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Journal of Pure and Applied Algebra 184 (2003) 105 – 138

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Structurable tori and extended affine Lie algebras of type BC_1

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Received 13 November 2002; received in revised form 19 February 2003
Communicated by R. Parimala

Abstract

Structurable n-tori are nonassociative algebras with involution that generalize the quantum n-tori with involution that occur as coordinate structures of extended affine Lie algebras. We show that the core of an extended affine Lie algebra of type BC_1 and nullity n is a central extension of the Kantor Lie algebra obtained from a structurable n-torus over $\mathbb{C}$. With this result as motivation, we prove general properties of structurable n-tori and show that they divide naturally into three classes. We classify tori in one of the three classes in general, and we classify tori in the other classes when $n=2$. It turns out that all structurable 2-tori are obtained from hermitian forms over quantum 2-tori with involution.

MSC: Primary: 17B60, 17A60; secondary: 17B65

1. Introduction

Extended affine Lie algebras (or EALAs for short) are complex Lie algebras that are higher nullity generalizations of affine Kac-Moody Lie algebras. Since EALAs were introduced by Høegh-Krohn and Torresani in [12], a detailed structure theory has been

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1 Research was supported by a grant from NSERC.
2 Research was supported by a PIMS Postdoctoral Fellowship (2001) and an NSERC Postdoctoral Fellowship (2002).

0022-4049/03/$ - see front matter
doi:10.1016/S0022-4049(03)00068-9

developed for these algebras [9,10,5,7,23]. Any EALA $\mathscr{L}$ contains an ideal called the core of $\mathscr{L}$, and the quotient $\mathscr{K}$ of this ideal by its centre is called the centreless core of $\mathscr{L}$. To a large extent the structure of the centreless core of an EALA determines the structure of the EALA itself (at least with an additional natural assumption of tameness). For example in the case of an affine Kac-Moody Lie algebra $\mathscr{L}$, $\mathscr{K}$ is a (possibly twisted) loop algebra and $\mathscr{L}$ is obtained from $\mathscr{K}$ by double extension.

The centreless core of an EALA is graded by a finite root system (possibly nonreduced), whose type is called the type of the EALA. So far the structure theory has given a complete description of the centreless core of EALAs of all types except the low rank nonreduced types BC_1 and BC_2. The basic tool that has been used is coordinatization, which involves constructing the centreless core from (in general nonassociative) algebras that coordinatize the root spaces. This approach is analogous to the approach that has been used to study finite dimensional simple Lie algebras over nonalgebraically closed fields of characteristic 0 [20,2].

This paper studies EALAs of type BC_1 and their coordinate algebras. The approach is again borrowed from the finite dimensional setting. In the study of finite dimensional simple Lie algebras of restricted type BC_1 the coordinate algebras are called structurable algebras. Structurable algebras are nonassociative algebras with involution that are defined by a multilinear identity of degree 4. It turns out that any finite dimensional simple Lie algebra of type BC_1 can be constructed using a Lie algebra construction of I.L. Kantor from a finite dimensional structurable division algebra $(\mathscr{A},^-)$ whose involution $^-$ is not the identity map. Furthermore the structure of finite dimensional structurable division algebras is quite well understood [4].

It is natural then to expect that infinite dimensional analogs of finite dimensional structurable division algebras will coordinatize EALAs of type BC_1. This turns out to be the case, and the infinite dimensional algebras with involution that are used for this purpose are called structurable tori.

To define structurable tori, let Λ be a free abelian group of rank n and let F be a field of characteristic $\neq 2$ or 3. Let $(\mathscr{A},^-)$ be a structurable algebra over F. Suppose further that $\mathscr{A} = \bigoplus_{\sigma\in\Lambda} \mathscr{A}^\sigma$ is Λ-graded as an algebra and that the involution $^-$ preserves the grading. $(\mathscr{A},^-)$ is called a *structurable n-torus* (or a structurable torus for short) if the following 3 properties hold:

(a) $\dim_F(\mathscr{A}^\sigma) \leq 1$ for all $\sigma\in\Lambda$;
(b) for each nonzero homogeneous element $x\in\mathscr{A}$ there exists $y\in\mathscr{A}$ so that $xy=yx=1$ and $[L_x,L_y]=[R_x,R_y]=0$;
(c) The support $\{\sigma\in\Lambda \mid \mathscr{A}^\sigma\neq 0\}$ of $\mathscr{A}$ generates the group Λ.

(Sections 3 and 4 contain more information about this definition and about the notion of invertibility occurring in (b).)

The best known example of a structurable n-torus is a quantum n-torus with involution. A quantum n-torus is an associative but noncommutative generalization of the algebra of Laurent polynomials in n-variables $t_1,\dots,t_n$ [17,9]. The grading is obtained by defining the degree of t_i to be σ_i, where $\{\sigma_1,\dots,\sigma_n\}$ is a basis for Λ, and the involution is assumed to be graded. Quantum tori with involution are classified in [25].

Nonassociative examples of structurable tori include Jordan tori with the identity map as the involution. Jordan tori are classified in [23], but they are not of primary interest here since they coordinatize EALAs of type A_1 rather than BC_1. For the purposes of this paper, the most important example of a nonassociative structurable torus is the structurable torus determined by a hermitian form on a free module over a quantum torus with involution. This example is introduced in Section 4.

Returning to the study of EALAs, we show in Section 5 of this paper that, as suggested above, the centreless core of an EALA of type BC_1 and nullity n can be constructed using the Kantor construction from a structurable n-torus $(\mathscr{A}, {}^-)$ over $\mathbb{C}$ that satisfies ${}^- \neq \mathrm{id}$. This effectively reduces the study of tame EALAs of type BC_1 to the study of structurable tori over $\mathbb{C}$.

In Sections 6–8, we develop the general properties of structurable tori. To briefly discuss some of the results that we obtain, assume that $(\mathscr{A}, {}^-)$ is a structurable n-torus over F and suppose that ${}^- \neq \mathrm{id}$. (Assuming that ${}^- \neq \mathrm{id}$ simply eliminates Jordan tori from the discussion.) Let $\mathscr{A}_-$ and $\mathscr{A}_+$ denote the space of skew-hermitian and hermitian elements of $\mathscr{A}$ relative to ${}^-$.

In Section 6 we deduce some simple preliminary results about $(\mathscr{A}, {}^-)$, and then in Section 7 we obtain properties of the support sets S, S_- and S_+ in Λ of the spaces $\mathscr{A}$, $\mathscr{A}_-$ and $\mathscr{A}_+$ respectively. These properties of the support sets are crucial in later sections and they are sufficient to give a classification of structurable 1-tori at this point. There is only one such torus with nonidentity involution, namely $(F[t^{\pm 1}], \natural)$ where $t^\natural = -t$. This corresponds to the fact that there is exactly one affine Kac-Moody Lie algebras of type BC_1, namely the Kac-Moody algebra constructed from the generalized Cartan matrix $A_2^{(2)}$ [14, Section 6.3].

In Section 8 we study the decomposition

$$\mathscr{A} = \mathscr{E} \oplus \mathscr{W}, \tag{1.1}$$

where $\mathscr{E}$ is the subalgebra of $\mathscr{A}$ generated by $\mathscr{A}_-$ and $\mathscr{W}$ is the unique graded complement of $\mathscr{E}$. The counterpart of this decomposition has played an important role in the finite dimensional structure theory [1,21]. One remarkable fact that emerges from our analysis of the decomposition (1.1) is the fact that $\mathscr{A}$ is a free module of *finite rank* over the centre $\mathscr{Z}$ of $(\mathscr{A}, {}^-)$. This fact will undoubtedly play a key role in future investigations, since it brings with it the finite dimensional theory as a tool.

In general we have $\mathscr{E}\mathscr{W} + \mathscr{W}\mathscr{E} \subset \mathscr{W}$, but we do not have always have $\mathscr{W}\mathscr{W} \subset \mathscr{E}$ as one might expect. In fact our study naturally divides into 3 cases:

I. $\mathscr{E} = \mathscr{A}$.
II. $\mathscr{E} \neq \mathscr{A}$ and $\mathscr{W}\mathscr{W} \subset \mathscr{E}$.
III. $\mathscr{E} \neq \mathscr{A}$ and $\mathscr{W}\mathscr{W} \not\subset \mathscr{E}$.

We say that $(\mathscr{A}, {}^-)$ has class I, II, or III accordingly. It turns out the properties of structurable tori in the 3 classes are quite different. For example we see that $\mathscr{E}$ is in general nonassociative if $(\mathscr{A}, {}^-)$ is of class I, $\mathscr{E}$ is associative in class II and $\mathscr{E}$ is associative and commutative in class III.

In Sections 6–8, we do not actually use the assumption that the grading group Λ is a finitely generated free abelian group. In Section 9, we exploit this assumption to obtain very detailed information about the structure of tori in each of the three classes. This information is sufficient to classify structurable n-tori in Class I if $n=2$, in Class II for general n, and in class III for $n=2$. Putting these results together we obtain a classification of structurable 2-tori. It turns out that all structurable 2-tori with $^- \neq \mathrm{id}$ can be constructed from a hermitian form over a quantum 2-torus with involution (as described in Section 4). Combining this with the work in Section 5 on EALAs, we have a classification of the centreless cores of all EALAs of type BC_1 and nullity 2.

In subsequent work we plan to use the results of this paper to further investigate structurable n-tori for $n \geqslant 3$. Because of the results of this paper, the tori in class II are well understood. In Class I, new examples occur that are infinite dimensional analogs of the tensor product of two composition algebras. We expect that the methods of this paper will be powerful enough to show that such tensor products are the only nonassociative examples in Class I. In class III, there are also new examples that occur, but the path to the classification here is not as clear.

Throughout this paper we assume that F is a field of characteristic $\neq 2$ or 3. All algebras (except Lie algebras) are assumed to be unital.

2. Structurable algebras

In this section, we recall the basic definitions, notation and facts that we will need for structurable algebras.

Let $(\mathscr{A}, ^-)$ be an algebra with involution over F. This means that $\mathscr{A}$ is an algebra over F and $^-$ is an antiautomorphism of $\mathscr{A}$ of period 2. For $x, y \in \mathscr{A}$, we define $V_{x,y} \in \mathrm{End}_F\, \mathscr{A}$ by

$$V_{x,y}(z) := (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$$

for $z \in \mathscr{A}$, and we set

$$\{x, y, z\} := V_{x,y}(z).$$

Put $T_x = V_{x,1}$ for $x \in \mathscr{A}$. Then $T_x(z) = xz + zx - z\bar{x}$. We say $(\mathscr{A}, ^-)$ is a *structurable algebra* if

$$[T_z, V_{x,y}] = V_{T_z x,y} - V_{x,T_{\bar{z}} y}$$

for $x, y, z \in \mathscr{A}$, where $[\cdot,\cdot]$ is the commutator.

Example 2.1. (a) Any alternative algebra (and in particular any associative algebra) with involution is a structurable algebra [19, p. 411].

(b) Suppose that $(\mathscr{A}, ^-)$ is a commutative algebra with involution equal to the identity map. Then $(\mathscr{A}, ^-)$ is a structurable algebra if and only if $\mathscr{A}$ is a Jordan algebra [1, Section 1].

(c) Suppose that $(\mathscr{E}, ^-)$ is an associative algebra with involution. Let $\mathscr{W}$ be a left $\mathscr{E}$-module with action denoted by $(e, w) \mapsto e \circ w$. Suppose that $\chi : \mathscr{W} \times \mathscr{W} \to \mathscr{E}$ is a hermitian form over $(\mathscr{E},^{-})$. That is

$$\chi(e\circ w_1,w_2)=e\chi(w_1,w_2),\qquad \chi(w_1,e\circ w_2)=\chi(w_1,w_2)\bar{e}$$

and

$$\chi(w_1,w_2)=\overline{\chi(w_2,w_1)}$$

for $w_1,w_2\in\mathscr{W}$ and $e\in\mathscr{E}$. Let $\mathscr{A}=\mathscr{E}\oplus\mathscr{W}$ with product and involution defined respectively by

$$(e_1+w_1)(e_2+w_2)=e_1e_2+\chi(w_2,w_1)+\overline{e_1}\circ w_2+e_2\circ w_1\quad\text{and}\quad\overline{e+w}=\bar{e}+w$$

for $e_1,e_2,e\in\mathscr{E}$ and $w_1,w_2,w\in\mathscr{W}$. The algebra with involution $(\mathscr{A},^{-})$ is a structurable algebra called the *structurable algebra associated with the hermitian form* χ. (See [1, Section 8].)

(d) There is a classical special case of example (c). Namely, suppose that $(\mathscr{E},^{-})$, $\mathscr{W}$ and χ are as in (c). Suppose further that $\mathscr{W}$ is a free module of rank 1 over $\mathscr{E}$ with basis w, and that $\mu:=\chi(w,w)$ is in the centre of $\mathscr{E}$. Let $(\mathscr{A},^{-})=(\mathscr{E}\oplus(\mathscr{E}\circ w),^{-})$ be the structurable algebra associated with χ. Then the multiplication on $(\mathscr{A},^{-})$ is given by

$$(e_1+e_2\circ w)(e_3+e_4\circ w)=e_1e_3+e_4\bar{e}_2\mu+(\bar{e}_1e_4+e_3e_2)\circ w$$

for $e_1,e_2,e_3,e_4\in\mathscr{E}$. Thus $\mathscr{A}$ is the algebra obtained by the *Cayley-Dickson process* from $(\mathscr{E},^{-})$ [18, p. 45]. (However the involution $^{-}$ on $\mathscr{A}$ is not the involution usually considered in this context.) It is easy to check (and well known) that $\mathscr{A}$ is associative if and only if $\mathscr{E}$ is commutative. Also, $\mathscr{A}$ is alternative if and only if $\{e\in\mathscr{E}\mid\bar{e}=e\}$ is contained in the centre of $\mathscr{E}$.

Assume now that $(\mathscr{A},^{-})$ is a structurable algebra.

For $\varepsilon=\pm$, we set

$$\mathscr{A}_\varepsilon=\{x\in\mathscr{A}\mid\bar{x}=\varepsilon x\}.$$

(Here $\pm$ is a convenient abbreviation for ±1.) In that case we have $\mathscr{A}=\mathscr{A}_+\oplus\mathscr{A}_-$. We call an element of $\mathscr{A}_+$ (resp. $\mathscr{A}_-$) *hermitian* (resp. *skew*).

We use the notation $(x,y,z):=(xy)z-x(yz)$ for the associator in $\mathscr{A}$ and $[x,y]=xy-yx$ for the commutator in $\mathscr{A}$. It is known [1, Proposition 1] that

$$(s,x,y)=-(x,s,y)=(x,y,s)\tag{2.2}$$

for $x,y\in\mathscr{A}$ and $s\in\mathscr{A}_-$, and

$$(a,b,c)-(c,a,b)=(b,a,c)-(c,b,a)\tag{2.3}$$

for $a,b,c\in\mathscr{A}_+$.

If $x\in\mathscr{A}$, we define U_x, L_x and R_x in $\mathrm{End}_F(\mathscr{A})$ by

$$U_x(y)=\{x,y,x\}=2(x\bar{y})x-(x\bar{x})y,$$

$L_xy=xy$ and $R_xy=yx$. Then, we have

$$U_{sx}=-L_sU_xL_s.\tag{2.4}$$

[6, Proposition 11.3] for $x,y\in\mathscr{A}$ and $s\in\mathscr{A}_-$.

For $x,y\in\mathscr{A}$, we define $D_{x,y}\in\mathrm{End}_F(\mathscr{A})$ by

$$D_{x,y}z=\tfrac{1}{3}\,[[x,y]+[\bar{x},\bar{y}],z]+(z,y,x)-(\bar{z},\bar{x},\bar{y}).$$

Then $D_{x,y}$ is a derivation of $(\mathscr{A},^-)$ (that is a derivation of $\mathscr{A}$ that commutes with $^-$) and we have

$$D_{x,y}=-D_{y,x},\quad D_{\bar{x},\bar{y}}=D_{x,y}\quad\text{and}\quad D_{xy,z}+D_{zx,y}+D_{yz,x}=0 \tag{2.5}$$

for $x,y,z\in\mathscr{A}$ [1, Section 3]. Also, if $a,b\in\mathscr{A}_+$, then by [1, Lemma 7] we have the following simple formula for $D_{a,b}$:

$$D_{a,b}=-\tfrac{1}{3}\,L_{[a,b]}+[L_a,L_b]. \tag{2.6}$$

Structurable algebras have been studied primarily because of their use in the construction of Lie algebras. Indeed, if $(\mathscr{A},^-)$ is a structurable algebra, the *Kantor Lie algebra* $K(\mathscr{A},^-)$ is constructed as follows [15,16,1]. Let $\mathscr{N}=(\mathscr{A},\mathscr{A}_-)$, $V_{\mathscr{A},\mathscr{A}}=\mathrm{span}_F\{V_{x,y}\mid x,y\in\mathscr{A}\}$ and $\mathscr{N}^\sim$ be a vector space copy of $\mathscr{N}$. Then $K(\mathscr{A},^-)=\mathscr{N}^\sim\oplus V_{\mathscr{A},\mathscr{A}}\oplus\mathscr{N}$ is a Lie algebra with product $[\,,]$ defined by

$$\begin{aligned}
&[V_{x,y},(z,s)]=(V_{x,y}(z),\psi(sy,x)),\\
&[V_{x,y},(z,s)^\sim]=(-V_{y,x}(z),-\psi(sx,y))^\sim,\\
&[(x,r),(y,s)]=(0,\psi(x,y)),\\
&[(x,r)^\sim,(y,s)^\sim]=(0,\psi(x,y))^\sim,\\
&[(x,r),(y,s)^\sim]=-(sx,0)^\sim+V_{x,y}+L_rL_s+(ry,0),
\end{aligned} \tag{2.7}$$

for $x,y,z\in\mathscr{A}$ and $s,r\in\mathscr{A}_-$, where

$$\psi(x,y)=x\bar{y}-y\bar{x}.$$

If $\mathscr{A}$ is a Jordan algebra and $^-$ is the identity map, $K(\mathscr{A},^-)$ is the classical Tits-Kantor-Koecher Lie algebra constructed from the Jordan algebra $\mathscr{A}$.

