# LIE $G$-TORI

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Abstract. We explain how the notion of Lie $G$-tori appeared naturally from finite-dimensional simple isotropic Lie algebras. Then we discuss the classification of Lie $G$-tori of type $C_\ell$.

Throughout this report, let $F$ be a field of characteristic 0.

## §1 finite-dimensional simple isotropic Lie algebras

The details of this section can be found in [S] and [ABG].

Let $L$ be a finite-dimensional Lie algebra. If there exists a nonzero ad-nilpotent element in $L$, then, by the theorem of Jacobson-Morozov, there exists a nonzero abelian ad-diagonalizable subalgebra, called a *split torus*. Such a Lie algebra is called *isotropic*.

Let $L$ be a finite-dimensional simple isotropic Lie algebra and let $\mathfrak{h}$ be a maximal split torus of $L$. Then $L$ decomposes into the root spaces, i.e.,

$$L = \bigoplus_{\mu \in \mathfrak{h}^*} L_\mu = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu,$$

where $\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$, $L_\mu = \{x \in L \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$, and

$$\Delta = \{\mu \in \mathfrak{h}^* \mid L_\mu \neq 0,\ \mu \neq 0\}$$

is called the set of *roots*. We note that $\mathfrak{h} \subset L_0$ and they are not equal in general. Also, $\dim_F L_\mu \geq 1$ for all $\mu \in \Delta$, but the equality $\dim_F L_\mu = 1$ does not hold in general. Thus the structure for $L$ is much more complicated than the one for finite-dimensional **split** simple Lie algebras. However, the set $\Delta$ becomes almost same as in the case for finite-dimensional split simple Lie algebras. Namely, $\Delta$ becomes a finite irreducible root system, i.e., $\mathrm{A}_\ell$ ($\ell \geq 1$), $\mathrm{B}_\ell$ ($\ell \geq 2$), $\mathrm{C}_\ell$ ($\ell \geq 2$), $\mathrm{D}_\ell$ ($\ell \geq 4$), $\mathrm{E}_\ell$ ($\ell = 6, 7, 8$), $\mathrm{F}_4$, $\mathrm{G}_2$, and $\mathrm{BC}_\ell$ ($\ell \geq 1$). So the only difference is the appearance of type $\mathrm{BC}_\ell$. ($\Delta$ is called reduced if $\Delta \neq \mathrm{BC}_\ell$.) We say that $L$ has relative type $\Delta$. Moreover, there is a very strong property which finite-dimensional split simple Lie algebras possess:

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**Division Property.** *For any $\mu \in \Delta$ and $0 \neq x \in L_\mu$, there exist $y \in \Delta_{-\mu}$ and $h \in \mathfrak{h}$ such that $[h, x] = 2x$, $[h, y] = -2y$ and $[x, y] = h$.*

Using this property and Serre's Theorem, one can show that there exists a split simple Lie subalgebra $\mathfrak{g}$ of $L$ containing $\mathfrak{h}$ as a split Cartan subalgebra of type $\Delta^{\text{red}}$, where

$$\Delta^{\text{red}} := \begin{cases} \Delta & \text{if } \Delta \text{ is reduced} \\ \{\mu \in \Delta \mid \mu \text{ is reduced, i.e., } \frac{1}{2}\mu \notin \Delta\} & \text{otherwise, i.e., if } \Delta = \mathrm{BC}_l. \end{cases}$$

In particular, $L$ is a so-called $\Delta$-graded Lie algebra with grading pair $(\mathfrak{g}, \mathfrak{h})$. For each $\Delta$, there is a certain way of constructing a $\Delta$-graded Lie algebra, which generalizes a construction of a finite-dimensional split simple Lie algebra.

For example, one can construct a $\Delta$-graded Lie algebra $sl_{\ell+1}(A)$ of type $\mathrm{A}_\ell$ from a unital associative algebra $A$, or a $\Delta$-graded Lie algebra $sp_{2\ell}(A, \sigma)$ of type $\mathrm{C}_\ell$ from a unital associative algebra $A$ with involution $\sigma$. The following are well-known:
(1) $sl_{\ell+1}(A)$ is a finite-dimensional simple Lie algebra of relative type $\mathrm{A}_\ell \Leftrightarrow A$ is a finite-dimensional division algebra.
(2) $sp_{2\ell}(A, \sigma)$ is a finite-dimensional simple Lie algebra of relative type $\mathrm{C}_\ell \Leftrightarrow A$ is a finite-dimensional division algebra with involution $\sigma$ (anti-automorphism of period 2).

## §2 Lie $G$-tori

The details of this section can found in [Y2], [Y3], and [ABG].

