_j+1)}) \prod_{i=j+1}^{k-1} \ \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)}) \\
&= \left[\prod_{i=1}^{j-1} \ \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\right]\varphi^{(\boldsymbol{\alpha})_j}\big(q_{jk}\varphi_j(q_{jk}^{(\alpha_j)})\big) \prod_{i=j+1}^{k-1} \ \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)}),
\end{aligned}$$

by Lemma 3.2(4). Consequently, after cancellations and notifying $q_{ii}=1$, we obtain

$$\psi_k(t_j r t^{(\boldsymbol{\alpha})_k}) = ABC t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k}$$

$$(*) \qquad = q_{jk}\varphi_j\varphi_k(r)\prod_{i=1}^{j} \varphi^{(\boldsymbol{\alpha})_i}\big(\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}\big)\prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)})t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k}.$$

On the other hand, we have

$$\begin{aligned}
\psi_k(t_j)\psi_k(rt^{(\boldsymbol{\alpha})_k}) &= q_{jk}t_j\varphi_k(r)\prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})t^{(\boldsymbol{\alpha})_k}\\
&= q_{jk}\varphi_j\Big[\varphi_k(r)\prod_{i=1}^{k-1} \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\Big]t_j t^{(\boldsymbol{\alpha})_k}\\
&= q_{jk}\varphi_j\varphi_k(r)\prod_{i=1}^{k-1} \varphi_j\varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k}.
\end{aligned}$$

We rewrite $D := \prod_{i=1}^{k-1} \varphi_j\varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)})$. To find an expression for D, we use the following lemma:

Lemma 3.6. *Let A be a unital associative algebra, $a_0=1, a_1, \ldots, a_k \in A$ units and $b_1, \ldots, b_k \in A$. Then we have*

$$(1) \qquad \prod_{i=1}^{k}\Big(I\big(\prod_{l=1}^{i-1} a_l\big)(b_i)\Big) = \prod_{i=1}^{k} a_i b_i\big(\prod_{l=1}^{k-1} a_l\big)^{-1}.$$

$$(2) \qquad \prod_{i=j+1}^{k}\Big(I\big(\prod_{l=1}^{j-1} a_l\big)(b_i)\Big) = I\big(\prod_{l=1}^{j-1} a_l\big)\big(\prod_{i=j+1}^{k} b_i\big).$$

Proof. (1) is straightforward and (2) is obvious. □

By Lemma 3.2(3), we have, for $i<j$,

$$\varphi_j\varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) = \mathrm{I}\Big(\prod_{l=1}^{i-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\Big)\big(\varphi^{(\boldsymbol{\alpha})_i}\varphi_j(q_{ik}^{(\alpha_i)})\big).$$

So, by Lemma 3.6(1), we get

$$\prod_{i=1}^{j} \varphi_j\varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) = \prod_{i=1}^{j} \varphi^{(\boldsymbol{\alpha})_i}\big(\varphi_j(q_{ik}^{(\alpha_i)})(q_{ij}^{(\alpha_i)})\big)\Big[\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\Big]^{-1}.$$

By Lemma 3.2(3), we have, for $j < i$,

$$\varphi_j \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) = \mathrm{I}\Big(\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\Big)\big(\varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)})\big).$$

So, by Lemma 3.6(2), we get

$$\prod_{i=j+1}^{k-1} \varphi_j \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) = \mathrm{I}\Big(\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\Big)\Big(\prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)})\Big).$$

Hence we get

$$\begin{aligned} D &= \prod_{i=1}^{k-1} \varphi_j \varphi^{(\boldsymbol{\alpha})_i}(q_{ik}^{(\alpha_i)}) \\ &= \prod_{i=1}^{j} \varphi^{(\boldsymbol{\alpha})_i}\big(\varphi_j(q_{ik}^{(\alpha_i)}) q_{ij}^{(\alpha_i)}\big) \prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)}) t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k} \Big[\prod_{l=1}^{j-1} \varphi^{(\boldsymbol{\alpha})_l}(q_{lj}^{(\alpha_l)})\Big]^{-1}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &\psi_k(t_j)\psi_k(rt^{(\boldsymbol{\alpha})_k}) \\ &\qquad = q_{jk}\varphi_j\varphi_k(r) \prod_{i=1}^{j} \varphi^{(\boldsymbol{\alpha})_i}\big(\varphi_j(q_{ik}^{(\alpha_i)}) q_{ij}^{(\alpha_i)}\big) \prod_{i=j+1}^{k-1} \varphi^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_i}(q_{ik}^{(\alpha_i)}) t^{(\boldsymbol{\alpha}+\boldsymbol{\varepsilon}_j)_k}, \end{aligned}$$