There is a notion of invertibility called conjugate invertibility in a structurable algebra $(\mathscr{A},^-)$. Indeed, if $x,y\in\mathscr{A}$ and $V_{x,y}=\mathrm{id}$, we say that x is *conjugate invertible* in $(\mathscr{A},^-)$ with *conjugate inverse* y. In that case the element y is unique [6, Lemma 6.1(ii)] and denoted by $\hat{x}$. If x is conjugate invertible, then so is $\hat{x}$ and we have $\hat{\hat{x}}=x$ [6, Lemma 6.1(iii)]. Also, we have

$$x \text{ is conjugate invertible } \Rightarrow U_x \text{ is invertible} \tag{2.8}$$

for $x\in\mathscr{A}$ [6, Proposition 5.1]. If x is conjugate invertible then the elements $(x,0)$ and $(\hat{x},0)^\sim$ generate a subalgebra of the Kantor Lie algebra $K(\mathscr{A},^-)$ that is isomorphic to $\mathrm{sl}_2(F)$. Indeed that fact was the motivation for the introduction of the conjugate inverse in the first place.

Conjugate inversion is very well-behaved for skew elements:

Lemma 2.9 ([6, Proposition 11.1]). *For* $s\in\mathscr{A}_-$,

$$s \text{ is conjugate invertible} \Leftrightarrow L_s \text{ is invertible} \Leftrightarrow R_s \text{ is invertible}.$$

Moreover, in this case, $\hat{s}\in\mathscr{A}_-$, $\hat{s}=-L_s^{-1}1$, $s\hat{s}=\hat{s}s=-1$ *and* $(\hat{s},s,x)=0$ *for* $x\in\mathscr{A}$.

Lemma 2.10 ([3, Proposition 2.6]). *If $x \in \mathscr{A}$ and $s \in \mathscr{A}_-$ are conjugate invertible, then the element $\psi(x, \{x, sx, x\})$ is conjugate invertible.*

Remark 2.11. (a) There is a standard notion of invertibility for alternative algebras (see for example [18, p. 38]). In an alternative algebra with involution $(\mathscr{A}, ^-)$ an element x is conjugate invertible if and only if $x\bar{x}$ is invertible, in which case $\hat{x} = (x\bar{x})^{-1}x$ [3, Theorem 3.4]. In particular, if x is invertible then x is conjugate invertible and $\hat{x} = \bar{x}^{-1}$.

(b) There is also a standard notion of invertibility in Jordan algebras (see [13, Section I.11]). In a Jordan algebra with identity involution an element x is conjugate invertible if and only if x is invertible, in which case $\hat{x} = x^{-1}$ [6, Section 7].

(c) Suppose that $(\mathscr{A}, ^-) = (\mathscr{E} \oplus \mathscr{W}, ^-)$ is the structurable algebra associated with a hermitian form $\chi : \mathscr{W} \times \mathscr{W} \to \mathscr{E}$ as in Example 2.1(c). Let $x = e + w \in \mathscr{A}$, where $e \in \mathscr{E}$ and $w \in \mathscr{W}$. Put $g = e\bar{e} - \chi(w, w) \in \mathscr{E}$. If g is invertible in $\mathscr{E}$, then one checks directly that x is conjugate invertible with conjugate inverse $\hat{x} = g^{-1}e - g^{-1} \circ w$.

3. Finely Λ-graded structurable algebras

Throughout the section, assume that Λ is an abelian group. As preparation for the introduction of structurable tori, we consider finely Λ-graded structurable algebras. We see that in such algebras conjugate invertibility of homogeneous elements has a very simple characterization.

We begin by recalling some terminology for Λ-graded algebras. A *Λ-graded algebra* is an algebra $\mathscr{A}$ together with a vector space decomposition

$$\mathscr{A} = \bigoplus_{\sigma \in \Lambda} \mathscr{A}^\sigma$$

so that $\mathscr{A}^\sigma \mathscr{A}^\tau \subset \mathscr{A}^{\sigma+\tau}$ for $\sigma, \tau \in \Lambda$. In that case, the space $\mathscr{A}^\sigma$ is called the *homogeneous space* of degree σ, and an element x of $\mathscr{A}^\sigma$ is said to be a *homogeneous element* of degree σ. Also, we let

$$\operatorname{supp} \mathscr{A} := \{\sigma \in \Lambda \mid \mathscr{A}^\sigma \neq 0\}$$

and call $\operatorname{supp} \mathscr{A}$ the *support* of $\mathscr{A}$. We denote the subgroup of Λ generated by $\operatorname{supp} \mathscr{A}$ by $\langle \operatorname{supp} \mathscr{A} \rangle$. Finally we say that the Λ-grading of $\mathscr{A}$ is *fine* if $\dim_F(\mathscr{A}^\sigma) \leqslant 1$ for all $\sigma \in \Lambda$. In that case, one has $\dim_F(\mathscr{A}^\sigma) = 1$ for all $\sigma \in \operatorname{supp} \mathscr{A}$.

A *Λ-graded algebra with involution* is a pair $(\mathscr{A}, ^-)$ such that $\mathscr{A}$ is a Λ-graded algebra and $^-$ is an involution of $\mathscr{A}$ that preserves the Λ-grading. As the name suggests, a *Λ-graded structurable algebra* is a Λ-graded algebra with involution that is also a structurable algebra.

Proposition 3.1. *Suppose that $(\mathscr{A}, ^-)$ is a Λ-graded structurable algebra so that the Λ-grading on $\mathscr{A}$ is fine. Let $x \in \mathscr{A}^\sigma$ be a homogeneous element of $\mathscr{A}$, where $\sigma \in \Lambda$. Then, $\bar{x} = \varepsilon x$, where $\varepsilon = \pm 1$. Also, the following statements are equivalent:*

(a) *x is conjugate invertible*
(b) *There exists $y \in \mathscr{A}$ so that*

$$xy = yx = 1 \quad \text{and} \quad [L_x, L_y] = [R_x, R_y] = 0. \tag{3.2}$$

Moreover, if statements (a) *and* (b) *hold, then the element y in* (b) *is unique and satisfies $y = \varepsilon\hat{x} \in \mathscr{A}^{-\sigma}$ and $\bar{y} = \varepsilon y$.*

Proof. Since the homogeneous spaces are 1-dimensional and stabilized by $^-$, we can choose $\varepsilon = \pm 1$ so that $\bar{x} = \varepsilon x$.

To prove the equivalence of (a) and (b), suppose first that (a) holds. So x is conjugate invertible. Then, from the uniqueness of the conjugate inverse, it follows that $\hat{x}$ is homogeneous of degree $-\sigma$. Hence $\hat{x}$ is either hermitian or skew. But from Lemma 2.9, we know that x is skew if and only if $\hat{x}$ is skew. So $\bar{\hat{x}} = \varepsilon\hat{x}$. Now let $y = \varepsilon\hat{x}$ in which case $\bar{y} = \varepsilon y$. Then we have $\varepsilon q = V_{x,y}q = (x\bar{y})q + (q\bar{y})x - (q\bar{x})y = \varepsilon((xy)q + (qy)x - (qx)y)$ and therefore

$$q = (xy)q + (qy)x - (qx)y \tag{3.3}$$

for $q \in \mathscr{A}$. Putting $q = 1$ in this equation gives $yx = 1$ and hence, applying the involution, $xy = 1$. But then (3.3) tells us that $[R_x, R_y] = 0$ and so, conjugating by the involution, $[L_x, L_y] = 0$. Thus (b) holds.

Conversely, suppose that (b) holds. Then there exists an element $y \in \mathscr{A}$ satisfying (3.2). Replacing y by its component in $\mathscr{A}^{-\sigma}$, we can assume that $y \in \mathscr{A}^{-\sigma}$. Then, since y is homogeneous, y is skew or hermitian. Next we claim that y is skew if and only if x is skew. Indeed suppose first that y is skew. Then, $U_1 = U_{yx} = -L_y U_x L_y$ by (2.4). But 1 is conjugate invertible and so U_1 is invertible. Therefore L_y is invertible and so, by Lemma 2.9, $x = L_y^{-1}1$ is skew. The reverse argument is similar and so we have the claim. Therefore $\bar{y} = \varepsilon y$. So for $p \in \mathscr{A}$ we have $V_{x,y}p = (x\bar{y})p + (p\bar{y})x - (p\bar{x})y = \varepsilon((xy)p + (py)x - (px)y) = \varepsilon(p + [R_x, R_y]p) = \varepsilon p$. Hence x is conjugate invertible and $y = \varepsilon\hat{x}$. Consequently (a) holds.

Finally, suppose that (a) and (b) hold. We have just seen that there is a choice of y so that (3.2) holds, $y = \varepsilon\hat{x} \in \mathscr{A}^{-\sigma}$ and $\bar{y} = \varepsilon y$. Hence it remains only to show that the element y satisfying (3.2) is unique. Therefore it is enough to show that if $v \in \mathscr{A}$ satisfies $xv = vx = 0$ and $[L_x, L_v] = [R_x, R_v] = 0$, then $v = 0$. Indeed for such a v we have $U_x v = 2(x\bar{v})x - (x\bar{x})v = 2\varepsilon(\overline{vx})x - \varepsilon(xx)v = -\varepsilon(xx)v = -\varepsilon(xv)x = 0$. But since x is conjugate invertible, U_x is invertible and so $v = 0$. $\square$

Remark 3.4. Suppose that $(\mathscr{A}, ^-)$ and x satisfy the hypotheses of Proposition 3.1. If $(\mathscr{A}, ^-)$ is an alternative algebra with involution or if $(\mathscr{A}, ^-)$ is a Jordan algebra with identity involution, then statement (b) is equivalent to the statement that x is invertible in the alternative or Jordan algebra $\mathscr{A}$ respectively. Moreover, in that case, $y = x^{-1}$. (Compare this with Remarks 2.11(a) and (b).)

4. Structurable tori

In this section we introduce the main objects of study in this paper—structurable tori. We assume that Λ is an abelian group.

As background, recall that an *associative* (*resp. alternative*) (*resp. Jordan*) Λ*-torus* is an associative (resp. alternative) (resp. Jordan) Λ-graded algebra $\mathscr{A}$ such that the Λ-grading of $\mathscr{A}$ is fine, each nonzero homogeneous element of $\mathscr{A}$ is invertible and $\langle \operatorname{supp} \mathscr{A} \rangle = \Lambda$. We now introduce the analogous notion for structurable algebras.

Definition 4.1. A *structurable* Λ*-torus* is a Λ-graded structurable algebra $(\mathscr{A}, ^-)$ over F that satisfies the following three conditions:

(a) the Λ-grading of $\mathscr{A}$ is fine;
(b) each nonzero homogeneous element x of $\mathscr{A}$ is conjugate invertible (or equivalently each such x satisfies condition (b) in Proposition 3.1);
(c) $\langle \operatorname{supp} \mathscr{A} \rangle = \Lambda$.

In the study of tori, the term Λ-torus is often abbreviated as n*-torus* if Λ is a free abelian group of rank $n \geqslant 1$. If n is understood from the context, n-torus is often further abbreviated as torus. In particular a structurable Λ-torus is called a *structurable* n*-torus* or *structurable torus* if Λ is a free abelian group of rank $n \geqslant 1$.

We now consider some examples of structurable Λ-tori.

Example 4.2. As the name suggests, an *associative* (*resp. alternative*) Λ*-torus with involution* is a pair $(\mathscr{A}, ^-)$ where $\mathscr{A}$ is an associative (resp. alternative) Λ-torus and $^-$ is an involution of $\mathscr{A}$ that preserves the Λ-grading. By Remark 3.4, any associative (resp. alternative) Λ-torus with involution is a structurable Λ-torus. Conversely (again by Remark 3.4), if $(\mathscr{A}, ^-)$ is a Λ-structurable torus and $\mathscr{A}$ is associative (resp. alternative), then $(\mathscr{A}, ^-)$ is an associative (resp. alternative) Λ-torus with involution. Associative n-tori with involution are precisely the quantum n-tori with involution (see Proposition 4.5 below). Alternative n-tori (but not yet alternative n-tori with involution) have been classified in [10].

Example 4.3. If $\mathscr{A}$ is a Jordan Λ-torus and $^-$ is the identity map, then $(\mathscr{A}, ^-)$ is a structurable Λ-torus. Conversely, if $(\mathscr{A}, ^-)$ is structurable Λ-torus and $^-$ is the identity map, then $\mathscr{A}$ is a Jordan Λ-torus. Jordan n-tori were classified in [23].

Suppose for the rest of the section that Λ is free abelian of rank $n \geqslant 1$. In that case associative Λ-tori with involution can be described concretely:

Example 4.4. Let $\mathbf{q} = (q_{ij})$ be an *elementary quantum* $n \times n$*-matrix*. This means that $\mathbf{q}$ is symmetric, $q_{ij} = \pm 1$ and $q_{ii} = 1$ for all $1 \leqslant i, j \leqslant n$. Also let $\mathbf{r} = (r_1, \dots, r_n)$ be an *elementary* n*-vector*. This means that $r_i = \pm 1$ for all $1 \leqslant i \leqslant n$. Suppose that $B = \{\sigma_1, \dots, \sigma_n\}$ is a basis for Λ.

Let $F_{\mathbf{q}}$ be the associative algebra over F with generators t_i and t_i^{-1} for $1 \leqslant i \leqslant n$, subject to the relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad \text{and} \quad t_j t_i = q_{ij} t_i t_j \quad \text{for } 1 \leqslant i, j \leqslant n.$$

Let $j_{\mathbf{r}} : F_{\mathbf{q}} \to F_{\mathbf{q}}$ be the involution on $F_{\mathbf{q}}$ satisfying

$$(t_i)^{j_{\mathbf{r}}} = r_i t_i \quad \text{for } 1 \leqslant i \leqslant n.$$

We define a grading on $F_{\mathbf{q}}$ by setting

$$t_B^\sigma = t_1^{k_1} \cdots t_n^{k_n} \quad \text{and} \quad (F_{\mathbf{q}})^\sigma = F t_B^\sigma$$

for $\sigma = k_1\sigma_1 + \cdots + k_n\sigma_n \in \Lambda$. Notice that the element t_B^σ and the Λ-grading just defined depend on the choice of the basis B. We call this Λ-grading of $F_{\mathbf{q}}$ the *toral* Λ*-grading determined by* B. We call the Λ-graded algebra with involution $(F_{\mathbf{q}}, j_{\mathbf{r}})$ a *quantum* Λ*-torus with involution*.

Since the involution $j_{\mathbf{r}}$ preserves the Λ-grading, we have

$$\Lambda = S_{\Lambda,B}(\mathbf{q},\mathbf{r}) \cup S_{\Lambda,B}(\mathbf{q},\mathbf{r})^c,$$

where

$$S_{\Lambda,B}(\mathbf{q},\mathbf{r}) = \{\sigma \in \Lambda \mid (t_B^\sigma)^{j_{\mathbf{r}}} = t_B^\sigma\} \quad \text{and}$$

$$S_{\Lambda,B}(\mathbf{q},\mathbf{r})^c = \Lambda \setminus S_{\Lambda,B}(\mathbf{q},\mathbf{r}) = \{\sigma \in \Lambda \mid (t_B^\sigma)^{j_{\mathbf{r}}} = -t_B^\sigma\}.$$

We have explicitly

$$S_{\Lambda,B}(\mathbf{q},\mathbf{r}) = \left\{ \sum k_i\sigma_i \in \Lambda \;\middle|\; \sum_{i\in I_{\mathbf{r}}} k_i + \sum_{(i,j)\in J_{\mathbf{q}}} k_i k_j \equiv 0 \pmod 2 \right\},$$

where $I_{\mathbf{r}} = \{i \mid 1 \leqslant i \leqslant n,\ r_i = -1\}$ and $J_{\mathbf{q}} = \{(i,j) \mid 1 \leqslant i < j \leqslant n,\ q_{ij} = -1\}$ [5, Section III.3]. Note that $0 \in S_{\Lambda,B}(\mathbf{q},\mathbf{r})$ and $S_{\Lambda,B}(\mathbf{q},\mathbf{r})$ is a union of cosets of 2Λ in Λ.