Among infinite-dimensional Lie algebras, the loop algebra $sl_{\ell+1}(F[t^{\pm 1}])$ is probably the most understandable algebra and has many applications. Note that $sl_{\ell+1}(F[t^{\pm 1}])$ has an $\mathrm{A}_\ell$-grading and also has a $\mathbb{Z}$-grading. We consider a generalization of both finite-dimensional isotropic simple Lie algebras and loop algebras. We note that the division property holds for homogeneous elements in the double-graded algebra $sl_{\ell+1}(F[t^{\pm 1}])$.

**Definition.** Let $G$ be an abelian group.

(1) A $\Delta$-graded Lie algebra $L = \oplus_{\mu \in \Delta \cup \{0\}} L_\mu$ with grading pair $(\mathfrak{g}, \mathfrak{h})$ is called $(\Delta, G)$-*graded* if $L = \oplus_{g \in G} L^g$ is a $G$-graded Lie algebra such that $\mathfrak{g} \subset L^0$. Then since $L^g$ is an $\mathfrak{h}$-submodule of $L$, we have
$$L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{g \in G} L_\mu^g,$$
where $L_\mu^g = L_\mu \cap L^g$. For convenience, we always assume that
$$\operatorname{supp} L := \{g \in G \mid L^g \neq 0\} \text{ generates } G.$$

(2) Let $Z(L)$ be the centre of $L$ and let $\mu^\vee \in \mathfrak{h}$ for $\mu \in \Delta$ be the coroot of $\mu$. Then $L$ is called a *division* $(\Delta, G)$*-graded Lie algebra* if for any $\mu \in \Delta$ and any $0 \neq x \in L_\mu^g$,
$$\text{there exists } y \in L_{-\mu}^{-g} \text{ such that } [x, y] \equiv \mu^\vee \text{ modulo } Z(L). \quad \text{(division property)}$$

(3) A division $(\Delta, G)$-graded Lie algebra $L = \oplus_{\mu \in \Delta \cup \{0\}} \oplus_{g \in G} L_\mu^g$ is called a *Lie $G$-torus of type* $\Delta$ if
$$\dim_F L_\mu^g \leq 1 \text{ for all } g \in G \text{ and } \mu \in \Delta. \quad \text{(1-dimensionality)}$$
If $G = \mathbb{Z}^n$, it is called a *Lie $n$-torus* or simply a *Lie torus.*

**Example.** Let $F[G]$ be a group algebra. Then $sl_{\ell+1}(F[G])$ is a Lie $G$-torus. More generally, if $F^t[G]$ is a twisted group algebra, then $sl_{\ell+1}(F^t[G])$ is a Lie $G$-torus.

When $G = \mathbb{Z}^n$, there is a more concrete description for $F^t[G]$, called a *quantum torus*: An $n\times n$ matrix $\boldsymbol{q} = (q_{ij})$ over $F$ such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ is called a *quantum matrix.* The *quantum torus* $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \dots , t_n^{\pm 1}]$ *determined by* $\boldsymbol{q}$ is defined as the associative algebra over $F$ with $2n$ generators $t_1^{\pm 1}, \dots , t_n^{\pm 1}$, and relations $t_i t_i^{-1} = t_i^{-1} t_i = 1$ and $t_j t_i = q_{ij} t_i t_j$ for all $1 \leq i, j \leq n$. Note that $F_{\boldsymbol{q}}$ is commutative if and only if $\boldsymbol{q} = \mathbf{1}$ where all the entries of $\mathbf{1}$ are 1. In this case, the quantum torus $F_{\mathbf{1}}$ becomes the algebra $F[t_1^{\pm 1}, \dots , t_n^{\pm 1}]$ of Laurent polynomials. Also, $F^t[\mathbb{Z}^n] \cong F_{\boldsymbol{q}}$ for some $\boldsymbol{q}$, and $F[\mathbb{Z}^n] \cong F_{\mathbf{1}}$.

We have two results corresponding to the ones in §1.
(1) Any Lie $G$-torus of type $\mathrm{A}_\ell$ for $\ell \geq 3$ is centrally isogeneous to $sl_{\ell+1}(F^t[G])$.
(2) Any Lie $G$-torus of type $\mathrm{C}_\ell$ for $\ell \geq 4$ is centrally isogeneous to $sp_{2\ell}(F^t[G], \sigma)$, where $\sigma$ is a graded involution of $F^t[G]$.

In the classification of extended affine Lie algebras, the classification of Lie tori (i.e., $G = \mathbb{Z}^n$) becomes the central issue (see [A-P] or [AG]). Thus we need the classification of $(F_{\boldsymbol{q}}, \sigma)$.

## §3 Classification of $(F_{\boldsymbol{q}}, \sigma)$

The details of this section can be found in [Y1].