which is exactly $(*)$. Hence we have shown (B) and constructed a crossed product algebra $R * \mathbb{Z}^k = R^{(k)}$ for $k = 1, \ldots, n$ from $(R, \boldsymbol{\varphi}, \boldsymbol{q})$.

Let us put $R_{\boldsymbol{\varphi},\boldsymbol{q}} := R^{(n)} = \oplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} Rt_{\boldsymbol{\alpha}}$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ and $t_{\boldsymbol{\alpha}} = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$. Since $\psi_k \mid_R = \varphi_k$ for $k = 1, \ldots, n$, we have $t_i r = \varphi_i(r) t_i$. Also, we have $t_j t_i = \psi_j(t_i) t_j = q_{ij} t_i t_j$ for $1 \le i < j \le n$, and so $t_j t_i = q_{ij} t_i t_j$ for all $1 \le i, j \le n$. Hence, our $R_{\boldsymbol{\varphi},\boldsymbol{q}}$ satisfies (3.4). The uniqueness of the multiplication on $R_{\boldsymbol{\varphi},\boldsymbol{q}}$ is clear since $R \cup \{t_1^{\pm1}, \ldots, t_n^{\pm1}\}$ is a generating set of $R_{\boldsymbol{\varphi},\boldsymbol{q}}$.

Now, one can easily check that $\psi_j^{\alpha_j}(t^{(\boldsymbol{\beta})_j}) = q_{\boldsymbol{\varepsilon}_j,\boldsymbol{\beta}}^{(\alpha_j)} t^{(\boldsymbol{\beta})_j}$. So for $rt_{\boldsymbol{\alpha}}, r't_{\boldsymbol{\beta}} \in R_{\boldsymbol{\varphi},\boldsymbol{q}}$, we get

$$\begin{aligned} rt_{\boldsymbol{\alpha}} r' t_{\boldsymbol{\beta}} &= r\varphi^{\boldsymbol{\alpha}}(r') t_{\boldsymbol{\alpha}} t_{\boldsymbol{\beta}} \\ &= r\varphi^{\boldsymbol{\alpha}}(r') t^{(\boldsymbol{\alpha})_n} t_n^{\alpha_n} t^{(\boldsymbol{\beta})_n} t_n^{\beta_n} \\ &= r\varphi^{\boldsymbol{\alpha}}(r') t^{(\boldsymbol{\alpha})_n} \psi_n^{\alpha_n}(t^{(\boldsymbol{\beta})_n}) t_n^{\alpha_n+\beta_n} \\ &= r\varphi^{\boldsymbol{\alpha}}(r') t^{(\boldsymbol{\alpha})_n} q_{\boldsymbol{\varepsilon}_n,\boldsymbol{\beta}}^{(\alpha_n)} t^{(\boldsymbol{\beta})_n} t_n^{\alpha_n+\beta_n} \\ &= r\varphi^{\boldsymbol{\alpha}}(r') \varphi^{(\boldsymbol{\alpha})_n}(q_{\boldsymbol{\varepsilon}_n,\boldsymbol{\beta}}^{(\alpha_n)}) t^{(\boldsymbol{\alpha})_n} t^{(\boldsymbol{\beta})_n} t_n^{\alpha_n+\beta_n} \\ &\cdots\cdots \\ &= r\varphi^{\boldsymbol{\alpha}}(r') \varphi^{(\boldsymbol{\alpha})_n}(r_{\boldsymbol{\varepsilon}_n,\boldsymbol{\beta}}^{(\alpha_n)}) \cdots \varphi^{(\boldsymbol{\alpha})_2}(q_{\boldsymbol{\varepsilon}_2,\boldsymbol{\beta}}^{(\alpha_2)}) t_1^{\alpha_1+\beta_1} \cdots t_n^{\alpha_n+\beta_n} \\ &= r\varphi^{\boldsymbol{\alpha}}(r') q_{\boldsymbol{\alpha},\boldsymbol{\beta}} t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}. \end{aligned}$$