Any quantum Λ-torus with involution is an associative Λ-torus with involution. Conversely, one easily shows the following [25, Proposition 2.1]:

Proposition 4.5. *Suppose that* $(\mathscr{E}, ^-)$ *is an associative* Λ*-torus with involution, where* Λ *is free abelian of rank* n*. Let* B *be a basis for* Λ*. Then there exist unique* $\mathbf{q}$ *and* $\mathbf{r}$ *so that* $(\mathscr{E}, ^-) \cong (F_{\mathbf{q}}, j_{\mathbf{r}})$ *as* Λ*-graded algebras with involution, where* $F_{\mathbf{q}}$ *has the toral* Λ*-grading determined by* B.

Our final example of a structurable n-torus will be the most important example for our purposes.

Example 4.6. As a starting point for this construction we assume that we are given:

(i) An elementary quantum $n \times n$-matrix $\mathbf{q}$ and an elementary n-vector $\mathbf{r}$.
(ii) A subgroup M of Λ so that $2\Lambda \subset M \subset \Lambda$ together with a basis B of M.
(iii) Elements $\rho_1, \ldots, \rho_m$ of Λ that are distinct and nonzero modulo M so that

$$\Lambda = \langle M, \rho_1, \ldots, \rho_m \rangle \quad \text{and} \quad 2\rho_1, \ldots, 2\rho_m \in S_{M,B}(\mathbf{q}, \mathbf{r}).$$

(iv) Nonzero scalars $a_1, \ldots, a_m \in F$.

Starting from these ingredients, we can construct a structurable torus as the structurable algebra of a hermitian form over $(F_{\mathbf{q}}, j_{\mathbf{r}})$.

First of all we have the quantum M-torus with involution $(F_{\mathbf{q}}, j_{\mathbf{r}})$, where $F_{\mathbf{q}}$ has the toral M-grading determined by the basis B. Note that the assumption in (iii) that $2\rho_1, \ldots, 2\rho_m \in S_{M,B}(\mathbf{q}, \mathbf{r})$ is equivalent to the assumption that

$$(t_B^{2\rho_i})^{j_{\mathbf{r}}} = t_B^{2\rho_i}$$

for $i = 1, \ldots, m$.

Let $\mathscr{W}$ be the free $F_{\mathbf{q}}$-module of rank m with $F_{\mathbf{q}}$-basis $w_1, \ldots, w_m$. So

$$\mathscr{W} = F_{\mathbf{q}} \circ w_1 \oplus \cdots \oplus F_{\mathbf{q}} \circ w_m.$$

Let $\chi : \mathscr{W} \times \mathscr{W} \to F_{\mathbf{q}}$ be the unique hermitian form on $\mathscr{W}$ over $(F_{\mathbf{q}}, j_{\mathbf{r}})$ so that

$$\chi(w_i, w_j) = \delta_{ij} a_i t_B^{2\rho_i}$$

for $1 \leqslant i, j \leqslant m$. We denote this hermitian form χ by

$$\langle\langle a_1 t_B^{2\rho_1}, \ldots, a_m t_B^{2\rho_m} \rangle\rangle.$$

Now let

$$(\mathscr{A}, {}^-) = (F_{\mathbf{q}} \oplus \mathscr{W}, {}^-)$$

be the structurable algebra associated with the hermitian form χ (see Example 2.1(c)). The algebra $\mathscr{A}$ has a unique Λ-grading so that

$$\deg t_B^{\sigma} = \sigma \quad \text{for } \sigma \in M \quad \text{and} \quad \deg w_i = \rho_i \quad \text{for } i = 1, \ldots, m.$$

In other words this Λ-grading of $\mathscr{A}$ extends the toral M-grading on $F_{\mathbf{q}}$ determined by B and satisfies $\deg w_i = \rho_i$ for $i = 1, \ldots, m$. $(\mathscr{A}, {}^-)$ is a Λ-graded structurable algebra and the support of $\mathscr{A}$ is given by

$$\operatorname{supp} \mathscr{A} = \bigcup_{i=0}^{m} (M + \rho_i), \tag{4.7}$$

where for convenience we have set $\rho_0 = 0$.

To see that $(\mathscr{A}, {}^-)$ is a Λ-structurable torus, we must check conditions (a), (b) and (c) in Definition 4.1. Indeed (a) is clear and (c) follows from (4.7). Finally, for (b), if x is a nonzero homogeneous element of $\mathscr{A}$, then $x \in F_{\mathbf{q}}^{\sigma}$ or $x \in F_{\mathbf{q}}^{\sigma} \circ w_i$ for some $\sigma \in M$ and $i \in \{1,\dots,m\}$, in which case x is conjugate invertible by Remark 2.11(c). Therefore $(\mathscr{A},^-)$ is a structurable Λ-torus.

We call $(\mathscr{A},^-)$ the *structurable Λ-torus of the hermitian form*

$$\langle\langle a_1 t_B^{2\rho_1},\dots,a_m t_B^{2\rho_m}\rangle\rangle$$

on the free module $\bigoplus_{i=1}^m F_{\mathbf{q}} \circ w_i$ over the quantum M-torus with involution $(F_{\mathbf{q}}, j_{\mathbf{r}})$.

Remark 4.8. If $\mathbf{q}=\mathbf{1}$ (all entries are 1) and $\mathbf{r}=(1,\dots,1)$, then the involution $^-$ on $\mathscr{A}$ is the identity. Hence $\mathscr{A}$ is a Jordan algebra. In fact $\mathscr{A}$ is a Jordan Λ-torus. It is the Jordan Λ-torus constructed from the semilattice $\bigcup_{i=0}^m (M+\rho_i)$ [23, Example 5.2].

Remark 4.9. Suppose in Example 4.9 we let $\rho_i'=\rho_i+\mu_i$, where $\mu_i \in M$, for $i=1,\dots,m$. Then one checks that the structurable Λ-torus of the hermitian form $\langle\langle a_1 t_B^{2\rho_1},\dots,a_m t_B^{2\rho_m}\rangle\rangle$ is isomorphic to the structurable Λ-torus of the hermitian form $\langle\langle a_1' t_B^{2\rho_1'},\dots,a_m' t_B^{2\rho_m'}\rangle\rangle$ for some nonzero $a_1',\dots,a_m' \in F$. So in this sense the construction depends only on the cosets of M in Λ represented by $\rho_1,\dots,\rho_m$.

5. The core of an EALA of type A_1 or BC_1

We were led to the study of structurable tori because of their connection with extended affine Lie algebras (EALAs). In this section, we describe this connection. For the sake of brevity, we will not repeat the basic definitions for EALAs here. Instead we will describe the properties of EALAs that we need. The interested reader can consult [9,5] or [7] for more background on this topic.

Throughout the section we assume that F is the field $\mathbb{C}$ of complex numbers.

Let $\mathscr{L}$ be an extended affine Lie algebra of type A_1 or BC_1 and nullity $n \geqslant 1$ over $\mathbb{C}$. Suppose that $\mathscr{C}$ is the core of $\mathscr{L}$. By definition, $\mathscr{C}$ is the subalgebra of $\mathscr{L}$ generated by the root spaces of $\mathscr{L}$ corresponding to nonisotropic roots. Let

$$\mathscr{K}=\mathscr{C}/\mathscr{Z}(\mathscr{C}),$$

where $\mathscr{Z}(\mathscr{C})$ is the centre of $\mathscr{C}$. Since $\mathscr{C}$ is perfect, $\mathscr{K}$ has trivial centre. We call $\mathscr{K}$ the *centreless core* of $\mathscr{L}$.

The Lie algebra $\mathscr{K}$ has two gradings that are important to us. First of all,

$$\mathscr{K}=\bigoplus_{\sigma\in\Lambda}\mathscr{K}^{\sigma}$$

is a Λ-graded Lie algebra, where Λ is a free abelian group of rank n and $\operatorname{supp}\mathscr{K}$ generates Λ [7, Proposition 1.28(c)].

Secondly, since $\mathscr{K}$ has type A_1 or BC_1, $\mathscr{K}$ contains an sl_2-triplet (e,h,f) so that

$$\mathrm{ad}_{\mathscr{K}}(h) \text{ is diagonalizable with eigenvalues contained in } \{-4,-2,0,2,4\} \tag{5.1}$$

[7, Proposition 1.28(d)]. Moreover, (e,h,f) can be chosen so that

$$e \in \mathscr{K}^0, \quad h \in \mathscr{K}^0 \quad \text{and} \quad f \in \mathscr{K}^0$$

[7, Proposition 1.28(i)]. We set

$$\mathfrak{g} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f \cong \mathrm{sl}_2(\mathbb{C})$$

and let $\mathfrak{h} = \mathbb{C}h$ be the corresponding Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$ and, for $\alpha \in \mathfrak{h}^*$, let $\mathscr{K}_\alpha = \{x \in \mathscr{K} \mid [k,x] = \alpha(k)x \text{ for } k \in \mathfrak{h}\}$. Let μ be the element of $\mathfrak{h}^*$ so that $\mu(h) = 2$, and let

$$\Delta = \{-2\mu, -\mu, \mu, 2\mu\} \subset \mathfrak{h}^*.$$

Then Δ is the root system of type BC_1 and, by (5.1), we have

$$\mathscr{K} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathscr{K}_\alpha = \mathscr{K}_{-2\mu} \oplus \mathscr{K}_{-\mu} \oplus \mathscr{K}_0 \oplus \mathscr{K}_\mu \oplus \mathscr{K}_{2\mu}.$$

Moreover, it is shown in [7, Proposition 1.28(d)] that

$$\text{the spaces } \mathscr{K}_\alpha,\ \alpha \in \Delta, \text{ generate the algebra } \mathscr{K}. \tag{5.2}$$

(A Lie algebra $\mathscr{K}$ containing an sl_2-triplet (e,h,f) so that (5.1) and (5.2) hold is called a BC_1-*graded Lie algebra*. See [8].)

Next, since $h \in \mathscr{K}^0$, the spaces $\mathscr{K}^\sigma$ are $\mathrm{ad}_{\mathscr{K}}(h)$-stable. Thus our two gradings are compatible:

$$\mathscr{K} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \bigoplus_{\sigma \in \Lambda} \mathscr{K}_\alpha^\sigma, \tag{5.3}$$

where $\mathscr{K}_\alpha^\sigma = \mathscr{K}_\alpha \cap \mathscr{K}^\sigma$ for $\alpha \in \mathfrak{h}^*, \sigma \in \Lambda$. Moreover, $[\mathscr{K}_\alpha^\sigma, \mathscr{K}_\beta^\tau] \subset \mathscr{K}_{\alpha+\beta}^{\sigma+\tau}$ for all $\alpha, \beta \in \mathfrak{h}^*$ and $\sigma, \tau \in \Lambda$. Also

$$\dim_{\mathbb{C}} \mathscr{K}_\alpha^\sigma \leqslant 1 \quad \text{for all } \alpha \in \Delta \text{ and } \sigma \in \Lambda \tag{5.4}$$

[7, Proposition 1.28(f)].

Finally we need the following lemma:

Lemma 5.5. *Suppose that* $0 \neq x \in \mathscr{K}_\mu^\sigma$ *where* $\sigma \in \Lambda$. *Then there exists* $y \in \mathscr{K}_{-\mu}^{-\sigma}$ *such that* $[x,y] = h$.

Proof. Our argument follows [24, Example 2.8(b)]. By [7, (1.18), (1.21), (1.26) and (1.27)] as well as [5, pp. 7–9 and Proposition I.2.1], there exists $y \in \mathscr{K}_{-\mu}^{-\sigma}$ such that $\mathrm{ad}([x,y])|_{\mathscr{K}_{p\mu}} = 2p\,\mathrm{id}|_{\mathscr{K}_{p\mu}}$ for $p \in \{-2,-1,0,1,2\}$. Hence $[x,y] - h$ is in the centre of $\mathscr{K}$, and so $[x,y] = h$. $\square$

Theorem 5.6. *Suppose $\mathscr{K}$ is the centreless core of an EALA $\mathscr{L}$ of type* A_1 *or* BC_1 *with nullity $n \geqslant 1$. Then $\mathscr{K}$ is isomorphic to the Kantor Lie algebra $K(\mathscr{A},^-)$ for some structurable n-torus $(\mathscr{A},^-)$.*

Proof. Since $\mathscr{K}$ is a BC_1-graded Lie algebra with trivial centre, it follows using [2, Theorem 4] that $\mathscr{K}$ can be identified with the Kantor Lie algebra $K(\mathscr{A},^-)$ for some structurable algebra $(\mathscr{A},^-)$ (see [8, Theorem 9.20]). In fact, this can be done so that $\mathscr{K}_{-2\mu} = (0,\mathscr{A}_-)\tilde{}$, $\mathscr{K}_{-\mu} = (\mathscr{A},0)\tilde{}$, $\mathscr{K}_0 = V_{\mathscr{A},\mathscr{A}}$, $\mathscr{K}_\mu = (\mathscr{A},0)$ and $\mathscr{K}_{2\mu} = (0,\mathscr{A}_-)$ with

$$e = (2,0), \quad h = 2V_{1,1} = 2L_1 \quad \text{and} \quad f = (1,0)\tilde{}.$$

Now we have $(\mathscr{A},0) = \mathscr{K}_\mu = \bigoplus_{\sigma\in\Lambda} \mathscr{K}_\mu^\sigma$ by (5.3). Hence we may choose subspaces $\mathscr{A}^\sigma$ of $\mathscr{A}$ for $\sigma\in\Lambda$ so that

$$(\mathscr{A}^\sigma,0) = \mathscr{K}_\mu^\sigma. \tag{5.7}$$

for $\sigma\in\Lambda$. Then $\mathscr{A} = \bigoplus_{\sigma\in\Lambda} \mathscr{A}^\sigma$ with

$$\dim_{\mathbb{C}} \mathscr{A}^\sigma \leqslant 1 \tag{5.8}$$

by (5.4). Note that $(1,0) = \frac{1}{2}e \in \mathscr{K}_\mu^0$ and so $1 \in \mathscr{A}^0$.

Observe from (2.7) that

$$(\{1,\mathscr{A}^\sigma,1\},0)\tilde{} = [[(\mathscr{A}^\sigma,0),f],f] \subset [[\mathscr{K}^\sigma,\mathscr{K}^0],\mathscr{K}^0] \subset \mathscr{K}^\sigma. \tag{5.9}$$

Thus, if $x\in\mathscr{A}^\sigma$, $y\in\{1,\mathscr{A}^\tau,1\}$ and $z\in\mathscr{A}^\rho$ for $\sigma,\tau,\rho\in\Lambda$, then $(x,0)\in\mathscr{K}^\sigma$, $(y,0)\tilde{}\in\mathscr{K}^\tau$ and $(z,0)\in\mathscr{K}^\rho$, and hence $(\{x,y,z\},0) = [[(x,0),(y,0)\tilde{}],(z,0)] \in \mathscr{K}^{\sigma+\tau+\rho}$. Therefore $\{\mathscr{A}^\sigma,\{1,\mathscr{A}^\tau,1\},\mathscr{A}^\rho\} \subset \mathscr{A}^{\sigma+\tau+\rho}$ and in particular,

$$\{\mathscr{A}^\sigma,1,\mathscr{A}^\rho\} = \{\mathscr{A}^\sigma,\{1,1,1\},\mathscr{A}^\rho\} \subset \mathscr{A}^{\sigma+\rho}.$$

Hence $2x - \bar{x} = \{x,1,1\} \in \{\mathscr{A}^\sigma,1,\mathscr{A}^0\} \subset \mathscr{A}^\sigma$ for $x\in\mathscr{A}^\sigma$, and so $\overline{\mathscr{A}^\sigma} \subset \mathscr{A}^\sigma$ for $\sigma\in\Lambda$. Since $x \mapsto 2x-\bar{x}$ is a bijection on $\mathscr{A}$ ($x \mapsto \frac{2}{3}x + \frac{1}{3}\bar{x}$ is the inverse map), we get $\{\mathscr{A}^\sigma,1,1\} = \mathscr{A}^\sigma$ for $\sigma\in\Lambda$. Thus $\{\mathscr{A}^\sigma,\mathscr{A}^\tau,\mathscr{A}^\rho\} \subset \{\mathscr{A}^\sigma,\{\mathscr{A}^\tau,1,1\},\mathscr{A}^\rho\} \subset \mathscr{A}^{\sigma+\tau+\rho}$. Using the identity

$$(x\bar{y})z = \tfrac{1}{2}(\{x,y,z\} + \{y,x,z\} + \{x,z,y\} - \{y,z,x\})$$

for $x,y,z\in\mathscr{A}$ (which is verified by expansion of the right hand side [11]), we obtain $(\mathscr{A}^\sigma\overline{\mathscr{A}^\tau})\mathscr{A}^\rho \subset \mathscr{A}^{\sigma+\tau+\rho}$ for $\sigma,\tau,\rho\in\Lambda$. Hence $\mathscr{A}^\sigma\mathscr{A}^\rho \subset \mathscr{A}^{\sigma+\rho}$ for all $\sigma,\rho\in\Lambda$. Thus $(\mathscr{A},^-)$ is a Λ-graded structurable algebra. The Λ-grading is fine by (5.8).