The existence of a graded involution on $F_{\boldsymbol{q}}$ forces the quantum matrix $\boldsymbol{q}$ to be elementary, i.e., $q_{ij} = \pm 1$. Moreover, there exists $l \geq 0$ such that $F_{\boldsymbol{q}} \cong F_{\boldsymbol{h}_{l,n}}$ where

$$\boldsymbol{h}_{l,n} = \overbrace{\boldsymbol{h} \times \cdots \times \boldsymbol{h}}^{l\text{-times}} \times \mathbf{1}_{n-2l} \quad \text{and} \quad \boldsymbol{h} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and the product $\times$ is defined as follows:

For square matrices $A_1, \dots, A_r$ of sizes $l_i$, $i = 1, \dots , r$, we define the square matrix $A_1 \times \cdots \times A_r$ of size $l_1 + \cdots + l_r$ to be

$$A_1 \times \cdots \times A_r = \begin{pmatrix} A_1 & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{1} & A_2 & \mathbf{1} & & \vdots \\ \mathbf{1} & \mathbf{1} & A_3 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{1} \\ \mathbf{1} & \cdots & \cdots & \mathbf{1} & A_r \end{pmatrix},$$

where the $\mathbf{1}$'s are matrices of suitable sizes whose entries are all 1.

For the classification of involutions, first note that an involution $\sigma$ on a quantum torus $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \dots , t_n^{\pm 1}]$ is determined by a vector $(a_1, \dots , a_n)$, where $a_i = \pm 1$, so that $\sigma(t_i) = a_i t_i$ for all $1 \leq i \leq n$. So we write $\sigma = (a_1, \dots , a_n)$. Also, $* := (1, \dots , 1)$ is called the *main involution.* Note that $*$ is the identity map if and only if $\boldsymbol{q} = \mathbf{1}$.

Using this notation, we can state the classification. Namely, we have

$$(F_{\boldsymbol{q}}, \sigma)) \cong (F_{\boldsymbol{h}_{l,n}}, *), \ (F_{\boldsymbol{h}_{l,n}}, \sigma_1), \quad \text{or} \quad (F_{\boldsymbol{h}_{l,n}}, \sigma_2),$$

where

$$\begin{aligned} &\sigma_1 = (1, \dots, 1, -1, 1, \dots, 1), \\ &\quad \text{only the } 2l+1 \text{ position is } -1, \text{ provided } n - 2l \geq 1, \\ &\sigma_2 = (1, \dots, 1, -1, -1, 1, \dots, 1), \\ &\quad \text{only the } 2l-1 \text{ and } 2l \text{ positions are } -1, \text{ provided } l \geq 1. \end{aligned}$$

For example, if $n = 1$, then $F_{\boldsymbol{q}} = F[t^{\pm 1}]$, and $\sigma$ has two choices, i.e., $\sigma = * = (1)$ (identity map) or $\sigma_1 = (-1)$. Note that $sp_{2\ell}(F[t^{\pm 1}], *) \cong sp_{2\ell}(F) \otimes_F F[t^{\pm 1}]$ is the untwisted loop algebra of type $\mathrm{C}_\ell^{(1)}$, and $sp_{2\ell}(F[t^{\pm 1}], \sigma_1)$ is the twisted loop algebra of type $\mathrm{A}_{2\ell-1}^{(2)}$ (Kac's label) or $\mathrm{C}_\ell^{(2)}$ (Moody's label, which is more natural in our point of view). The twisting comes from a nontrivial involution of $F_{\boldsymbol{q}}$ in our context.

If $n = 2$, we already have 3 nontrivial twistings (and 1 trivial one). Namely, they are

$$(F[t_1^{\pm 1}, t_2^{\pm 1}], \sigma_1), \ (F_{\boldsymbol{h}}[t_1^{\pm 1}, t_2^{\pm 1}], *), \quad \text{or} \quad (F_{\boldsymbol{h}}[t_1^{\pm 1}, t_2^{\pm 1}], \sigma_2)$$

(and $(F[t_1^{\pm 1}, t_2^{\pm 1}], *)$). Note that $* = (1,1)$, $\sigma_1 = (1, -1)$, and $\sigma_2 = (-1, -1)$. As far as the author knows, the three twisted double-loop algebras

$$sp_{2\ell}(F[t_1^{\pm 1}, t_2^{\pm 1}], \sigma_1), \ sp_{2\ell}(F_{\boldsymbol{h}}[t_1^{\pm 1}, t_2^{\pm 1}], *), \quad \text{and} \quad sp_{2\ell}(F_{\boldsymbol{h}}[t_1^{\pm 1}, t_2^{\pm 1}], \sigma_2)$$

have not been studied at all. Note that $sp_{2\ell}(F[t_1^{\pm 1}, t_2^{\pm 1}], *) \cong sp_{2\ell}(F) \otimes_F F[t_1^{\pm 1}, t_2^{\pm 1}]$ is the untwisted double-loop algebra, or equivalently, a toroidal Lie algebra of type $\mathrm{C}_\ell$.

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[Y2] and [Y3] can be obtained from http://mathematik.uibk.ac.at/mathematik/jordan/