Conversely, for any crossed product algebra $R * \mathbb{Z}^n = (R, \mathbb{Z}^n, \tau, \sigma) = \oplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} R\overline{\boldsymbol{\alpha}}$, we take a new R-basis $\{t_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{Z}^n\}$ of $R * \mathbb{Z}^n$ where $t_{\boldsymbol{\alpha}} = \overline{\boldsymbol{\varepsilon}}_1{}^{\alpha_1} \cdots \overline{\boldsymbol{\varepsilon}}_n{}^{\alpha_n}$. We set $q_{ij} := \tau(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i)$ for $1 \leq i \leq j \leq n$, $q_{ji} := q_{ij}^{-1}$ and $\varphi_i := \sigma_{\boldsymbol{\varepsilon}_i}$. Note that $\tau(\boldsymbol{\varepsilon}_i, \boldsymbol{\varepsilon}_j) = 1$. Then one can check that the triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ is a $\mathbb{Z}^n$-grading triple:

(G1) is clear. Let $t_i := \overline{\boldsymbol{\varepsilon}}_i$ for $i = 1, \ldots, n$. Then, for $i \leq j$ and $r \in R$, we have $t_j t_i r = \varphi_j \varphi_i(r) t_j t_i = \varphi_j \varphi_i(r) q_{ij} t_i t_j$ and $t_j t_i r = q_{ij} t_i t_j r = q_{ij} \varphi_i \varphi_j(r) t_i t_j$. Hence, $\varphi_j \varphi_i(r) q_{ij} = q_{ij} \varphi_i \varphi_j(r)$, i.e., (G2) holds. For $i \leq j \leq k$, we have $t_k t_j t_i = t_k q_{ij} t_i t_j = \varphi_k(q_{ij}) q_{ik} t_i t_k t_j = \varphi_k(q_{ij}) q_{ik} \varphi_i(q_{jk}) t_i t_j t_k$ and $t_k t_j t_i = q_{jk} t_j t_k t_i = q_{jk} \varphi_j(q_{ik}) t_j t_i t_k = q_{jk} \varphi_j(q_{ik}) q_{ij} t_i t_j t_k$. Hence, $\varphi_k(q_{ij}) q_{ik} \varphi_i(q_{jk}) = q_{jk} \varphi_j(q_{ik}) q_{ij}$, i.e., (G3) holds.

Finally, it is clear that $R * \mathbb{Z}^n = \oplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} R t_{\boldsymbol{\alpha}}$ satisfies (3.4). Therefore, we obtain $R * \mathbb{Z}^n = R_{\boldsymbol{\varphi}, \boldsymbol{q}}$. $\square$

Thus the following is clear:

Corollary 3.7. *Let $(D, \boldsymbol{\varphi}, \boldsymbol{q})$ be a division $\mathbb{Z}^n$-grading triple. Then, $D_{\boldsymbol{\varphi}, \boldsymbol{q}}$ is a division $\mathbb{Z}^n$-graded algebra. Conversely, for any division $\mathbb{Z}^n$-graded algebra A, there exists a division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$ such that $A = D_{\boldsymbol{\varphi}, \boldsymbol{q}}$.*

Remark. What we have shown in Theorem 3.3 can be written in the following way:

Let $B := \{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n\}$ and $C := \{(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i) \mid 1 \leq i < j \leq n\}$. Suppose that maps

$$\sigma : B \longrightarrow \mathrm{Aut}_F(R) \quad \text{and} \quad \tau : C \longrightarrow U(R)$$

satisfy

(a) $\quad \sigma_{\boldsymbol{\varepsilon}_j} \sigma_{\boldsymbol{\varepsilon}_i} = \mathrm{I}\big(\tau(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i)\big) \sigma_{\boldsymbol{\varepsilon}_i} \sigma_{\boldsymbol{\varepsilon}_j}$ and