Next we have $\{1,\mathscr{A}^\sigma,1\} \subset \mathscr{A}^\sigma$, and since $x \mapsto \{1,x,1\}$ is a bijection on $\mathscr{A}$, we get $\{1,\mathscr{A}^\sigma,1\} = \mathscr{A}^\sigma$. Thus, by (5.9), $(\mathscr{A}^\sigma,0)\tilde{} \subset \mathscr{K}^\sigma$, and hence $(\mathscr{A}^\sigma,0)\tilde{} \subset \mathscr{K}_{-\mu}^\sigma$. Since $\dim_{\mathbb{C}} \mathscr{A}^\sigma = \dim_{\mathbb{C}} \mathscr{K}_\mu^\sigma = \dim_{\mathbb{C}} \mathscr{K}_{-\mu}^{-\sigma}$ by (5.4) and Lemma 5.5, we obtain

$$(\mathscr{A}^\sigma,0)\tilde{} = \mathscr{K}_{-\mu}^\sigma \quad \text{for all } \sigma\in\Lambda. \tag{5.10}$$

Now Λ is generated as a group by $\operatorname{supp}\mathscr{K}$. Also $\mathscr{K}$ is generated as an algebra by $\mathscr{K}_\mu$ and $\mathscr{K}_{-\mu}$ (since $K(\mathscr{A},^-)$ is generated by $(\mathscr{A},0)$ and $(\mathscr{A},0)\tilde{}$ by (2.7)). Hence Λ

is generated as a group by $\operatorname{supp}\mathscr{K}_{\mu}\cup\operatorname{supp}\mathscr{K}_{-\mu}$, where $\operatorname{supp}\mathscr{K}_{\varepsilon\mu}=\{\sigma\in\Lambda \mid \mathscr{K}^{\sigma}_{\varepsilon\mu}\neq 0\}$, for $\varepsilon=\pm$. Consequently, it follows from (5.7) and (5.10) that Λ is generated as a group by $\operatorname{supp}\mathscr{A}$.

Finally let $0\neq x\in\mathscr{A}^{\sigma}$. By Lemma 5.5 and (5.10), there exists $y\in\mathscr{A}^{-\sigma}$ such that $[(x,0),(y,0)\tilde{\ }]=\frac{1}{2}h$. Thus $V_{x,y}=L_1=\mathrm{id}$ and so x is conjugate invertible. Therefore $(\mathscr{A},^{-})$ is a structurable Λ-torus. □

Remark 5.11. Suppose that $\mathscr{L}$, $\mathscr{K}$ and $(\mathscr{A},^{-})$ are as in Theorem 5.6.

(a) $\mathscr{L}$ has type A_1 if and only if the involution $^{-}$ on $\mathscr{A}$ is the identity map (in which case $\mathscr{A}$ is a Jordan algebra).

(b) The Lie algebra $K(\mathscr{A},^{-})$ has a unique Λ-grading so that $(x_\sigma,0)$ and $(x_\sigma,0)\tilde{\ }$ are homogeneous of degree σ in $K(\mathscr{A},^{-})$ for $\sigma\in\Lambda$, $x_\sigma\in\mathscr{A}^{\sigma}$. Furthermore, the isomorphism in the proposition preserves Λ-gradings.

(c) The Lie algebra $K(\mathscr{A},^{-})$ has the natural structure of a BC_1-graded Lie algebra (using the sl_2-triplet $(2,0),2L_1,\ (1,0)\tilde{\ }$). The isomorphism in the proposition preserves this structure (the sl_2-triplets correspond).

Remark 5.12. The converse of Theorem 5.6 is true. More precisely, if $(\mathscr{A},^{-})$ is a structurable n-torus, there exists an EALA $\mathscr{L}$ of type A_1 or BC_1 and nullity n so that the centreless core of $\mathscr{L}$ is isomorphic to $K(\mathscr{A},^{-})$. Since this is a special case of a more general result proved in [26], we omit further discussion here.

6. Preliminary facts about structurable Λ-tori

With the previous section as motivation, we now begin our study of structurable tori. Since no extra effort is required we will work from now on over an arbitrary field F of characteristic $\neq 2$ or 3. Although our main interest in this paper is in the case where Λ is a free abelian group of finite rank, we won't need that assumption for some time. Thus for the next three sections we assume that Λ is an abelian group and that $(\mathscr{A},^{-})=(\bigoplus_{\sigma\in\Lambda}\mathscr{A}^{\sigma},^{-})$ is a structurable Λ-torus over F.

In this section we collect a few easy facts and definitions for $(\mathscr{A},^{-})$. First of all, we introduce the notion of inverse for nonzero homogeneous elements of $\mathscr{A}$.

Definition 6.1. Let $0\neq x\in\mathscr{A}^{\sigma}$, where $\sigma\in\operatorname{supp}\mathscr{A}$. Let x^{-1} be the unique element of $\mathscr{A}$ so that

$$xx^{-1}=x^{-1}x=1 \quad\text{and}\quad [L_x,L_{x^{-1}}]=[R_x,R_{x^{-1}}]=0$$

(see Proposition 3.1). x^{-1} is called the *inverse* of x.

Proposition 6.2. *Let $0\neq x\in\mathscr{A}^{\sigma}$, where $\sigma\in\operatorname{supp}\mathscr{A}$. Choose $\varepsilon=\pm 1$ so that $\bar{x}=\varepsilon x$. Then,*

(a) $0\neq x^{-1}\in\mathscr{A}^{-\sigma}$ *and* $\overline{x^{-1}}=\varepsilon x^{-1}$.
(b) $(x^{-1})^{-1}=x$.

(c) $x^{-1} = \varepsilon\hat{x}$.
(d) $x^2 \neq 0$.

Proof. (a), (b) and (c) follow immediately from the definition of x^{-1} and Proposition 3.1. For (d), observe that $x^{-1}x^2 = x^{-1}(xx) = x(x^{-1}x) = x$ and so $x^2 \neq 0$. □

If K is a subset of Λ, we set

$$\mathscr{A}^K = \sum_{\sigma\in K} \mathscr{A}^\sigma. \tag{6.3}$$

Then, $\mathscr{A}^K$ is a graded subspace of $\mathscr{A}$ that is stabilized by $^-$. Furthermore, if K is a subgroup of Λ and we denote the restriction of $^-$ to $\mathscr{A}^K$ by $^-$, then $(\mathscr{A}^K, ^-)$ is a K-graded structurable algebra.

The following propositions are clear:

Proposition 6.4. *Suppose that K is a subgroup of Λ that is generated by a subset of* $\operatorname{supp}\mathscr{A}$. *Then, $(\mathscr{A}^K, ^-)$ is a structurable K-torus.*

Proposition 6.5. *$\mathscr{A}$ is graded left (resp. right) simple. That is any homogeneous left (resp. right) ideal of $\mathscr{A}$ is 0 or $\mathscr{A}$.*

We next prove an easy technical lemma. For this some terminology is convenient. Suppose that Λ_0 and Λ_1 are subsets of Λ such that Λ_0 is a subgroup of Λ and $\Lambda_0 + \Lambda_1 \subset \Lambda_1$. (We are not assuming that Λ_1 contains Λ_0 or that Λ_1 is a subgroup of Λ.) Then Λ_1 is a union of cosets of Λ_0 in Λ. Hence we can choose a set $\{\rho_i\}_{i\in I}$ of elements of Λ_1 so that elements ρ_i, $i \in I$, represent distinct cosets of Λ_0 in Λ and so that

$$\Lambda_1 = \bigcup_{i\in I} (\Lambda_0 + \rho_i).$$

In that case we call the set $\{\rho_i\}_{i\in I}$ a *cross-section* of Λ_1 relative to Λ_0.

Lemma 6.6. *Suppose that Λ_0 and Λ_1 are subsets of S such that Λ_0 is a subgroup of Λ and $\Lambda_0 + \Lambda_1 \subset \Lambda_1$. Suppose that for each $\sigma \in \Lambda_0$ and each nonzero $a \in \mathscr{A}^\sigma$, L_a is invertible. Let $0 \neq x_i \in \mathscr{A}^{\rho_i}$ for $i \in I$, where $\{\rho_i\}_{i\in I}$ is a cross-section of Λ_1 relative to Λ_0. Then each element $x \in \mathscr{A}^{\Lambda_1}$ can be expressed uniquely in the form $\sum_{i\in I} a_i x_i$, where $a_i \in \mathscr{A}^{\Lambda_0}$ for $i \in I$ and all but finitely many of the elements a_i are 0.*

Proof. To show that x can be expressed in the indicated form we can assume that x is homogeneous. Thus $x \in \mathscr{A}^\tau$, where $\tau \in \Lambda_1$. Write $\tau = \sigma + \rho_i$, where $\sigma \in \Lambda_0$ and $i \in I$. Choose $0 \neq a \in \mathscr{A}^\sigma$. Then $x = a(L_a^{-1}x)$ and $L_a^{-1}x \in L_a^{-1}\mathscr{A}^{\sigma+\rho_i} \subset \mathscr{A}^{\rho_i} = Fx_i$ as desired.

For uniqueness suppose that $\sum_{i\in I} a_i x_i = 0$, where $a_i \in \mathscr{A}^{\Lambda_0}$ for $i \in I$ and all but finitely many of the elements a_i are 0. Then $a_i x_i = 0$ for $i \in I$, since the elements ρ_i are distinct mod Λ_0. Thus $a_i = 0$ for $i \in I$. □

Next we consider the centre of $(\mathscr{A}, ^-)$. Let

$$\mathscr{Z} = \mathscr{Z}(\mathscr{A}, ^-) = \{a \in \mathscr{A}_+ \mid [a, \mathscr{A}] = (a, \mathscr{A}, \mathscr{A}) = (\mathscr{A}, a, \mathscr{A}) = (\mathscr{A}, \mathscr{A}, a) = 0\}$$

be the *centre* of $(\mathscr{A}, ^-)$, which is a homogeneous subalgebra of $\mathscr{A}$. One checks that

$$\mathscr{Z} = \{a \in \mathscr{A}_+ \mid [a, \mathscr{A}] = (a, \mathscr{A}, \mathscr{A}) = 0\}.$$

We put

$$\Gamma = \Gamma(\mathscr{A}, ^-) = \{\sigma \in S \mid \mathscr{A}^\sigma \subset \mathscr{Z}\}$$

and we have

$$\mathscr{Z} = \mathscr{A}^\Gamma.$$

Proposition 6.7. *We have*:

(a) *If* $0 \neq a \in \mathscr{A}^\sigma$, *where* $\sigma \in \Gamma$, *then* L_a *and* R_a *are invertible with* $L_a^{-1} = L_{a^{-1}}$ *and* $R_a^{-1} = R_{a^{-1}}$.
(b) Γ *is a subgroup of* Λ.
(c) $\mathscr{Z}$ *is a commutative associative* Γ-*torus.*
(d) $\Gamma + S \subset S$.
(e) $\mathscr{A}$ *is a free* $\mathscr{Z}$-*module with multiplication action and with basis* $\{x_i\}_{i \in I}$, *where* $0 \neq x_i \in \mathscr{A}^{\rho_i}$ *for* $i \in I$ *and* $\{\rho_i\}_{i \in I}$ *is a cross-section of* S *relative to* Γ.

Proof. (a) If $y \in \mathscr{A}$, we have $a(a^{-1}y) = (aa^{-1})y = y$ and $a^{-1}(ay) = (a^{-1}a)y = y$. Hence $L_a L_{a^{-1}} = L_{a^{-1}} L_a = \mathrm{id}$. Applying $^-$, we get $R_a R_{a^{-1}} = R_{a^{-1}} R_a = \mathrm{id}$.

(b) Certainly $0 \in \Gamma$ since $1 \in \mathscr{Z}$. Next let $\sigma \in \Gamma$. Choose $0 \neq a \in \mathscr{A}^\sigma$. Then, if $y \in \mathscr{A}$, we have, using (a), $[a^{-1}, ay] = a^{-1}(ay) - (ay)a^{-1} = a^{-1}(ay) - (ya)a^{-1} = y - y = 0$. Also, for $y, z \in \mathscr{A}$, $(a^{-1}, ay, z) = (a^{-1}(ay))z - a^{-1}((ay)z) = (a^{-1}(ay))z - a^{-1}(a(yz)) = yz - yz = 0$. Thus $[a^{-1}, a\mathscr{A}] = (a^{-1}, a\mathscr{A}, \mathscr{A}) = 0$. So $[a^{-1}, \mathscr{A}] = (a^{-1}, \mathscr{A}, \mathscr{A}) = 0$. But by Proposition 6.2(a), $a^{-1} \in \mathscr{A}_+$. So $a^{-1} \in \mathscr{Z}$. Hence $-\sigma \in \Gamma$. Suppose $\sigma, \tau \in \Gamma$. Choose $0 \neq a \in \mathscr{A}^\sigma$ and $0 \neq b \in \mathscr{A}^\tau$. Then $ab \neq 0$ by (a). So $\mathscr{A}^{\sigma+\tau} \subset \mathscr{Z}$. Hence $\sigma + \tau \in \Gamma$.

(c) follows from (b) and Proposition 6.4. (d) follows from (a). Finally (e) follows from (a), (d) and Lemma 6.6 (with $\Lambda_0 = \Gamma$ and $\Gamma_1 = S$). □

The group Γ is called the *central grading group of* $(\mathscr{A}, ^-)$.

7. The support sets S, S_- and S_+

Suppose again that $(\mathscr{A}, ^-)$ is a structurable Λ-torus over F, where Λ is an abelian group. Here, and in the rest of the paper, we use the notation

$$S := \operatorname{supp} \mathscr{A}$$

and

$$S_\varepsilon = S_\varepsilon(\mathscr{A}, ^-) := \{\sigma \in S \mid \mathscr{A}^\sigma \subset \mathscr{A}_\varepsilon\}$$

for $\varepsilon = \pm$. Then we have

$$S = S_+ \cup S_- \quad \text{(disjoint)}.$$

In this section, we derive some properties of S, S_- and S_+ and deduce some consequences of those properties.

Let

$$\Lambda_- = \Lambda_-(\mathscr{A}, ^-) := \langle S_- \rangle.$$

If $S_- = \emptyset$, we take $\Lambda_- = 0$. (If $S_- = \emptyset$, the statements involving S_- or Λ_- in the next proposition and its corollary are trivially true.)

Proposition 7.1. *We have*

(a) $0 \in S$, $-S = S$, $2S + S \subset S$ *and* $\langle S \rangle = \Lambda$.
(b) $0 \notin S_-$, $-S_- = S_-$, $2S_- + S_- \subset S_-$ *and* $\langle S_- \rangle = \Lambda_-$.
(c) $S + S_- \subset S$, $2S \subset S_+$, $S_- \cap 2S = \emptyset$ *and* $4S + S_- \subset S_-$.

Proof. (a) and (b): Since $1 \in \mathscr{A}^0$, we have $0 \in S$ and $0 \notin S_-$. By Proposition 6.2(a), we have $-S = S$ and $-S_- = S_-$ (also $-S_+ = S_+$). For $0 \neq x \in \mathscr{A}^\sigma$ and $0 \neq y \in \mathscr{A}^\tau$, since x is conjugate invertible and $^-$ is graded, $0 \neq U_x y = 2(x\bar{y})x - (x\bar{x})y \in \mathscr{A}^{2\sigma+\tau}$ by (2.8), and hence $2S + S \subset S$. If s and t are nonzero homogeneous skew elements, then $(st)s$ is skew (since $(s,t,s) = 0$ by (2.2)) and $(st)s \neq 0$ (by Lemma 2.9). Thus $2\sigma + \tau \in S_-$ for $\sigma, \tau \in S_-$.

(c): The first inclusion follows from the fact that L_s is invertible for a skew conjugate invertible element s (see Lemma 2.9). By Proposition 6.2(d), we have $2S \subset S_+$, and so $S_- \cap 2S = \emptyset$. Let $0 \neq s \in \mathscr{A}^\tau$ for $\tau \in S_-$. Then $0 \neq \psi(x, \{x, sx, x\}) \in \mathscr{A}^{4\sigma+\tau}$ is skew by Lemma 2.10, and so the fourth inclusion holds. □

Remark 7.2. When Λ is a free abelian group of finite rank, a subset S (resp. S_-) satisfying 7.1(a) (resp. 7.1(b)) is called a *semilattice* (resp. *translated semilattice*) in [5]. In Proposition 7.1 we have independently shown the properties of the pair (S, S_-) which were previously shown in the context of extended affine root systems of type BC_1 [5, Chapter 2].

