STOKES' THEOREM. *Let A be a vector field in xyz-space. Let S be an oriented surface with boundary curve C, which is a simple closed curve. If C is positively oriented relative to S, then*

$$\int_C A \cdot dr = \int_S rotA \cdot dS.$$

We prove this for a special case where S is the rectangular determined by the vectors $a\boldsymbol{i}$ and $b\boldsymbol{j}$ in the plane $z = c$. (We use $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ as the fundamental vectors in xyz-space, and $a, b, c > 0$). Let

$$D = \{(u, v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$$

be the square region, and parametrize S by

$$S : r(u, v) = (au, bv, c)$$

on D. Then the boundary C of S is the union of

$$\begin{aligned} C_1 &: r_1(t) = (at, 0, c), \\ C_2 &: r_2(t) = (a, bt, c), \\ C_3 &: r_3(t) = (a - at, b, c), \\ C_4 &: r_4(t) = (0, b - bt, c), \end{aligned}$$

for $0 \le t \le 1$. Then, $C = C_1 + C_2 + C_3 + C_4$ is positively oriented relative to S since

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (a, 0, 0) \times (0, b, 0) = ab\boldsymbol{k}.$$

Let

$$A = (A_1, A_2, A_3).$$

Note that $r_1'(t) = (a, 0, 0)$, $r_2'(t) = (0, b, 0)$, $r_3'(t) = (-a, 0, 0)$ and $r_4'(t) = (0, -b, 0)$. Hence,

$$\begin{aligned} \int_C A \cdot dr &= \int_{C_1} A \cdot dr + \int_{C_2} A \cdot dr + \int_{C_3} A \cdot dr + \int_{C_4} A \cdot dr \\ &= \int_0^1 aA_1(r_1(t))\ dt + \int_0^1 bA_2(r_2(t))\ dt - \int_0^1 aA_1(r_3(t))\ dt - \int_0^1 bA_2(r_4(t))\ dt \\ &= \int_0^1 aA_1(at, 0, c)\ dt + \int_0^1 bA_2(a, bt, c)\ dt - \int_0^1 aA_1(a - at, b, c)\ dt - \int_0^1 bA_2(0, b - bt, c)\ dt \\ &= \int_0^a A_1(x, 0, c)\ dx + \int_0^b A_2(a, y, c)\ dy - \int_0^a A_1(x, b, c)\ dx - \int_0^b A_2(0, y, c)\ dy \\ &= -\int_0^a \Big(\int_0^b \frac{\partial A_1}{\partial y}(t, y, c)\ dy \Big)\ dx + \int_0^b \Big(\int_0^a \frac{\partial A_2}{\partial x}(x, t, c)\ dx \Big)\ dy \quad \text{(Fundamental Theorem of Calculus)} \\ &= \int_0^b \int_0^a \Big(\frac{\partial A_2}{\partial x}(x, y, c) - \frac{\partial A_1}{\partial y}(x, y, c) \Big)\ dxdy \end{aligned}$$

Now, we compute $\int_S rotA \cdot dS$. Recall that

$$rotA = \Big(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}, \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \Big).$$

Thus

$$\int_S rotA \cdot dS = \iint_D \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}, \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\big(r(u,v)\big) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right)\, dudv$$

$$= \int_0^1 \int_0^1 ab\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)(au, bv, c)\, dudv$$

$$= \int_0^b \int_0^a \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)(x, y, c)\, dxdy.$$

Therefore, we have shown $\int_C A \cdot dr = \int_S rotA \cdot dS$. $\square$

EXERCISE 1. Let $A = (2x - y, -yz^2, -y^2z)$, S the half sphere of radius a centered at the origin, and C the boundary of S. Prove the Stokes' Theorem for this situation.

The following theorem, so-called Green's Theorem, is a corollary of Stokes Theorem.

GREEN'S THEOREM. *Let $A = (A_1, A_2)$ be a vector field in xy-plane. Let C be a simple closed curve with counterclockwise orientation. Let D be the region surrounded by C. Then*

$$\int_C A \cdot dr = \iint_D \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\, dxdy.$$

PROOF. Embed everything in xyz-space, and parametrize D by x and y, say $r(x, y)$, so that $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ has the same direction as $\boldsymbol{k}$. Applying Stokes' Theorem for $(A_1, A_2, 0)$ and the surface D with boundary C in xyz-space, we get this theorem. $\square$

Green's Theorem is crucial on a theory of vector fields in a plane. Also, Green's Theorem is used to prove the Cauchy's Integral Theorem in line integrals of complex functions. Namely, if $f(z)$ is a differentiable function in a simply connected region R and C is any closed curve contained in R, then $\int_C f(z)dz = 0$.

EXERCISE 2. Let $A = (-y, x)$ be a vector field in xy-plane. Let C be the circle of radius a centered at the origin. Prove Green's Theorem in this situation.

EXERCISE 3. Let $A = (xy + y^2, x^2)$ be a vector field in xy-plane. Let D be the region enclosed by $y = x$ and $y = x^2$. Prove Green's Theorem in this situation.

EXERCISE 4. (1) Let $A = \frac{1}{d^3}(x, y, z)$, $d = \sqrt{x^2 + y^2 + z^2}$, be a vector field in xyz-space. Let C be any closed curve which dose not pass the origin. Then show $\int_C A \cdot dr = 0$.

(2) Let $A = \frac{1}{d^3}(x, y)$, $d = \sqrt{x^2 + y^2}$, be a vector field in xy-plane. Let C be any closed curve which dose not pass the origin. Then show $\int_C A \cdot dr = 0$.

REMARK 1. The vector field A in Exercise 4 is conservative and one of the potentials is given by $f = -\frac{1}{d}$. (Check $\nabla f = A$ by yourself!) Note that the vector field $A = (x, y, z)$ or $A = (x, y)$ is also conservative. Find a potential!

EXERCISE 5. Let $A = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ be a vector field in xy-plane. Let C be any closed curve which dose not pass the origin. Prove the following two statements.

(i) If C does not enclose the origin, then $\int_C A \cdot dr = 0$.

(ii) If C does enclose the origin with positive orientation, then $\int_C A \cdot dr = 2\pi$.

REMARK 2. The vector field A in Exercise 5 is conservative and one of the potentials is given by $f = -\arctan(\frac{x}{y})$. (Check $\nabla f = A$ by yourself!) But why $\int_C A \cdot dr \neq 0$?