(b) $\quad \sigma_{\boldsymbol{\varepsilon}_k}\big(\tau(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i)\big) \tau(\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_i) \sigma_{\boldsymbol{\varepsilon}_i}\big(\tau(\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_j)\big) = \tau(\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_j) \sigma_{\boldsymbol{\varepsilon}_j}\big(\tau(\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_i)\big) \tau(\boldsymbol{\varepsilon}_j, \boldsymbol{\varepsilon}_i)$

for all $1 \leq i < j < k \leq n$. Then there exist unique action $\tilde{\sigma} : \mathbb{Z}^n \longrightarrow \mathrm{Aut}_F(R)$ and twisting $\tilde{\tau} : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow U(R)$ such that $\tilde{\sigma}\mid_B = \sigma$, $\tilde{\tau}\mid_C = \tau$ and

(c) $\quad \tilde{\tau}(\alpha_1 \boldsymbol{\varepsilon}_1 + \cdots + \alpha_i \boldsymbol{\varepsilon}_i, \alpha_j \boldsymbol{\varepsilon}_j + \cdots + \alpha_n \boldsymbol{\varepsilon}_n) = 1 \quad$ for all $1 \leq i \leq j \leq n$.

Conversely, for any crossed product algebra $R * \mathbb{Z}^n$, we can use the diagonal basis change so that the action and twisting satisfy (a), (b) and (c).

In a certain situation, the condition (G3) of a $\mathbb{Z}^n$-grading triple is not needed. We use the notation $[a, b] = aba^{-1}b^{-1}$ for $a, b \in U(R)$.

Lemma 3.8. *Let R be a unital associative algebra over F, $\boldsymbol{\varphi} = \big(I(d_1), \ldots, I(d_n)\big)$ an n-tuple of inner automorphisms φ_i of R for some $d_1, \ldots, d_n \in U(R)$ and $\boldsymbol{q} = (q_{ij})$ an $n \times n$ matrix over F. Suppose that a triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ satisfies (G1) and (G2). Then, $(R, \boldsymbol{\varphi}, \boldsymbol{q})$ is a $\mathbb{Z}^n$-grading triple.*

Proof. We only need to check (G3). By (G1) and (G2), we have, for all $1 \leq i, j \leq n$, $I(d_j) I(d_i) = I(q_{ij}) I(d_i) I(d_j)$. So for all $r \in R$, $d_j d_i r d_i^{-1} d_j^{-1} = q_{ij} d_i d_j r d_j^{-1} d_i^{-1} q_{ji}$ and hence $rd_i^{-1}d_j^{-1}q_{ij}d_id_j = d_i^{-1}d_j^{-1}q_{ij}d_id_jr$, i.e., $d_i^{-1}d_j^{-1}q_{ij}d_id_j =: c_{ij}$ is in the centre of R. Note that $c_{ji}^{-1} = c_{ij}$. Thus we have

$$q_{ij} = c_{ij}[d_j, d_i].$$

Using this identity, we get (G3): for all $1 \le i < j < k \le n$,

$$\begin{aligned} &q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{kj})q_{ki} \\ =&c_{jk}[d_k, d_j]d_jc_{ik}[d_k, d_i]d_j^{-1}c_{ij}[d_j, d_i]d_ic_{kj}[d_j, d_k]d_i^{-1}c_{ki}[d_i, d_k] \\ =&d_kc_{ij}[d_j, d_i]d_k^{-1} = \varphi_k(q_{ij}). \quad \square \end{aligned}$$

By this lemma, if R is a finite dimensional central simple associative algebra, the defining identities of a $\mathbb{Z}^n$-grading triple are just (G1) and (G2).