Corollary 7.3. *We have*

(a) $2\Lambda + S \subset S$, $2\Lambda_- + S_- \subset S_-$, $\Lambda_- + S \subset S$ *and* $4\Lambda + S_- \subset S_-$.
(b) *S is a union of cosets of 2Λ in Λ including the coset 2Λ; S_- is a union of cosets of $2\Lambda_-$ in Λ_-; S is a union of cosets of Λ_- in Λ including the coset Λ_-; and S_- is a union of cosets of 4Λ in Λ.*
(c) *If $S_- \neq \emptyset$, then $4\Lambda \subset \Lambda_- \subset \Lambda$.*

Proof. (a) Since $\langle S \rangle = \Lambda$ and $-S = S$, any element of Λ can be written as $\sum_i \sigma_i$ for some $\sigma_i \in S$. Then for any $\sigma \in S$, using the inclusion $2S + S \subset S$ from Proposition 7.1 inductively, $2\sum_i \sigma_i + \sigma = \sum_i 2\sigma_i + \sigma \in S$. Hence $2\Lambda + S \subset S$. Similarly, since $\langle S_- \rangle = \Lambda_-$, using the inclusions $2S_- + S_- \subset S_-$ and $S_- + S \subset S$ from Proposition 7.1

inductively, we get $2\Lambda+S_- \subset S_-$ and $\Lambda_-+S \subset S$. Also, using the inclusion $4S+S_- \subset S_-$ from Proposition 7.1, we obtain $4\Lambda + S_- \subset S_-$.

(b) follows from (a) and $0 \in S$.

(c) Choose any $\sigma \in S_-$. Then by (a), $4\Lambda + \sigma \subset S_- \subset \Lambda_-$. Since $-\sigma \in S_-$, we get $4\Lambda + \sigma - \sigma \subset \Lambda_-$. $\square$

For technical reasons later in the paper we record the following:

Corollary 7.4. *Suppose that Λ is a free abelian group of rank n.*

(a) *There exists a basis of Λ contained in S.*
(b) *If $S_- \neq \emptyset$, there exists a basis of Λ_- contained in S_-.*
(c) *If $n = 2$, then $\Lambda_- \neq 2\Lambda$.*

Proof. (a) and (b): These follow from Propositions 7.1(a) and 7.1(b), respectively using the argument in [5, Proposition 1.11, Chapter II].

(c) Suppose that $\Lambda_- = 2\Lambda$. By (b), there exists a basis $\{\sigma_1, \sigma_2\}$ of 2Λ such that $\sigma_i \in S_-$. Let $\sigma_i' := \frac{1}{2}\sigma_i$ for $i = 1, 2$. Then $\{\sigma_1', \sigma_2'\}$ is a basis of Λ and so $\Lambda = 2\Lambda \cup (2\Lambda + \sigma_1') \cup (2\Lambda + \sigma_2') \cup (2\Lambda + \sigma_1' + \sigma_2')$. But S is a union of cosets of 2Λ in Λ and $\sigma_i' \notin S$ for $i = 1, 2$ by Proposition 7.1(c). Thus S is contained in $2\Lambda \cup (2\Lambda + \sigma_1' + \sigma_2')$ and hence S cannot generate Λ, which is a contradiction. $\square$

We close this section with the classification of structurable Λ-tori when Λ is cyclic.

Theorem 7.5. *Let $\Lambda = \langle\sigma\rangle$ be a cyclic group and let $(\mathscr{A}, ^-)$ be a structurable Λ-torus. Then*

(a) *$S = \Lambda$ and hence we may select $0 \neq x \in \mathscr{A}^\sigma$.*
(b) *$\mathscr{A}$ is generated as an algebra by x and x^{-1}.*
(c) *$\mathscr{A}$ is commutative and associative.*
(d) *Suppose $|\Lambda| = \infty$ (that is $(\mathscr{A}, ^-)$ is a structurable 1-torus). Then*

$$(\mathscr{A}, ^-) \simeq (F[t^{\pm 1}], *) \quad \text{or} \quad (F[t^{\pm 1}], \natural) \tag{7.6}$$

as Λ-graded algebras with involution, where on the right hand side $t^* = t$, $t^\natural = -t$ and $\deg(t) = \sigma$.

(e) *Suppose $|\Lambda| = m < \infty$. Then for some nonzero $a \in F$*

$$(\mathscr{A}, ^-) \simeq \begin{cases} (F[t]/I_a, *) \text{ or } (F[t]/I_a, \natural) & \text{if } m \text{ is even,} \\ (F[t]/I_a, *) & \text{if } m \text{ is odd} \end{cases} \tag{7.7}$$

as Λ-graded algebras with involution, where on the right hand side I_a is the principal ideal $(t^m - a)$, $(t + I_a)^* = t + I_a$, $(t + I_a)^\natural = -t + I_a$ and $\deg(t + I_a) = \sigma$.

Proof. (a) By Corollary 7.3(b), S is the union of cosets of 2Λ in Λ, $0\in S$ and $\langle S\rangle=\Lambda$. Hence, since Λ is cyclic, $S=\Lambda$. In particular $\sigma\in S$ and so we can choose $0\neq x\in\mathscr{A}^\sigma$.

(b) and (c): Let $\mathscr{B}$ be the subalgebra of $\mathscr{A}$ generated by x and x^{-1}. For the proof of (b) and (c) it suffices to show that $\mathscr{B}$ is associative (and therefore also commutative). Indeed if $\mathscr{B}$ is associative then we have $0\neq x^n\in\mathscr{A}^{n\sigma}$ for all $n\in\mathbb{Z}$ which implies that $\mathscr{B}=\mathscr{A}$.

To show that $\mathscr{B}$ is associative, suppose first that $S_-=\emptyset$. Then $\mathscr{A}$ is a Jordan algebra and so the algebra generated by any invertible element and its inverse is associative [13, Theorem 11, Section I.11]. So we may assume that $S_-\neq\emptyset$. Then $n\sigma\in S_-$ for some $n\in\mathbb{Z}$. But, by Corollary 7.3(a), $S_-+4\Lambda\subset S_-$. Hence we can choose n so that $-1\leqslant n\leqslant 2$. But by Proposition 7.1(c), $S_-\cap 2S=\emptyset$ and so $S_-\cap 2\Lambda=\emptyset$. So $n\neq 0$ or 2. Hence σ or $-\sigma$ is in S_-. Since $S_-=-S_-$, we have σ and $-\sigma$ in S_-. Thus x and x^{-1} are skew. But according to a theorem of Smirnov [22], a structurable algebra generated by two skew elements is associative. Consequently $\mathscr{B}$ is associative.

(d) Suppose that $|\Lambda|=\infty$. We have $\bar{x}=\varepsilon x$, where $\varepsilon=\pm 1$. Then it follows from (b) and (c) that we have an isomorphism as in (7.6) with $x^n\mapsto t^n$ for all $n\in\mathbb{Z}$.

(e) Suppose finally that $|\Lambda|=m<\infty$. Again we have $\bar{x}=\varepsilon x$, where $\varepsilon=\pm 1$. Also $0\neq x^m\in\mathscr{A}^{m\sigma}=\mathscr{A}^0$, and so $x^m=a1$ for some $0\neq a\in F$. Furthermore $a1=\overline{a1}=\overline{x^m}=(\bar{x})^m=\varepsilon^m x^m=\varepsilon^m a1$ and so $\varepsilon^m=1$. Thus $\varepsilon=1$ if m is odd. It now follows that we have an isomorphism as in (7.7) with $x^n\mapsto t^n+I_a$ for all $n\in\mathbb{Z}$. $\square$

Corollary 7.8. *Suppose that Λ is an abelian group, $(\mathscr{A},^-)$ is a structurable Λ-torus and $0\neq x\in\mathscr{A}^\sigma$, where $\sigma\in S$. Let $\mathscr{B}$ be the subalgebra of $\mathscr{A}$ that is generated by x and x^{-1}. Then, $\mathscr{B}=\mathscr{A}^{\mathbb{Z}\sigma}$ (see (6.3) for the notation) and $(\mathscr{B},^-)$ is an associative commutative $\mathbb{Z}\sigma$-torus with involution.*

Proof. $(\mathscr{A}^{\mathbb{Z}\sigma},^-)$ is a structurable $\mathbb{Z}\sigma$-torus by Proposition 6.4. The corollary follows by applying Theorem 7.5(b) and (c) to this $\mathbb{Z}\sigma$-torus. $\square$

8. The decomposition $\mathscr{A}=\mathscr{E}\oplus\mathscr{W}$

Suppose again that $(\mathscr{A},^-)$ is a structurable Λ-torus where Λ is an abelian group.

If $S_-=\emptyset$, the involution $^-$ is the identity map and hence $\mathscr{A}$ is a Jordan Λ-torus. Jordan tori are well understood. (When Λ is free of finite rank, they were classified in [23].) So in this section we assume that $S_-\neq\emptyset$.

Let

$$\mathscr{E}=\mathscr{A}^{\Lambda_-}\quad\text{and}\quad\mathscr{W}=\mathscr{A}^{\Lambda\setminus\Lambda_-}.$$

Then, by Proposition 6.4, $(\mathscr{E},^-)$ is a structurable Λ_--torus. Furthermore,

$$\mathscr{A}=\mathscr{E}\oplus\mathscr{W},$$

$$\mathscr{E}\mathscr{E}\subset\mathscr{E}\quad\mathscr{E}\mathscr{W}+\mathscr{W}\mathscr{E}\subset\mathscr{W},$$

$$\mathscr{A}_-\subset\mathscr{E}\quad\text{and}\quad\mathscr{W}\subset\mathscr{A}_+.$$

We denote the restriction of $^-$ to $\mathscr{E}$ also by $^-$.

The decomposition $\mathscr{A} = \mathscr{E} \oplus \mathscr{W}$ just described is the analog for structurable tori of the decomposition that has been used to study finite dimensional simple structurable algebras [1,21]. We now investigate this decomposition, omitting proofs when they are the same as in the finite dimensional case.

Proposition 8.1. *$\mathscr{E}$ is generated as an algebra by $\mathscr{A}_-$. Moreover* $\operatorname{supp}\mathscr{E} = \Lambda_-$.

Proof. Let $\sigma \in \Lambda_-$. Then $\sigma = \sigma_1 + \cdots + \sigma_k$, where $\sigma_i \in S_-$ with $k \geqslant 1$. Choose $0 \neq s_i \in \mathscr{A}^{\sigma_i}$ for each i. Then $L_{s_1} \cdots L_{s_{k-1}} s_k$ is a nonzero element of $\mathscr{A}^\sigma$, and so $\mathscr{A}^\sigma \neq 0$ and $\mathscr{A}^\sigma$ lies in the algebra generated by $\mathscr{A}_-$. □

Proposition 8.2. *We have $\mathscr{E} = \mathscr{A}_- \oplus \operatorname{span}_F\{st + ts \mid s, t \in \mathscr{A}_-\}$ and so $\Lambda_- = S_- \cup (S_- + S_-)$.*

Proof. The first statement is proved in [1, Lemma 14] (where $\mathscr{E}$ is defined to be the subalgebra of $\mathscr{A}$ generated by $\mathscr{A}_-$). The second follows from the first. □

Proposition 8.3. *If $e, e_1, e_2 \in \mathscr{E}$, $w \in \mathscr{W}$ and $x \in \mathscr{A}$, we have*

(a) $ew = w\bar{e}$
(b) $e_1(e_2 w) = (e_2 e_1)w$ *and* $(we_1)e_2 = w(e_2 e_1)$
(c) $e(wx) = w(\bar{e}x)$ *and* $(xw)e = (x\bar{e})w$.

Proof. See the proof of Lemma 21 of [1] (which uses the first statement of Proposition 8.2) for the proof of (a) and the first equations in (b) and (c). The other equations follow applying $^-$. □

As in [1], define an action of $\mathscr{E}$ on $\mathscr{W}$ by

$$e \circ w = \bar{e}w.$$

Also, define $\chi : \mathscr{W} \times \mathscr{W} \to \mathscr{E}$ and $\xi : \mathscr{W} \times \mathscr{W} \to \mathscr{W}$ by

$$w_2 w_1 = \chi(w_1, w_2) + \xi(w_1, w_2),$$

where $\chi(w_1, w_2) \in \mathscr{E}$ and $\xi(w_1, w_2) \in \mathscr{W}$.

Proposition 8.4. (a) *$\mathscr{W}$ is an associative $\mathscr{E}$-module*: $e_1 \circ (e_2 \circ w) = (e_1 e_2) \circ w$.

(b) *χ and ξ are $\mathscr{E}$-sesquilinear*:

$$e\chi(w_1, w_2) = \chi(e \circ w_1, w_2), \qquad \chi(w_1, w_2)e = \chi(w_1, \bar{e} \circ w_2),$$

$$e\xi(w_1, w_2) = \xi(e \circ w_1, w_2) \quad \text{and} \quad \xi(w_1, w_2)e = \xi(w_1, \bar{e} \circ w_2).$$

(c) *ξ is symmetric*: $\xi(w_1, w_2) = \overline{\xi(w_2, w_1)}$.
(d) *χ is hermitian*: $\chi(w_1, w_2) = \overline{\chi(w_2, w_1)}$.
(e) *χ is nondegenerate*: $\chi(w_0, \mathscr{W}) = 0 \Rightarrow w_0 = 0$.

Proof. (a) and (b) follow from Proposition 8.3(b) and (c). On the other hand, (c) and (d) are clear. To prove (e), we can assume $w_0 \in \mathscr{A}^\sigma$, where $\sigma \in \Lambda \setminus \Lambda_-$. Suppose $w_0 \neq 0$. Since $w_0 \mathscr{W} \subset \mathscr{W}$ and $w_0^{-1} \in \mathscr{A}^{-\sigma} \subset \mathscr{W}$, we get $1 = w_0 w_0^{-1} \in \mathscr{W}$, which is a contradiction. □

Corollary 8.5. *Suppose $\mathscr{E} \neq \mathscr{A}$.*

(a) *$(\mathscr{E}, ^-)$ is an associative Λ_--torus with involution.*
(b) *If e is a nonzero homogeneous element of $\mathscr{E}$, then L_e and R_e are invertible with $(L_e)^{-1} = L_{e^{-1}}$ and $(R_e)^{-1} = R_{e^{-1}}$.*

Proof. (a) Let $\mathfrak{a} := \{e \in \mathscr{E} \mid e \circ \mathscr{W} = 0\}$. Then $\mathfrak{a}$ is a homogeneous ideal of $\mathscr{E}$ containing $(\mathscr{E}, \mathscr{E}, \mathscr{E})$ since $\mathscr{W}$ is an associative $\mathscr{E}$-module. Further, $1 \notin \mathfrak{a}$ since $\mathscr{W} \neq 0$. Then $\mathfrak{a} = 0$ (by Proposition 6.5) and so $(\mathscr{E}, \mathscr{E}, \mathscr{E}) = 0$.

(b) It is enough to prove the statement about L_e. Since $\mathscr{E}$ is an associative Λ_--torus, $L_e|_{\mathscr{E}}$ is invertible with inverse $L_{e^{-1}}|_{\mathscr{E}}$. On the other hand, by Proposition 8.3(b), $L_e|_{\mathscr{W}}$ is invertible with inverse $L_{e^{-1}}|_{\mathscr{W}}$. □

Let $\mathscr{Z} = \mathscr{Z}(\mathscr{A}, ^-) = \bigoplus_{\sigma \in \Gamma} \mathscr{A}^\sigma$ be the centre of $(\mathscr{A}, ^-)$, where $\Gamma = \Gamma(\mathscr{A}, ^-)$ is the central grading group of $(\mathscr{A}, ^-)$.

Proposition 8.6. *Let $\mathscr{Z}(\mathscr{E}, ^-)$ be the centre of $(\mathscr{E}, ^-)$ and let $\Gamma(\mathscr{E}, ^-)$ be the central grading group of $(\mathscr{E}, ^-)$. Then $\mathscr{Z} = \mathscr{Z}(\mathscr{E}, ^-)$ and $\Gamma = \Gamma(\mathscr{E}, ^-)$.*

Proof. It suffices to show that $\mathscr{Z} = \mathscr{Z}(\mathscr{E}, ^-)$.

"$\supset$". Suppose $e \in \mathscr{Z}(\mathscr{E}, ^-)$. Then $[e, w] = 0$ for $e \in \mathscr{E}$ and $w \in \mathscr{W}$ by Proposition 8.3(a). We must show in addition that

$$(e, f, w) = (e, w, f) = (e, u, w) = 0$$

for $f \in \mathscr{E}$ and $u, w \in \mathscr{W}$. Now using Proposition 8.3, we have $(e, f, w) = (ef)w - e(fw) = (ef - fe)w = 0$ and $(e, w, f) = (ew)f - e(wf) = \bar{f}(ew) - e(\bar{f}w) = (e\bar{f} - \bar{f}e)w = 0$. Also, again using Proposition 8.3, $(e, u, w) = (eu)w - e(uw) = (ue)w - e(uw) = (uw)e - e(uw) = -[e, uw] = -[e, \chi(w, u) + \xi(w, u)] = 0$.