Remark 3.9. (1) For a $\mathbb{Z}^n$-grading triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$, if $\boldsymbol{\varphi} = \mathbf{1} := (\mathrm{id}, \dots, \mathrm{id})$, then the crossed product algebra $R_{\mathbf{1},\boldsymbol{q}}$ has the trivial action by Theorem 3.3. So, $R_{\mathbf{1},\boldsymbol{q}} = R^t[\mathbb{Z}^n]$ is a twisted group algebra.

(2) For a $\mathbb{Z}^n$-grading triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$, if $\boldsymbol{q} = \mathbf{1}_n = \mathbf{1} := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$, then a crossed product algebra $R_{\boldsymbol{\varphi},\mathbf{1}}$ has the trivial twisting by Theorem 3.3. So, $R_{\boldsymbol{\varphi},\mathbf{1}} = R\mathbb{Z}^n$ is a skew group algebra.

(3) By (G2), $(R, \boldsymbol{\varphi}, \mathbf{1})$ is a $\mathbb{Z}^n$-grading triple if and only if

$$(*) \qquad \varphi_j\varphi_i = \varphi_i\varphi_j \quad \text{for all } i, j.$$

Finally, we give some examples.

Example. (1) Let $F_{\boldsymbol{q}}$ be an arbitrary quantum torus and R an arbitrary associative algebra. Then it is easy to see that $R \otimes_F F_{\boldsymbol{q}}$ is a predivision $\mathbb{Z}^n$-graded associative algebra (division $\mathbb{Z}^n$-graded if R is a division algebra) and is isomorphic to $R_{\mathbf{1},\boldsymbol{q}}$. Note also if R is a field, then this example becomes a quantum torus over R. Conversely, for a division $\mathbb{Z}^n$-grading triple $(D, \boldsymbol{\varphi}, \boldsymbol{q})$, if $\boldsymbol{\varphi} = \mathbf{1}$, then $I(q_{ij}) = \mathrm{id}$ for all q_{ij}, by (G2). Hence q_{ij} is in the centre of D, say K, and we can show that $D_{\mathbf{1},\boldsymbol{q}} \cong D \otimes_K K_{\boldsymbol{q}}$. Therefore, $D_{\boldsymbol{\varphi},\boldsymbol{q}}$ is a tensor product with D and some quantum torus if and only if $\boldsymbol{\varphi} = \mathbf{1}$.

(2) Let $Q = \langle \boldsymbol{i}, \boldsymbol{j} \rangle$ be a quaternion algebra over a field, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the standard generators, $\boldsymbol{\varphi} = \boldsymbol{\varphi}_3 = \big(\mathrm{I}(\boldsymbol{i}), \mathrm{I}(\boldsymbol{j}), \mathrm{I}(\boldsymbol{ij})\big)$ and $\mathbf{1} = \mathbf{1}_3$. Then one can easily check $(*)$ in Remark 3.9(3), and hence $Q_{\boldsymbol{\varphi},\mathbf{1}}$ is a predivision $\mathbb{Z}^3$-graded associative algebra.

(3) Let $K = \mathbb{Q}(\zeta_5)$ be a cyclotomic extension of $\mathbb{Q}$ (the field of rational numbers) where $\zeta := \zeta_5$ is a primitive 5th root of unity, and φ the automorphism of K defined by $\varphi(\zeta) = \zeta^2$. Let $\boldsymbol{\varphi} = (\varphi, \varphi^2, \varphi^3)$ and

$$\boldsymbol{q} = \begin{pmatrix} 1 & \zeta & \zeta^2 \\ \zeta^{-1} & 1 & \zeta^{-1} \\ \zeta^3 & \zeta & 1 \end{pmatrix}.$$

Then one can easily check that $(K, \boldsymbol{\varphi}, \boldsymbol{q})$ is a division $\mathbb{Z}^3$-grading triple, and hence $K_{\boldsymbol{\varphi},\boldsymbol{q}}$ is a division $\mathbb{Z}^3$-graded associative algebra over $\mathbb{Q}$.