"$\subset$". Since $\mathscr{Z}$ is homogeneous, we have $\mathscr{Z} = (\mathscr{Z} \cap \mathscr{E}) \oplus (\mathscr{Z} \cap \mathscr{W})$. Clearly, $\mathscr{Z} \cap \mathscr{E} \subset \mathscr{Z}(\mathscr{E}, ^-)$. Thus it is enough to show that $\mathscr{Z} \cap \mathscr{W} = 0$. So let $w_0 \in \mathscr{Z} \cap \mathscr{W}$, with w_0 homogeneous. Then $w_0 w = w w_0$ for all $w \in \mathscr{W}$. Hence $\chi(w, w_0) = \chi(w_0, w)$ and so $\chi(w, w_0) \in \mathscr{E}_+$ for all $w \in \mathscr{W}$. Thus $\mathfrak{a} := \chi(\mathscr{W}, w_0)$ is a homogeneous left ideal of $\mathscr{E}$ that is contained in $\mathscr{E}_+$. Since $\mathscr{E}_+ \neq \mathscr{E}$, this implies that $\mathfrak{a} = 0$ by Proposition 6.5. So since χ is nondegenerate, $w_0 = 0$. □

Corollary 8.7. *If $\sigma \in S_-$ and $0 \neq s \in \mathscr{A}^\sigma$, then $s^2 \in \mathscr{Z}$.*

Proof. Let $\sigma, \tau \in S_-$, $0 \neq s \in \mathscr{A}^\sigma$ and $0 \neq t \in \mathscr{A}^\tau$. Then $0 \neq s(st) = s^2 t \in \mathscr{A}^{2\sigma + \tau}$. But $2\sigma + \tau \in S_-$ by Proposition 7.1(b). So $\overline{s^2 t} = -s^2 t$. Thus $-ts^2 = -s^2 t$, and so $[s^2, t] = 0$. Hence $[s^2, \mathscr{A}_-] = 0$. Thus, by the proof of Lemma 24 in [1], we have $[s^2, \mathscr{E}] = (s^2, \mathscr{E}, \mathscr{E}) = 0$, and, by Proposition 8.6, we obtain $s^2 \in \mathscr{Z}$. □

Theorem 8.8. (a) *We have*

$$8\Lambda \subset 2\Lambda_- \subset \Gamma \subset \Lambda_- \subset \Lambda. \tag{8.9}$$

(b) *If Λ is finitely generated then $\mathscr{A}$ is a free $\mathscr{Z}$-module of finite rank.*

Proof. (a) Observe that by Corollary 8.7, we have $2S_- \subset \Gamma$, and so $2\Lambda_- \subset \Gamma$. But $4\Lambda \subset \Lambda_-$ by Proposition 7.3(c), and hence $8\Lambda \subset 2\Lambda_-$. By Proposition 8.6, we have $\Gamma \subset \Lambda_-$.

(b) By Proposition 6.7(e), $\mathscr{A}$ is a free $\mathscr{Z}$-module of rank c, where c is the cardinality of a cross section of S relative to Γ. But c is less than or equal to $(\Lambda : \Gamma)$ and Λ/Γ is a finite group by (a). $\square$

It will be convenient in the next section to place structurable Λ-tori into three mutually exclusive classes.

Definition 8.10. Suppose that $(\mathscr{A}, ^-)$ is a Λ-torus with $S_- \neq \emptyset$. We say that $(\mathscr{A}, ^-)$ has *class* I, II, or III if the corresponding condition listed below holds:

I. $\mathscr{E} = \mathscr{A}$.
II. $\mathscr{E} \neq \mathscr{A}$ and $\mathscr{W}\mathscr{W} \subset \mathscr{E}$.
III. $\mathscr{E} \neq \mathscr{A}$ and $\mathscr{W}\mathscr{W} \not\subset \mathscr{E}$.

We now see that (8.9), which plays an important role in the rest of the paper, can be strengthened for each of these classes.

Proposition 8.11. *Suppose that $(\mathscr{A}, ^-)$ is of class I. Then $2\Lambda \subset \Gamma \subset \Lambda$.*

Proof. This follows from (8.9) since $\Lambda_- = \Lambda$ in this case. $\square$

Proposition 8.12. *Suppose that $(\mathscr{A}, ^-)$ is of class II. Then $2\Lambda \subset \Lambda_- \subset \Lambda$.*

Proof. It is enough to show that $2S \subset \Lambda_-$. To see this let $\sigma \in S$. If $\sigma \in S_-$, then $2\sigma \in \Lambda_-$. Hence we can assume that $\sigma \in \Lambda \setminus \Lambda_-$ and so $\mathscr{A}^\sigma \subset \mathscr{W}$. Then, using Proposition 6.2(d), we have $\mathscr{A}^{2\sigma} = \mathscr{A}^\sigma \mathscr{A}^\sigma \subset \mathscr{W}\mathscr{W} \subset \mathscr{E}$ and so $2\sigma \in \Lambda_-$. $\square$

Proposition 8.13. *Suppose that $(\mathscr{A}, ^-)$ is of class III. Choose $\sigma_0 \in S_-$ and $0 \neq s_0 \in \mathscr{A}^{\sigma_0}$. Then*:

(a) *$(\mathscr{E}, ^-)$ is a commutative associative Λ_--torus with involution.*
(b) *$\mathscr{Z} = \mathscr{Z}(\mathscr{E}, ^-) = \mathscr{E}_+$. Consequently, if x is a homogeneous element of $\mathscr{A}$, then $x^2 \in \mathscr{E} \Leftrightarrow x^2 \in \mathscr{Z}$.*
(c) *$\mathscr{E}_- = s_0\mathscr{Z}$, $s_0^2 \in \mathscr{Z}$ and $\mathscr{E} = \mathscr{Z} \oplus s_0\mathscr{Z}$.*
(d) *$S_- = \sigma_0 + \Gamma$, $2\sigma_0 \in \Gamma$ and $\Lambda_- = \Gamma \cup (\sigma_0 + \Gamma)$.*
(e) *$(\Lambda_- : \Gamma) = 2$.*
(f) *$4\Lambda \subset \Gamma \subset \Lambda_- \subset \Lambda$.*

Proof. (a) By Corollary 8.5, $\mathscr{E}$ is associative. We must show that $\mathscr{E}$ is commutative. By (2.3), we have

$$(w_1,w_2,w_3)-(w_3,w_1,w_2)=(w_2,w_1,w_3)-(w_3,w_2,w_1) \tag{8.14}$$

for $w_1,w_2,w_3\in\mathscr{W}$. Let $p_{\mathscr{E}}:\mathscr{A}\to\mathscr{E}$ be the projection onto $\mathscr{E}$ along $\mathscr{W}$. Then

$$\begin{aligned} p_{\mathscr{E}}((w_1,w_2,w_3)) &= p_{\mathscr{E}}((w_1w_2)w_3-w_1(w_2w_3))\\ &= p_{\mathscr{E}}(\xi(w_2,w_1)w_3-w_1\xi(w_3,w_2))\\ &= \chi(w_3,\xi(w_2,w_1))-\chi(\xi(w_3,w_2),w_1). \end{aligned}$$

Hence, by (8.14),

$$\begin{aligned} &\chi(w_3,\xi(w_2,w_1))-\chi(\xi(w_3,w_2),w_1)-\chi(w_2,\xi(w_1,w_3))+\chi(\xi(w_2,w_1),w_3)\\ &\quad=\chi(w_3,\xi(w_1,w_2))-\chi(\xi(w_3,w_1),w_2)-\chi(w_1,\xi(w_2,w_3))+\chi(\xi(w_1,w_2),w_3). \end{aligned}$$

Since ξ is symmetric, this yields

$$\chi(w_1,\xi(w_3,w_2))-\chi(w_2,\xi(w_3,w_1))=\chi(\xi(w_3,w_2),w_1)-\chi(\xi(w_3,w_1),w_2). \tag{8.15}$$

Now define

$$\Delta(w_1,w_2,w_3)=\chi(w_1,\xi(w_3,w_2))-\chi(w_2,\xi(w_3,w_1))\in\mathscr{E}$$

for $w_1,w_2,w_3\in\mathscr{W}$. Then (8.15) tells us that $\overline{\Delta(w_1,w_2,w_3)}=\Delta(w_1,w_2,w_3)$, and hence $\Delta(\mathscr{W},\mathscr{W},\mathscr{W})$ is contained in $\mathscr{E}_+$. Next if $e\in\mathscr{E}$, then $\chi(w_i,\xi(e\circ w_3,w_j))=\chi(w_i,\bar{e}\circ\xi(w_3,w_j))=\chi(w_i,\xi(w_3,w_j))e$. Thus

$$\Delta(w_1,w_2,e\circ w_3)=\Delta(w_1,w_2,w_3)e,$$

and so $\Delta(\mathscr{W},\mathscr{W},\mathscr{W})$ is a homogeneous right ideal of $\mathscr{E}$ that is contained in $\mathscr{E}_+$. Since $\mathscr{E}_+\neq\mathscr{E}$, we have $\Delta(\mathscr{W},\mathscr{W},\mathscr{W})=0$ (by Proposition 6.5). Therefore

$$\chi(w_1,\xi(w_3,w_2))=\chi(w_2,\xi(w_3,w_1))$$

for $w_1,w_2,w_3\in\mathscr{W}$. This equation allows us to argue as in [21, Section 4]. In fact if $e\in\mathscr{E}$, we have

$$\begin{aligned} e\chi(w_1,\xi(w_3,w_2)) &= \chi(e\circ w_1,\xi(w_3,w_2))=\chi(w_2,\xi(w_3,e\circ w_1))\\ &= \chi(w_2,\xi(w_3,w_1)\bar{e})=\chi(w_2,e\xi(w_3,w_1))\\ &= \chi(w_2,\bar{e}\circ\xi(w_3,w_1))=\chi(w_2,\xi(w_3,w_1))e\\ &= \chi(w_1,\xi(w_3,w_2))e. \end{aligned}$$

Hence $[\mathscr{E},\chi(\mathscr{W},\xi(\mathscr{W},\mathscr{W}))]=0$. But $\chi(\mathscr{W},\xi(\mathscr{W},\mathscr{W}))\neq 0$ since $\xi(\mathscr{W},\mathscr{W})\neq 0$ and χ is nondegenerate. So $\chi(\mathscr{W},\xi(\mathscr{W},\mathscr{W}))$ is a nonzero homogeneous ideal of $\mathscr{E}$. Thus by Proposition 6.5, $\chi(\mathscr{W},\xi(\mathscr{W},\mathscr{W}))=\mathscr{E}$ and therefore $[\mathscr{E},\mathscr{E}]=0$.

(b) follows from (a) and Proposition 8.6.

(c) By (b), it is enough to show that $\mathscr{E}_- = s_0\mathscr{Z}$. Clearly $s_0\mathscr{Z} \subset \mathscr{E}_-$ and so we only have to prove that $\mathscr{E}_- \subset s_0\mathscr{Z}$. For this let $\sigma \in S_-$ and $s \in \mathscr{A}^\sigma$. Then s_0^{-1} and s commute and so $s_0^{-1}s \in \mathscr{E}_+$. Thus $s \in s_0\mathscr{E}_+ = s_0\mathscr{Z}$.

(d) and (e) follow from (c).

(f) It suffices to show that $4S \subset \Gamma$. Let $\sigma \in S$. Then $\sigma_0 + 4\sigma \in S_- + 4S \subset S_-$, by Proposition 7.1(c). Thus, by (c), $\sigma_0 + 4\sigma = \sigma_0 + \tau$ for some $\tau \in \Gamma$. Hence $4\sigma = \tau \in \Gamma$. □

9. Structurable *n*-tori

We assume in this section that Λ is a free abelian group of rank n, and we suppose that $(\mathscr{A}, ^-)$ is a structurable Λ-torus. In other words, $(\mathscr{A}, ^-)$ is a structurable n-torus. We use the notation of the previous sections, and we assume that $S_- \neq \emptyset$ (except in our final summary Theorem 9.22).

Our plan in this section is to classify the algebras of class II in general, as well as the algebras of class I and III when $n=2$. (The case $n=1$ is done in Theorem 7.5(d).)

There are particular elementary quantum matrices and elementary vectors that occur frequently enough in this section to warrant special notation. We use the notation $\mathbf{1}$ for the $n \times n$ elementary quantum matrix all of whose entries are 1. If $n \geqslant 2$, we use the notation $\mathbf{h}$ for the $n \times n$-elementary quantum matrix (h_{ij}) with $h_{12} = h_{21} = -1$ and all other entries equal to 1. Finally if $\mathbf{q}$ is an elementary $n \times n$ quantum matrix we use the notation

$$\natural = j_{(-1,-1,\dots,-1)} \quad \text{and} \quad \# = j_{(-1,1,\dots,1)}$$

for the involutions of $F_{\mathbf{q}}$ determined by the elementary n-vectors

$$(-1,-1,\dots,-1) \quad \text{and} \quad (-1,1,\dots,1)$$

respectively. (See Example 4.4 for the definition of $j_{\mathbf{r}}$).

9.1. Class I

For any $n \geqslant 1$ we have the following:

Lemma 9.1. *Suppose that $(\mathscr{A}, ^-)$ is structurable n-torus of class I. Then*:

(a) *There exists a basis $\{\sigma_1, \dots, \sigma_n\}$ of Λ such that $\sigma_i \in S_-$.*
(b) *Let s_i be a nonzero element of degree σ_i for $1 \leqslant i \leqslant n$. Then $\mathscr{A}$ is generated as an algebra over F by $s_1, s_1^{-1}, \dots, s_n, s_n^{-1}$, and $\mathscr{A}$ is generated as an algebra over $\mathscr{Z}$ by $s_1, \dots, s_n$.*

Proof. Since $\Lambda = \Lambda_-$, (a) follows from Corollary 7.4(b). For (b), we have $s_i^{-1} \in \mathscr{A}^{-\sigma_i}$ and

$$L_{s_i}^{-1} = L_{s_i^{-1}}. \tag{9.2}$$

Hence the nonzero element

$$L_{s_1}^{k_1} \cdots L_{s_n}^{k_n} 1$$

has degree $k_1\sigma_1 + \cdots + k_n\sigma_n$ for all $k_1,\dots,k_n \in \mathbb{Z}$. Thus

$$\mathscr{A} = \operatorname{span}_F\{L_{s_1}^{k_1}\cdots L_{s_n}^{k_n}1 \mid k_1,\dots,k_n \in \mathbb{Z}\}. \tag{9.3}$$

Consequently by (9.2) and (9.3), $\mathscr{A}$ is generated by s_1, $s_1^{-1},\dots,s_n$, s_n^{-1} as an algebra over F. We have $z_i := s_i^2 \in \mathscr{Z}$ by Proposition 8.11. Then $s_i(z_i^{-1}s_i) = 1$ and $s_is_i^{-1} = 1$. So, since L_{s_i} is invertible, $s_i^{-1} = z_i^{-1}s_i$ for all $1 \leqslant i \leqslant n$, which shows (b). □

We can now prove:

Proposition 9.4. *Suppose that* $(\mathscr{A},^-)$ *is a structurable* 2*-torus of class I. Then* $(\mathscr{A},^-) \simeq (F_{\mathbf{q}},\natural)$ *as* Λ*-graded algebras with involution, where* $\mathbf{q} = \mathbf{1}$ *or* $\mathbf{h}$ *and* $F_{\mathbf{q}}$ *has a toral* Λ*-grading.*

Proof. Suppose that $n = 2$ and that σ_1,σ_2 and s_1,s_2 are as in Lemma 9.1(b). As we have mentioned earlier, O. Smirnov [22] has shown that any structurable algebra over a field of characteristic $\neq 2$ or 3 that is generated by 2 skew elements is associative. Moreover, the same argument, word for word, works over a commutative associative base ring containing $\frac{1}{6}$. Since $\mathscr{A}$ is generated as an algebra over $\mathscr{Z}$ by s_1,s_2, it follows that $\mathscr{A}$ is associative. Thus $\mathscr{A}$ is an associative algebra over F that is generated by s_1, s_1^{-1}, s_2, s_2^{-1}. Also, $\overline{s_1s_2} = \pm s_1s_2$. Hence $s_2s_1 = \pm s_1s_2$. So by Proposition 4.5, we have $(\mathscr{A},^-) = (F_{\mathbf{q}},\natural)$, where $\mathbf{q} = \mathbf{1}$ or $\mathbf{h}$ and $F_{\mathbf{q}}$ has the toral Λ-grading determined by $\{\sigma_1,\sigma_2\}$. □

9.2. Class II

Our general description of class II structurable n-tori is the following:

Theorem 9.5. *Suppose that* $(\mathscr{A},^-)$ *is a structurable* Λ*-torus of class II. Then* $(\mathscr{A},^-)$ *is isomorphic as a* Λ*-graded algebra with involution to the structurable* Λ*-torus* $(F_{\mathbf{q}} \oplus \mathscr{W},^-)$ *of a hermitian form* $\langle\langle a_1t_B^{2\rho_1},\dots,a_mt_B^{2\rho_m}\rangle\rangle$ *on a free module* $\mathscr{W} = \bigoplus_{i=1}^m F_{\mathbf{q}} \circ w_i$ *of rank* $m \geqslant 1$ *over the quantum* M*-torus with involution* $(F_{\mathbf{q}},\natural)$*, where* $\mathbf{q}$*,* M*,* B*,* $\rho_1,\dots,\rho_m$ *and* $a_1,\dots,a_m$ *satisfy the conditions* (a)–(d) *in Example* 4.6 (*with* $\mathbf{r} = (-1,\dots,-1)$).