(4) Let $\mathbb{H} = \langle \boldsymbol{i}, \boldsymbol{j} \rangle$ be Hamilton's quaternion over $\mathbb{R}$ (the field of real numbers), i.e., the unique quaternion division algebra over $\mathbb{R}$. Put $\boldsymbol{k} := \boldsymbol{i}\boldsymbol{j}$. Let $\boldsymbol{\varphi} = \big(\mathrm{I}(d_1), \mathrm{I}(d_2), \mathrm{I}(d_3)\big)$ where $d_1 = 1 + \boldsymbol{i}$, $d_2 = 1 + \boldsymbol{j}$ and $d_3 = 1 + \boldsymbol{k}$. We put $q_{ij} = 2[d_j, d_i]$ for $1 \le i < j \le 3$, $q_{ji} = q_{ij}^{-1}$ and $q_{ii} = 1$. Then, $(\mathbb{H}, \boldsymbol{\varphi}, \boldsymbol{q})$ satisfies (G1) and (G2), and

$$\boldsymbol{q} = \begin{pmatrix} 1 & 1 - \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} & 1 - \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} \\ (1 - \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})^{-1} & 1 & 1 - \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} \\ (1 - \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})^{-1} & (1 - \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})^{-1} & 1 \end{pmatrix}.$$

By Lemma 3.8, this is a division $\mathbb{Z}^3$-grading triple and hence $\mathbb{H}_{\boldsymbol{\varphi},\boldsymbol{q}}$ is a division $\mathbb{Z}^3$-graded associative algebra over $\mathbb{R}$.

§ 4 Conclusion

By 1.8, Example 2.8(c), Example 2.10, Proposition 2.13, Theorem 3.3 and Corollary 3.7, one can summarize our results as follows:

Corollary. (i) *Any predivision (resp. division) $A_l\mathbb{Z}^n$-graded Lie algebra over F for $l \ge 3$ is an $A_l\mathbb{Z}^n$-cover of $psl_{l+1}(R_{\boldsymbol{\varphi},\boldsymbol{q}})$ for some (resp. division) $\mathbb{Z}^n$-grading triple $(R, \boldsymbol{\varphi}, \boldsymbol{q})$. Conversely, any $psl_{l+1}(R_{\boldsymbol{\varphi},\boldsymbol{q}})$ for $l \ge 1$ is a predivision (resp. division) $A_l\mathbb{Z}^n$-graded Lie algebra.*

(ii) *Any predivision (resp. division) $\Delta\mathbb{Z}^n$-graded Lie algebra over F for $\Delta = D$ or E is a $\Delta\mathbb{Z}^n$-cover of $\mathfrak{g} \otimes_F K[z_1^{\pm}, \dots, z_n^{\pm}]$ where $\mathfrak{g}$ is a finite dimensional split simple Lie algebra over F of type D or E and K is a unital commutative associative algebra over F (resp. K is a field extension of F). Conversely, for any finite dimensional split simple Lie algebra $\mathfrak{g}$ over F of any type Δ, $\mathfrak{g} \otimes_F K[z_1^{\pm}, \dots, z_n^{\pm}]$ is a predivision (resp. division) $\Delta\mathbb{Z}^n$-graded Lie algebra.*

References

[1] B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. vol. **126**, no. 603, Amer. Math. Soc., Providence, RI, 1997.

[2] B. Allison and Y. Gao, *The root system and the core of an extended affine Lie algebra*, (to appear).

[3] S. Berman and R.V. Moody, *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy*, Invent. Math. **108** (1992), 323–347.
[4] S. Berman, Y. Gao and Y. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras*, J. Funct. Anal. **135** (1996), 339–389.
[5] G. Benkart and E. Zelmanov, *Lie algebras graded by finite root systems and intersection matrix algebras*, Invent. Math. **126** (1996), 1–45.
[6] R.V. Moody and A. Pianzola, *Lie algebras with triangular decomposition*, John Wiley, New York, 1995.
[7] E. Neher, *Lie algebras graded by 3-graded root systems*, Amer. J. Math. **118** (1996), 439–491.
[8] D. S. Passman, *Infinite crossed products*, Academic press, San Diego, 1989.
[9] Y. Yoshii, *The coordinate algebra of extended affine Lie algebras of type A_1*, (to be submitted).

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