Proof. Let $M = \Lambda_-$. Then, by Proposition 8.12, we have $2\Lambda \subset M \subset \Lambda$. Moreover, M is a proper subgroup of Λ since $\mathscr{W} \neq 0$.

By Corollary 8.5, $(\mathscr{E},^-)$ is an associative M-torus with involution. Choose a basis $B = \{\sigma_1,\dots,\sigma_n\}$ of M contained in S_- (see Corollary 7.4(b)). Then, by Proposition 4.5, there exist unique $\mathbf{q}$ and $\mathbf{r}$ so that $(\mathscr{E},^-) \cong (F_{\mathbf{q}}, j_{\mathbf{r}})$ as M-graded algebras with involution, where $F_{\mathbf{q}}$ has the toral M-grading determined by B. We use this isomorphism to identify $(\mathscr{E},^-) = (F_{\mathbf{q}}, j_{\mathbf{r}})$. Since $\sigma_1,\dots,\sigma_n \in S_-$, we have $(t_i)^{j_{\mathbf{r}}} = -t_i$ for all i and so $\mathbf{r} = (-1,-1,\dots,-1)$. Thus

$$(\mathscr{E},^-) = (F_{\mathbf{q}},\natural).$$

Also, by Proposition 8.4, $\mathscr{W}$ is an associative $\mathscr{E}$-module and χ is a nondegenerate hermitian form on $\mathscr{W}$. Moreover, since $\mathscr{W}\mathscr{W} \subset \mathscr{E}$ and since $we = \bar{e}w = e \circ w$ for

$e \in \mathscr{E}$ and $w \in \mathscr{W}$, we have $(e_1 + w_1)(e_2 + w_2) = e_1e_2 + w_1w_2 + e_1w_2 + w_1e_2 = e_1e_2 + \chi(w_2, w_1) + \bar{e}_1 \circ w_2 + e_2 \circ w_1$ for $e_1, e_2 \in \mathscr{E}$ and $w_1, w_2 \in \mathscr{W}$. Also since $\mathscr{W} \subset \mathscr{A}_+$, we have $\overline{e + w} = \bar{e} + w$ for $e \in \mathscr{E}$ and $w \in \mathscr{W}$. Therefore $(\mathscr{A}, ^-)$ is the structurable algebra of the hermitian form χ. It remains to describe the module $\mathscr{W}$ and the form χ precisely.

First, by Corollary 7.3(a), we have $M + S \subset S$. Hence (see Section 6) we may choose a cross-section $\{\rho_i\}_{i=0}^m$ of S relative to M, where $\rho_0 = 0$. Furthermore, since $\Lambda = \langle S \rangle$, we have $\Lambda = \langle M, \rho_1, \ldots, \rho_m \rangle$. Also, since $M \neq \Lambda$, we have $m \geqslant 1$.

Next, since $2\Lambda \subset M$, we have $2\rho_i \in M$ for all i. Also $2\rho_i \in S_+$ for all i by Proposition 7.1(c). So $(t_B^{2\rho_i})^\natural = t_B^{2\rho_i}$ and therefore $2\rho_1, \ldots, 2\rho_m \in S_{M,B}(\mathbf{q}, \natural)$.

If $1 \leqslant i \leqslant m$, then $\rho_i \in S \setminus M$ and so we may choose nonzero w_i so that $\mathscr{A}^{\rho_i} = Fw_i$. Then $w_i \in \mathscr{W}$. Furthermore it follows from Lemma 6.6 (with $\Lambda_0 = M$ and $\Lambda_1 = S \setminus M$) that $\mathscr{W}$ is a free $\mathscr{E}$-module with basis $\{w_1, \ldots, w_m\}$.

Finally $\chi(w_i, w_i) = w_i^2 \neq 0$ in $\mathscr{A}^{2\rho_i}$ for $1 \leqslant i \leqslant m$ (by Proposition 6.2(d)). But $\mathscr{A}^{2\rho_i} = Ft_B^{2\rho_i}$ and so $\chi(w_i, w_i) = a_i t_B^{2\rho_i}$, where $0 \neq a_i \in F$. On the other hand, if $1 \leqslant i \neq j \leqslant m$, then $\rho_i - \rho_j \notin M$. But $2\rho_j \in 2\Lambda \subset M$. Hence $\rho_i + \rho_j \notin M$ for $1 \leqslant i \neq j \leqslant m$. But $\chi(w_i, w_j) \in \mathscr{A}^{\rho_i + \rho_j} \cap \mathscr{E}$ and $\mathscr{E}$ has support M. Thus $\chi(w_i, w_j) = 0$ for $1 \leqslant i \neq j \leqslant m$. Therefore $\chi = \langle\langle a_1 t_B^{2\rho_1}, \ldots, a_m t_B^{2\rho_m} \rangle\rangle$. $\square$

The quantum Λ-torus with involution $(F_{\mathbf{h}}, \#)$ will play an important role in the rest of this paper (and we think also in the future development of the theory). This torus can be thought of as the toral analog of a finite dimensional quaternion algebra with nonstandard involution. One checks easily that $(F_{\mathbf{h}}, \#)$ (with any toral grading) is of class II and satisfies $(\Lambda : \Gamma) = 4$. We now see using Theorem 9.5 that these properties characterize $(F_{\mathbf{h}}, \#)$.

Corollary 9.6. *Suppose that $(\mathscr{A}, ^-)$ is a structurable n-torus of class II or III and suppose that $(\Lambda : \Gamma) = 4$. Then $n \geqslant 2$, $(\mathscr{A}, ^-)$ is of class II, and $(\mathscr{A}, ^-) \simeq (F_{\mathbf{h}}, \#)$ as Λ-graded algebras, where $F_{\mathbf{h}}$ has the toral Λ-grading determined by some basis B of Λ.*

Proof. We have $\Gamma \subset \Lambda_- \subset \Lambda$ by (8.9). Moreover $\Gamma \neq \Lambda_-$ and $\Lambda_- \neq \Lambda$. Thus $(\Lambda : \Lambda_-) = 2$ and $(\Lambda_- : \Gamma) = 2$.

If $n = 1$ we have $(\mathscr{A}, ^-) \simeq (F[t^{\pm 1}], \natural)$ by Theorem 7.5(d) and so $\Lambda_- = \Lambda$, a contradiction. Thus $n \geqslant 2$. Also, because $(\Lambda : \Lambda_-) = 2$, we have

$$(\Lambda \setminus \Lambda_-) + (\Lambda \setminus \Lambda_-) \subset \Lambda_-.$$

Hence $\mathscr{W}\mathscr{W} \subset \mathscr{E}$ and so $(\mathscr{A}, ^-)$ is of class II.

Next, since $(\Lambda_- : \Gamma) = 2$, we have $(\Lambda_- \setminus \Gamma) + (\Lambda_- \setminus \Gamma) \subset \Gamma$. Thus, if $\sigma, \tau \in S_-$, we have $\sigma + \tau \notin S_-$ and so $[\mathscr{A}^\sigma, \mathscr{A}^\tau] \subset \mathscr{A}^{\sigma+\tau} \cap \mathscr{A}_- = 0$. Consequently, by Proposition 8.1, $\mathscr{E}$ is commutative.

We now apply Theorem 9.5 to the class II torus $(\mathscr{A}, ^-)$. Since $(\Lambda : \Lambda_-) = 2$, the integer m appearing in the conclusion of this theorem is 1. Thus, by Example 2.1(d), $\mathscr{A}$ is obtained from $(\mathscr{E}, ^-)$ by means of the Cayley-Dickson process. Hence, since $\mathscr{E}$ is commutative, $\mathscr{A}$ is associative. Thus $(\mathscr{A}, ^-)$ is an associative torus with involution. Consequently also $S = \Lambda$.

Next since $(\Lambda:\Lambda_-)=2$ and $\Lambda=S$, we have $2(\Lambda\setminus\Lambda_-)\subset\Lambda_-\cap(2S)\subset\Lambda_-\cap S_+$ (by Proposition 7.1(c)). But $\Lambda_-\cap S_+\subset\Gamma$ since $(\Lambda_-:\Gamma)=2$. Therefore $2(\Lambda\setminus\Lambda_-)\subset\Gamma$. Hence Λ/Γ is not cyclic and therefore Λ/Γ is the Klein 4-group.

So we can choose a basis $B=\{\sigma_1,\dots,\sigma_n\}$ for Λ so that $\{2\sigma_1,2\sigma_2,\sigma_3,\dots,\sigma_n\}$ is a basis for Γ. Then, since $\Gamma+S_-\subset S_-$, we have $\{\sigma_1,\sigma_2,\sigma_1+\sigma_2\}\cap S_-\neq\emptyset$. Exchanging the roles of σ_1 and σ_2 if necessary we can assume that $\{\sigma_1,\sigma_1+\sigma_2\}\cap S_-\neq\emptyset$. Furthermore, replacing σ_1 by $\sigma_1+\sigma_2$ if necessary, we can assume that $\sigma_1\in S_-$. Then $\sigma_2\notin S_-$ and $\sigma_1+\sigma_2\notin S_-$ since $\Lambda_-\neq\Lambda$.

Now choose $t_i\in\mathscr{A}^{\sigma_i}$ for $i=1,\dots,n$. Then the elements $t_3,\dots,t_n$ are central. Also $\bar t_1=-t_1$ and $\bar t_i=t_i$ for $i\geqslant 2$. Finally, we have $0\neq t_1t_2\in\mathscr{A}^{\sigma_1+\sigma_2}$ and so, since $\sigma_1+\sigma_2\in S_+$, $t_1t_2=\overline{t_1t_2}=\bar t_2\bar t_1=-t_2t_1$. Thus we have identified $(\mathscr{A},^-)=(F_{\mathbf{h}},\#)$, where $F_{\mathbf{h}}$ has the toral B-grading. □

Using Corollary 9.6 we can give a very simple description of 2-tori of class II.

Corollary 9.7. *Suppose that $(\mathscr{A},^-)$ is a structurable 2-torus of class II. Then $(\mathscr{A},^-)\simeq(F_{\mathbf{h}},\#)$ as Λ-graded algebras with involution, where $F_{\mathbf{h}}$ has the toral Λ-grading determined by some basis B of Λ.*

Proof. We have the conclusion and the notation of Theorem 9.5. The elementary quantum matrix q has the form

$$\mathbf{q}=\begin{pmatrix}1 & q_{12}\\ q_{12} & 1\end{pmatrix},\quad \text{where } q_{12}=\pm1.$$

If $q_{12}=-1$, then the centre of $(F_{\mathbf{q}},\natural)$ is $F[t_1^{\pm2},t_2^{\pm2}]$ and so $S_{\Lambda_-,B}(\mathbf{q},\natural)=2\Lambda_-$. But $2\rho_1,\dots,2\rho_m\in S_{\Lambda_-,B}(\mathbf{q},\natural)$. Hence $2\rho_1,\dots,2\rho_m$ are all 0 modulo $2\Lambda_-$. So $\rho_1,\dots,\rho_m$ are all 0 modulo Λ_-, which is a contradiction. Hence $q_{12}=1$. Thus the centre of $(F_{\mathbf{q}},\natural)$ is generated by $t_1^{\pm2},t_2^{\pm2}$ and t_1t_2. Therefore $(\Lambda_-:\Gamma)=2$.

Next by Proposition 8.12, we have $2\Lambda\subset\Lambda_-\subset\Lambda$. But $\Lambda_-\neq\Lambda$ and $\Lambda_-\neq2\Lambda$ by Corollary 7.4(c). So, since $(\Lambda:2\Lambda)=4$, we have $(\Lambda:\Lambda_-)=2$. Thus $(\Lambda:\Gamma)=4$ and we're done by Corollary 9.6. □

9.3. Class III

We will shortly see that structurable tori constructed from hermitian forms over $(F_{\mathbf{h}},\#)$ give all structurable 2-tori of class III. We also expect that these tori will play an important role in description of tori of class III when $n\geqslant2$. For these reasons we prove the following proposition for general n that characterizes structurable tori constructed from hermitian forms over $(F_{\mathbf{h}},\#)$.

Proposition 9.8. *Suppose that $(\mathscr{A},^-)$ is a structurable n-torus of class III. Suppose that there exists a subgroup M of Λ satisfying the following conditions:*

(a) *$\Lambda_-\subset M\subset S$ and $(M:\Lambda_-)=2$*

(b) $\mathscr{A}^{S\setminus M}\mathscr{A}^{S\setminus M} \subset \mathscr{A}^M$

(c) *For each* $\sigma \in S \setminus M$ *there exists* $\tau \in S \setminus M$ *so that* $\mathscr{A}^\sigma \mathscr{A}^\tau \not\subset \mathscr{E}$.

Then $n \geqslant 2$ *and* $(\mathscr{A}, ^-)$ *is isomorphic as a* Λ*-graded algebra with involution to the structurable* Λ*-torus* $(F_{\mathbf{h}} \oplus \mathscr{V}, ^-)$ *of a hermitian form* $\langle\langle a_1 t_B^{2\rho_1}, \ldots, a_m t_B^{2\rho_m}\rangle\rangle$ *on a free module* $\mathscr{V} = \bigoplus_{i=1}^m F_{\mathbf{h}} \circ v_i$ *of rank* $m \geqslant 1$ *over the quantum* M*-torus with involution* $(F_{\mathbf{h}}, \#)$, *where* M, B, $\rho_1, \ldots, \rho_m$ *and* $a_1, \ldots, a_m$ *satisfy the conditions* (b)–(d) *in Example* 4.6 (*with* $\mathbf{q} = \mathbf{h}$ *and* $\mathbf{r} = (-1, 1, \ldots, 1)$).

Proof. Let $\mathscr{B} = \mathscr{A}^M$. By Proposition 6.4, $(\mathscr{B}, ^-)$ is a structurable M-torus.

By (8.9), $\Gamma \subset \Lambda_-$ and so

$$\Gamma \subset \Lambda_- \subset M.$$

So, by Proposition 8.6, we have $\Gamma = \Gamma(\mathscr{B}, ^-)$ and $\Lambda_- = \Lambda_-(\mathscr{B}, ^-)$. Thus $\Lambda_-(\mathscr{B}, ^-) = \Lambda_- \neq M$ and so $(\mathscr{B}, ^-)$ has class II or III. Also, by Proposition 8.13(e), $(\Lambda_- : \Gamma) = 2$ and so we have

$$(M : \Gamma) = 4.$$

Thus we can apply Corollary 9.6. This tells us that $n \geqslant 2$, $(\mathscr{B}, ^-)$ has class II and that we can identify

$$(\mathscr{B}, ^-) = (F_{\mathbf{h}}, \#)$$

as M-graded algebras, where $F_{\mathbf{h}}$ has the toral M-grading determined by some basis B of M. In particular, $\mathscr{B}$ is associative.

Next let $\mathscr{V} = \mathscr{A}^{S\setminus M}$. Then,

$$\mathscr{A} = \mathscr{B} \oplus \mathscr{V}$$

and

$$\mathscr{B}\mathscr{B} \subset \mathscr{B}, \quad \mathscr{B}\mathscr{V} + \mathscr{V}\mathscr{B} \subset \mathscr{V} \quad \text{and} \quad \mathscr{V}\mathscr{V} \subset \mathscr{B}.$$

(The last inclusion is assumption (b).) Also, let $\mathscr{F} = \mathscr{A}^{M\setminus\Lambda_-}$ in which case we have

$$\mathscr{B} = \mathscr{E} \oplus \mathscr{F} \quad \text{and} \quad \mathscr{F} \subset \mathscr{A}_+.$$

We now claim that if u is a nonzero homogeneous element of $\mathscr{V}$, there exists a nonzero homogeneous element $w \in \mathscr{V}$ so that

$$uw \notin \mathscr{E} \tag{9.9}$$

and

$$D_{u^2,w} = 0. \tag{9.10}$$

Indeed, if $u^2 \in \mathscr{Z}$, then (9.10) holds for all $w \in \mathscr{V}$ and so the claim follows from our assumption (c). Thus we can assume that $u^2 \notin \mathscr{Z}$. Then $u^2 \notin \mathscr{E}$ by Proposition 8.13(b). Setting $w = u$ we have $D_{u^2,w} = D_{u^2,u} = 0$ by (2.5). This proves the claim.

We next prove that

$$bv = v\bar{b} \tag{9.11}$$

$$b(uv) = u(\bar{b}v) \tag{9.12}$$

$$b_1(b_2u) = (b_2b_1)u \tag{9.13}$$

for $b, b_1, b_2 \in \mathscr{B}$ and $u, v \in \mathscr{V}$.

For (9.11), observe that $bv \in \mathscr{V} \subset \mathscr{A}_+$ and so $bv = \overline{bv} = \bar{v}\bar{b} = v\bar{b}$.

To prove (9.12) we can assume that b, u and v are homogeneous. If $b \in \mathscr{E}$ then (9.12) follows from Proposition 8.3(c). Thus we can assume that $b \in \mathscr{F}$. We can also assume that $u \neq 0$. Choose an nonzero homogeneous element $w \in \mathscr{V}$ so that (9.9) and (9.10) hold. By (2.5) we have $D_{uw,u}+D_{u^2,w}+D_{wu,u}=0$ and $D_{wu,u}=D_{\overline{wu},\bar{u}}=D_{\bar{u}\bar{w},\bar{u}}=D_{uw,u}$. Consequently (9.10) implies that $D_{uw,u} = 0$. Set

$$f = uw.$$

Then $D_{f,u} = 0$ and, by (9.9), f is a nonzero homogeneous element of $\mathscr{F}$. But by (9.11) we have $uf = fu$, and hence, by (2.6), we have $[L_f, L_u] = D_{f,u} = 0$. So

$$f(uv) - u(fv) = 0. \tag{9.14}$$

Finally, since $M \setminus \Lambda_-$ is a coset of Λ_- in Λ, it follows from Lemma 6.6 (with $\Lambda_0 = \Lambda_-$ and $\Lambda_1 = M \setminus \Lambda_-$) that $\mathscr{F} = \mathscr{E}f$. Hence $b = ef$ for some $e \in \mathscr{E}$. Thus we have

$$\begin{aligned}
b(uv) &= (ef)(uv) \\
&= e(f(uv)) \quad (\text{since } \mathscr{B} \text{ is associative and } uv \in \mathscr{B}) \\
&= e(u(fv)) \quad (\text{by } (9.14)) \\
&= u(\bar{e}(fv)) \quad (\text{by Proposition 8.3(c)}) \\
&= u((fv)e) \quad (\text{by Proposition 8.3(a)}) \\
&= u((f\bar{e})v) \quad (\text{by Proposition 8.3(c)}) \\
&= u(\bar{b}v),
\end{aligned}$$

and so (9.12) holds.

To prove (9.13) we can assume that b_1, b_2 and u are homogeneous. Let $w = b_1(b_2u) - (b_2b_1)u$. Then

$$w^2 = w(b_1(b_2u)) - w((b_2b_1)u)) = \bar{b}_1(\bar{b}_2(wu)) - (\bar{b}_1\bar{b}_2)(wu)$$

by (9.12). Since $\mathscr{B}$ is associative and $wu \in \mathscr{B}$ it follows that $w^2 = 0$. Because w is homogeneous we have $w = 0$. Hence (9.13) holds.

We now define an action of $\mathscr{B}$ on $\mathscr{V}$ by $b \circ v := \bar{b}v$ for $b \in \mathscr{B}$ and $v \in \mathscr{V}$. It follows from (9.13) that $\mathscr{V}$ is an associative $\mathscr{B}$-module under this action. Further define $\theta : \mathscr{V} \times \mathscr{V} \to \mathscr{B}$ by $\theta(u,v) = vu$. Then, by (9.12), θ is a hermitian form on $\mathscr{V}$ over $(\mathscr{B}, ^-)$. Using (9.11) it follows that $(\mathscr{A}, ^-)$ is the structurable algebra of θ. So, as in the proof of Theorem 9.5, all that remains is to describe the module $\mathscr{V}$ and the form θ precisely.

If b is a nonzero homogeneous element of $\mathscr{B}$ then it follows from (9.13) that L_b is invertible with inverse $L_{b^{-1}}$. So we have $M + S \subset S$. Thus we may choose a cross-section $\{\rho_i\}_{i=0}^m$ of S relative to M, where $\rho_0 = 0$. Then $\Lambda = \langle M, \rho_1, \ldots, \rho_m \rangle$. Also, since $(\mathscr{A}, ^-)$ has class III, $M \neq \Lambda$ and so $m \geqslant 1$.

Next, since $\mathscr{V}\mathscr{V} \subset \mathscr{B}$, we have $0 \neq v^2 \in \mathscr{B}$ for all nonzero homogeneous elements of $\mathscr{V}$. Thus $2(S \setminus M) \subset M$ and so $2S \subset M$. Therefore

$$2\Lambda \subset M \subset \Lambda.$$

Consequently, arguing as in the proof of Theorem 9.5, we obtain $2\rho_1, \ldots, 2\rho_m \in S_{M,B}(\mathbf{h}, \#)$.

For $1 \leqslant i \leqslant m$, we choose nonzero v_i so that $\mathscr{A}^{\rho_i} = F v_i$. Then, by Lemma 6.6 (with $\Lambda_0 = M$ and $\Lambda_1 = S \setminus M$), we see that $\mathscr{V}$ is a free $\mathscr{B}$-module with basis $\{v_1, \ldots, v_m\}$. Also, as in the proof of Theorem 9.5, we see that there exists nonzero scalars $a_1, \ldots, a_m$ so that $\theta(v_i, v_j) = \delta_{ij} a_i t_B^{2\rho_i}$ for $i = 1, \ldots, m$. Hence $\theta = \langle\langle a_1 t_B^{2\rho_1}, \ldots, a_m t_B^{2\rho_m} \rangle\rangle$. $\square$

Remark 9.15. Suppose that $(\mathscr{A}, ^-)$ is a structurable n-torus of class III. Suppose that there exists a subgroup M of Λ satisfying assumption (a) in the theorem. To show that assumptions (b) and (c) hold, it is sufficient to show the following:

(b)$'$ $((S \setminus M) + (S \setminus M)) \cap S \subset M$

(c)$'$ $(2(S \setminus M)) \cap \Gamma = \emptyset$.

Indeed (b)$'$ clearly implies (b). On the other hand (c)$'$ implies (c). To see this suppose that (c)$'$ holds and that $0 \neq u \in \mathscr{A}^\sigma$, where $\sigma \in S \setminus M$. Then $2\sigma \notin \Gamma$ by (c)$'$ and so $u^2 \notin \mathscr{Z}$. Thus $u^2 \notin \mathscr{E}$ by Proposition 8.13(b) and so we can take $\tau = \sigma$ in (c).

The advantage of (b)$'$ and (c)$'$ over (b) and (c) is that (b)$'$ and (c)$'$ can be checked knowing only information about support sets.

We can now use Proposition 9.8 to describe structurable 2-tori of class III.

Proposition 9.16. *Suppose that $(\mathscr{A}, ^-)$ is a structurable 2-torus of class III. Then $(\mathscr{A}, ^-)$ is isomorphic as a Λ-graded algebra with involution to the structurable Λ-torus $(F_\mathbf{h} \oplus \mathscr{V}, ^-)$ of a hermitian form $\langle\langle t_B^{2\rho_1}, \ldots, t_B^{2\rho_m} \rangle\rangle$ on a free module $\mathscr{V} = \bigoplus_{i=1}^m F_\mathbf{h} \circ v_i$ of rank m over the quantum M-torus with involution $(F_\mathbf{h}, \#)$, where M is a subgroup of Λ, $B = \{\sigma_1, \sigma_2\}$ is a basis for M, and the elements $\rho_1, \ldots, \rho_m$ satisfy $\Lambda = \langle M, \rho_1, \ldots, \rho_m \rangle$ and either*

(i) *$m = 1$ and $2\rho_1 = \sigma_2$ or*

(ii) *$m = 2$, $2\rho_1 = \sigma_2$ and $2\rho_2 = \sigma_1 + \sigma_2$.*

Proof. First of all, by Proposition 8.13(f), we have

$$4\Lambda \subset \Gamma \subset \Lambda_- \subset \Lambda. \tag{9.17}$$

Hence $(\Lambda:\Gamma)$ is a divisor of 16. But also

$$(\Lambda_-:\Gamma)=2 \tag{9.18}$$

by Proposition 8.13(e), and $(\Lambda:\Lambda_-)\neq 1$ or 2 since $(\mathscr{A},^-)$ has class III. Thus $(\Lambda:\Gamma)=$ 8 or 16. Since Λ/Γ has 2 generators and exponent 4, we have

$$\Lambda/\Gamma \simeq \begin{cases} \mathbb{Z}/(2)\oplus\mathbb{Z}/(4) & \text{if } (\Lambda:\Gamma)=8 \\ \mathbb{Z}/(4)\oplus\mathbb{Z}/(4) & \text{if } (\Lambda:\Gamma)=16. \end{cases} \tag{9.19}$$

Now let

$$M=\{\sigma\in\Lambda: 2\sigma\in\Gamma\}.$$

So M/Γ is the subgroup of Λ/Γ consisting of the elements of Λ/Γ of period 2. Then from (9.19) we have

$$(M:\Gamma)=4. \tag{9.20}$$

We claim next that

$$M=\Lambda_-+2\Lambda. \tag{9.21}$$

Indeed we have $\Lambda_-\subset M$ by (9.18), and $2\Lambda\subset M$ since $4\Lambda\subset\Gamma$. Thus $\Lambda_-+2\Lambda\subset M$. Hence for (9.21) it suffices to show that $(\Lambda_-+2\Lambda)/\Gamma$ has order at least 4. Suppose that contrary. Then $(\Lambda_-+2\Lambda)/\Gamma$ has order 2 and so $(\Lambda_-+2\Lambda)/\Gamma=\Lambda_-/\Gamma$. Hence $2\Lambda\subset\Lambda_-$. But $2\Lambda\neq\Lambda_-$ by Corollary 7.4(c), and hence Λ_- has index 1 or 2 in Λ which contradicts our assumption that $(\mathscr{A},^-)$ has class III. So we have (9.21).

Now $\Lambda_-+S\subset S$ and $2\Lambda+S\subset S$ by Corollary 7.3(a). So by (9.21) we have

$$M+S\subset S.$$

Thus S is a union of the cosets of M in Λ including the trivial coset M.

In order to apply Proposition 9.8, we next show that M satisfies the following conditions:

(a) $\Lambda_-\subset M\subset S$ and $(M:\Lambda_-)=2$
(b)′ $((S\setminus M)+(S\setminus M))\cap S\subset M$
(c)′ $(2(S\setminus M))\cap\Gamma=\emptyset$.

(See Remark 9.15.) Indeed (a) follows from (9.18), (9.20) and (9.21). Also by definition of M we have $(2(\Lambda\setminus M))\cap\Gamma=\emptyset$ and so (c′) holds. To prove (b′), suppose first that $(\Lambda:\Gamma)=8$. Then $(\Lambda:M)=2$, and so $(\Lambda\setminus M)+(\Lambda\setminus M)\subset M$, which implies (b′). Suppose second that $(\Lambda:\Gamma)=16$. Then by (9.17) we have $\Gamma=4\Lambda$ and so $M=2\Lambda$. If $S=\Lambda$, then $S_-\subset M=2S$, which contradicts the fact that $S_-\cap(2S)=\emptyset$ (see Proposition 7.1(c)). So $S\neq\Lambda$. Thus S is the union of exactly three cosets of $M=2\Lambda$ in Λ, and therefore $S\setminus M$ is the union of exactly two cosets of M in Λ, and so we have (b)′.

Thus we have the conclusion of Proposition 9.8 with $n=2$. Denote the elements of the basis B of M by σ_1 and σ_2. Then

$$S_{M,B}(F_{\mathbf{h}},\#)=2M\cup(\sigma_2+2M)\cup(\sigma_1+\sigma_2+2M).$$

But by (c) in Example 4.6, the elements $2\rho_1, \ldots, 2\rho_m$ are distinct and nonzero elements of $2M$. Hence, by Remark 4.9, we may assume that $m=1$ and $2\rho_1=\sigma_2$, or that $m=1$ and $2\rho_1=\sigma_1+\sigma_2$, or that $m=2$, $2\rho_1=\sigma_2$ and $2\rho_2=\sigma_1+\sigma_2$. Suppose first that $m=1$ and $2\rho_1=\sigma_2$. Then $v_1^2=a_1 t_B^{2\rho_1}=a_1 t^{\sigma_2}=a_1 t_2$. Replacing the generator t_2 of $F_{\mathbf{h}}$ by $a_1 t_2$, we may assume that $a_1=1$ and so we are done (in particular conclusion (i) holds). Suppose next that $m=1$ and $2\rho_1=\sigma_1+\sigma_2$. Then replacing the generator t_2 of F_h by $a_1 t_1 t_2$ and replacing the basis B of M by $\{\sigma_1, \sigma_1+\sigma_2\}$, we are again done (with conclusion (i) again). Finally suppose that $m=2$, $2\rho_1=\sigma_2$ and $2\rho_2=\sigma_1+\sigma_2$. Then replacing the generators t_1 and t_2 by $\frac{a_2}{a_1}t_1$ and $a_1 t_2$ respectively, we may assume that $a_1=1$ and $a_2=1$ and we are done (with conclusion (ii)). $\square$

Putting together the results of this section we obtain our main result that classifies structurable 2-tori. (See the beginning of this section for the notation.)

Theorem 9.22. *Let $(\mathscr{A},^-)$ be a structurable Λ-torus, where Λ is a free abelian group of rank 2. Then $(\mathscr{A},^-)$ is a Jordan Λ-torus with identity involution, or $(\mathscr{A},^-)$ is isomorphic as Λ-graded algebras with involution to one of the following five structurable tori*:

(a) $(F_{\mathbf{1}}, \natural)$ *with a toral Λ-grading*
(b) $(F_{\mathbf{h}}, \natural)$ *with a toral Λ-grading*
(c) $(F_{\mathbf{h}}, \#)$ *with a toral Λ-grading*
(d) *The structurable torus $(F_{\mathbf{h}} \oplus (F_{\mathbf{h}} \circ v),^-)$ of the hermitian form $\langle\langle t_2\rangle\rangle$ on the free module $F_{\mathbf{h}} \circ v$ of rank 1 over $(F_{\mathbf{h}}, \#)$, where the Λ-grading on $F_{\mathbf{h}} \oplus (F_{\mathbf{h}} \circ v)$ satisfies* $\deg(t_1)=\tau_1$, $\deg(t_2)=2\tau_2$ *and* $\deg(v)=\tau_2$ *for some basis $\{\tau_1,\tau_2\}$ of Λ.*
(e) *The structurable torus $(F_{\mathbf{h}} \oplus (F_{\mathbf{h}} \circ v_1) \oplus (F_{\mathbf{h}} \circ v_2),^-)$ of the hermitian form $\langle\langle t_2, t_1 t_2\rangle\rangle$ on the free module $(F_{\mathbf{h}} \circ v_1) \oplus (F_{\mathbf{h}} \circ v_2)$ of rank 2 over $(F_{\mathbf{h}}, \#)$, where the Λ-grading on $F_{\mathbf{h}} \oplus (F_{\mathbf{h}} \circ v_1) \oplus (F_{\mathbf{h}} \circ v_2)$ satisfies* $\deg(t_1)=2\tau_1$, $\deg(t_2)=2\tau_2$, $\deg(v_1)=\tau_2$ *and* $\deg(v_2)=\tau_1+\tau_2$ *for some basis $\{\tau_1,\tau_2\}$ of Λ.*

The tori in (a) *and* (b) *are of class I, the tori in* (c) *are of class II and the tori in* (d) *and* (e) *are of class III.*

Proof. If $(\mathscr{A},^-)$ has class I, then we have (a) or (b) by Proposition 9.4. If $(\mathscr{A},^-)$ has class II, we have (c) by Corollary 9.7. Finally, suppose that $(\mathscr{A},^-)$ has class III. Then we have the conclusion of Proposition 9.16. If $m=1$ and $2\rho_1=\sigma_2$, we let $\tau_1=\sigma_1$ and $\tau_2=\frac{1}{2}\sigma_2$, and we have (d). If $m=2$, $2\rho_1=\sigma_2$ and $2\rho_2=\sigma_1+\sigma_2$, we let $\tau_1=\frac{1}{2}\sigma_1$ and $\tau_2=\frac{1}{2}\sigma_2$, and we have (e). $\square$

Remark 9.23. Suppose that $F=\mathbb{C}$ and that $\mathscr{L}$ is an EALA of type BC_1 and nullity 2 over F (see Section 5). Let $\mathscr{K}=\mathscr{C}/\mathscr{Z}(\mathscr{C})$, where $\mathscr{C}$ is the core of $\mathscr{L}$. By Theorem 5.6, Remark 5.11 and Theorem 9.22, we know that $\mathscr{K}=K(\mathscr{A},^-)$ as Λ-graded Lie algebras where Λ is a free abelian group of rank 2 and $(\mathscr{A},^-)$ is one of the five structurable tori listed in (a)–(e) of Theorem 9.22. Using this fact and the results of [7], it is not difficult to calculate the 5 possible root system of $\mathscr{L}$. It turns out that the root systems of EALAs of type BC_1 and nullity 2 are precisely the same as the extended affine root systems of type BC_1 and nullity 2 (which are classified in [5, II, Table II.4.77]).

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